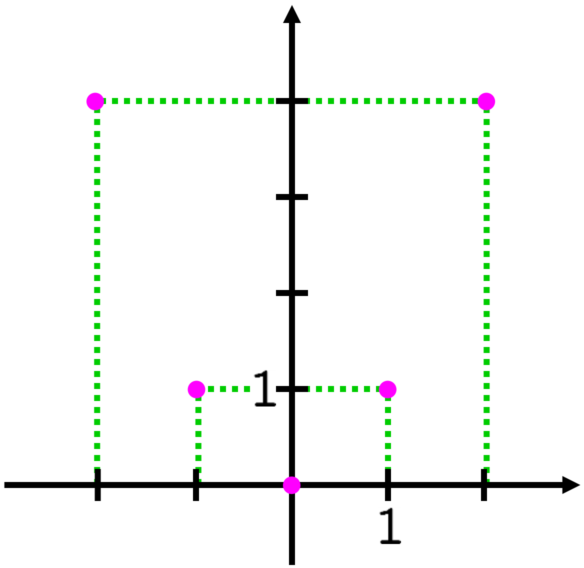
CALCULUS
Elementary graphing

Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function.

The **graph of** f is $\{(x, f(x)) \,|\, x \in D\}$;
it is a subset of $\mathbb{R}^2$, and so can be visualized as a subset of a (coordinatized) plane.

e.g.: For $f(x) = x^2$, the points
$(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 4)$
are all "on" the graph of f, *viz.*:
"elements of"

coordinatized plane:



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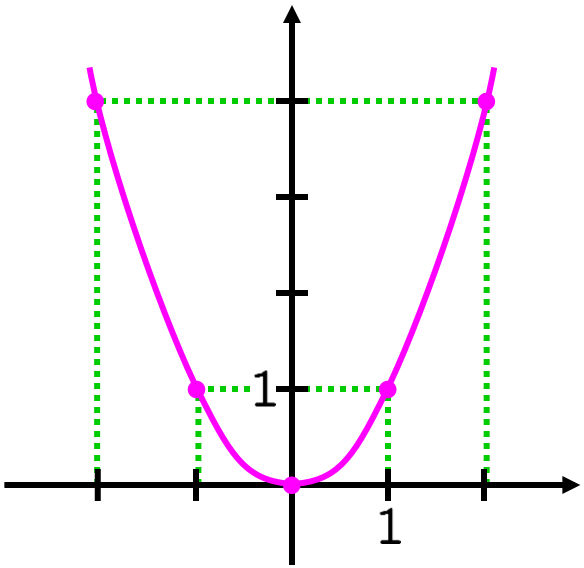

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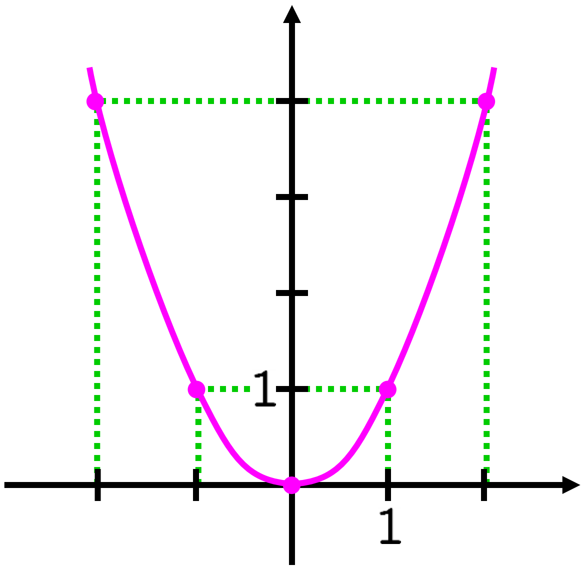
Picture is twice as high as wide;

OK, but we somtimes change the units on one axis (or both) for a better picture. . .

e.g.: For $f(x) = x^2$, the full graph of f is

coordinatized plane:



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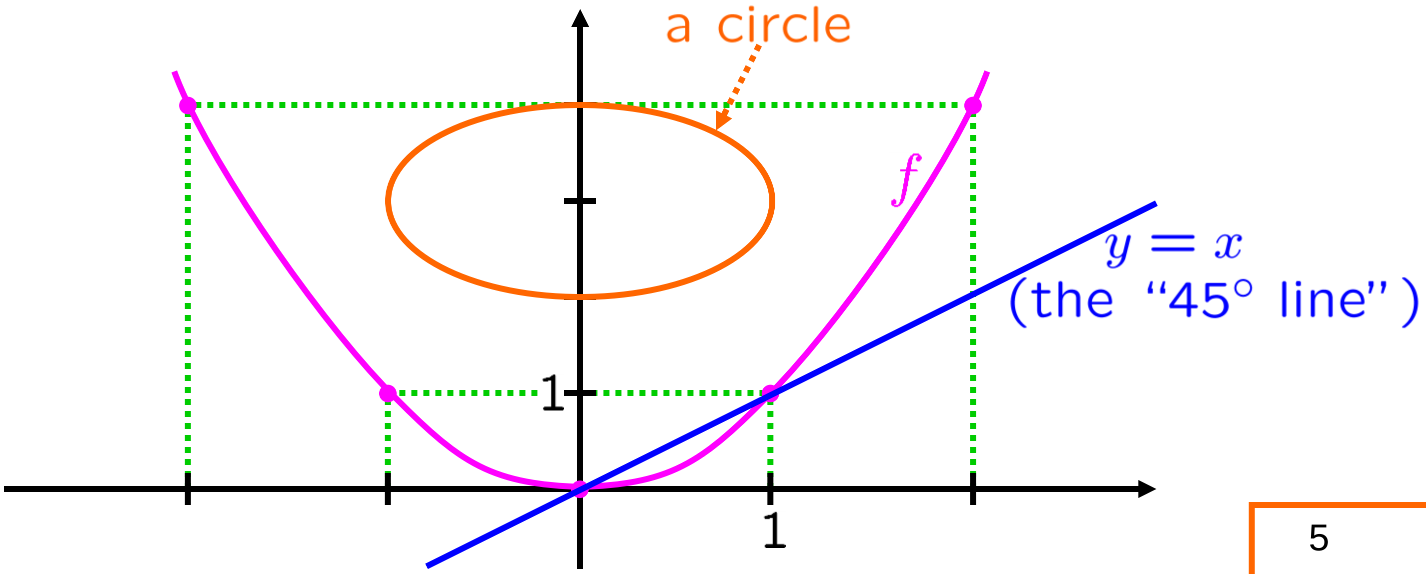
it is a subset of $\mathbb{R}^2$, and so can be visualized as a subset of a (coordinatized) plane.

Useful, but beware of distortions. . .

e.g.: For $f(x) = x^2$, the full graph of f is

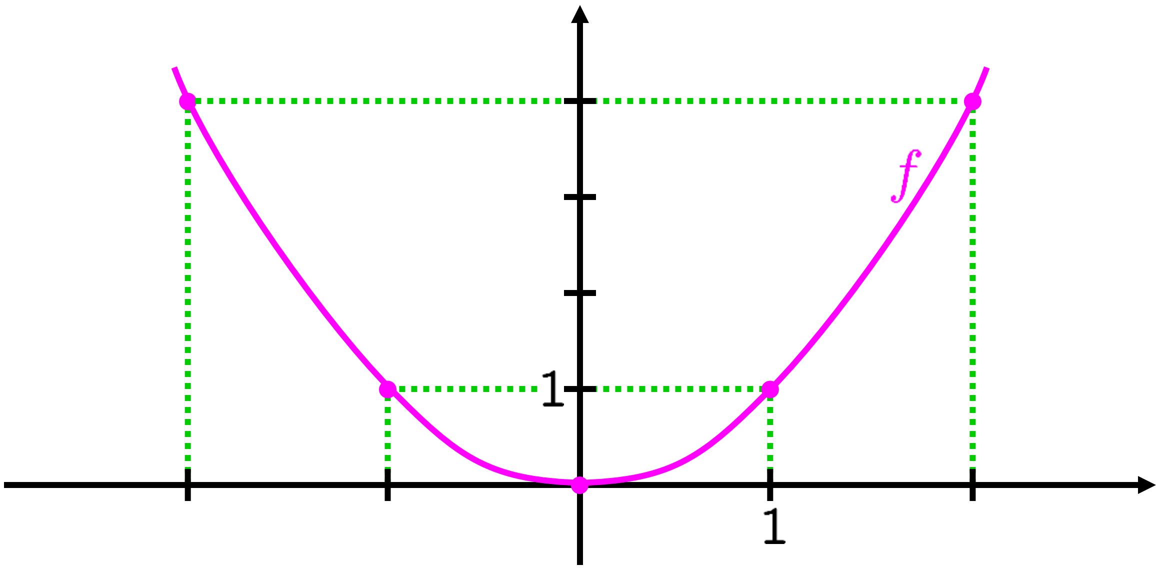


Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function.
Let $C \subseteq D$.

Def'n: The **restriction of f to C** is the function

$$f|C : C \to \mathbb{R} \text{ defined by } (f|C)(x) = f(x).$$

e.g.: For $f(x) = x^2$, the full graph of f is



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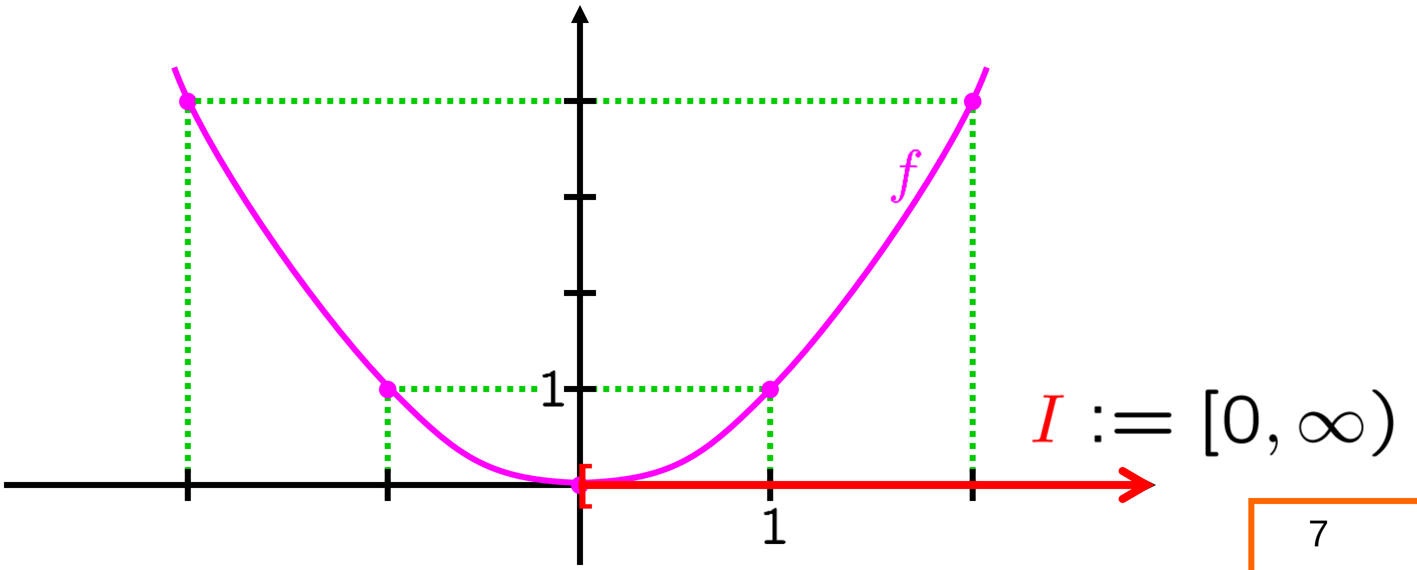

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Let $C \subseteq D$.

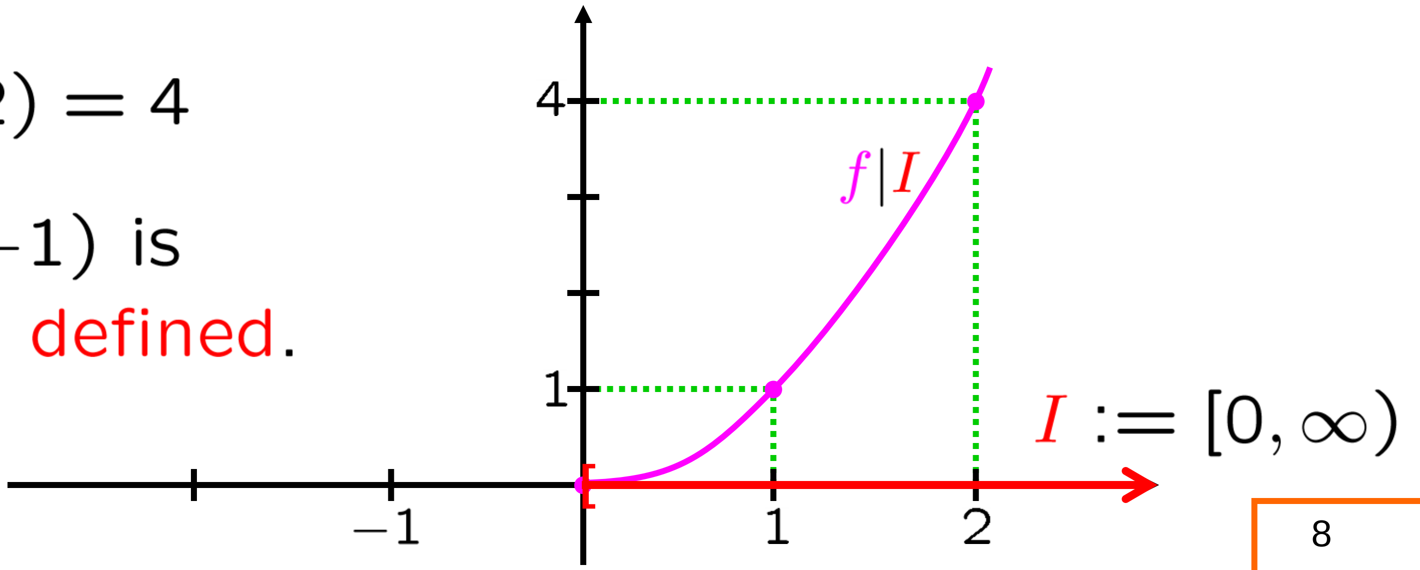
Def'n: The **restriction of f to C** is the function
$f|C : C \to \mathbb{R}$ defined by $(f|C)(x) = f(x)$.

Next subtopic: Translations and dilations

e.g.: For $f(x) = x^2$, the full graph of f is

$(f|I)(2) = 4$

Note: $(f|I)(-1)$ is not defined.



Spp

General problem:

Graph some equation in x and y.

Given a number a.

Replace x by $x-a$ in the equation.

Graph the new equation.

Example:

Graph $x^2 = 9 - y^2$. $\Leftrightarrow x^2 + y^2 = 9$

$\Leftrightarrow \sqrt{(x-0)^2 + (y-0)^2} = 3$

$\Leftrightarrow \mathrm{dist}((x,y),(0,0)) = 3$

Replace x by $x-2$.

Graph $(x-2)^2 = 9 - y^2$.

In $\mathbb{R}$: $\mathrm{dist}(a,s) = |s-a| = \sqrt{(s-a)^2}$

In $\mathbb{R}^2$: $\mathrm{dist}((a,b),(s,t)) = \sqrt{(s-a)^2 + (t-b)^2}$

In $\mathbb{R}^3$: $\mathrm{dist}((a,b,c),(s,t,u))$

$$= \sqrt{(s-a)^2 + (t-b)^2 + (u-c)^2}$$

Example:

Graph $x^2 = 9 - y^2$. $\Leftrightarrow$ $\text{dist}((x,y),(0,0)) = 3$

Replace x by $x - 2$.

Graph $(x-2)^2 = 9 - y^2$.

Example:

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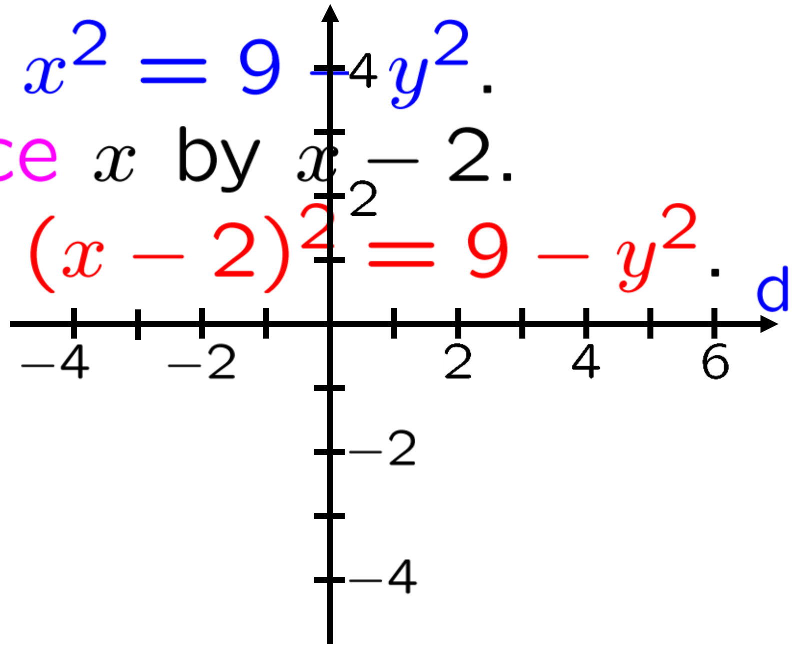


Translation

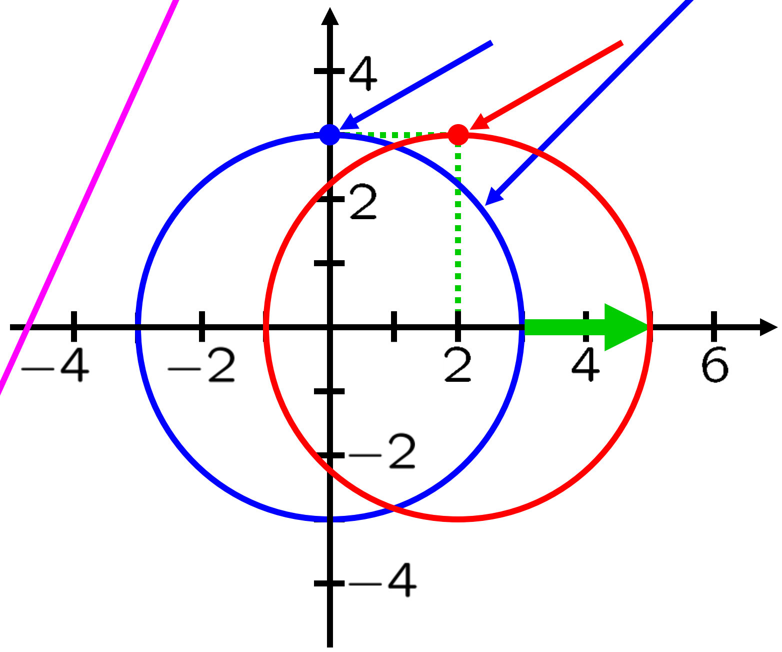
Example: $x :\rightarrow 0, y :\rightarrow 3$ $\Leftrightarrow$ $\text{dist}((x,y),(0,0)) = 3$

Graph $x^2 = 9 - y^2$.

Replace x by $x - 2$.

Graph $(x-2)^2 = 9 - y^2$.

$x :\rightarrow 2, y :\rightarrow 3$

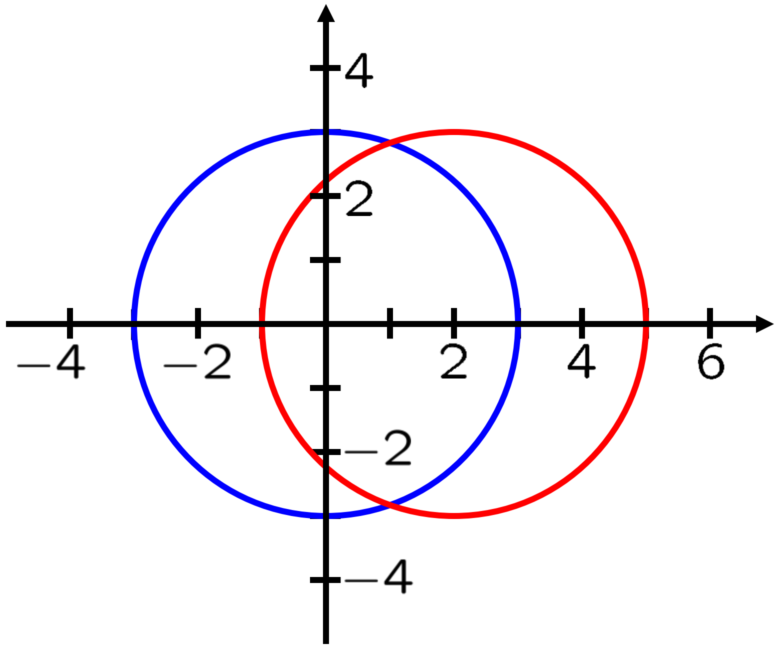


Shift old graph 2 units to right to get new graph.

Translation

General problem:

Graph some equation in x and y.
Given a number a.
Replace $\boxed{x}$ by $\boxed{x-a}$ in the equation.
Graph the new equation.



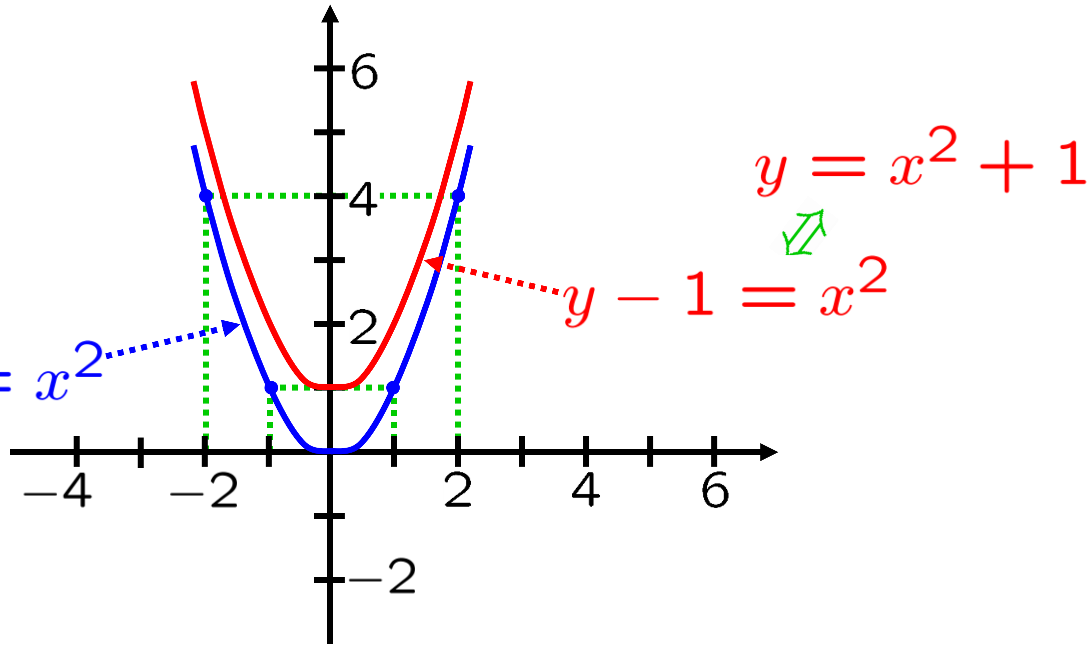
Shift old graph a units to right
to get new graph.

Translation

General problem:

Graph some equation in x and y.
Given a number a.
Replace $\boxed{y}$ by $\boxed{y-a}$ in the equation.
Graph the new equation.

Replace
y by $y-1$.

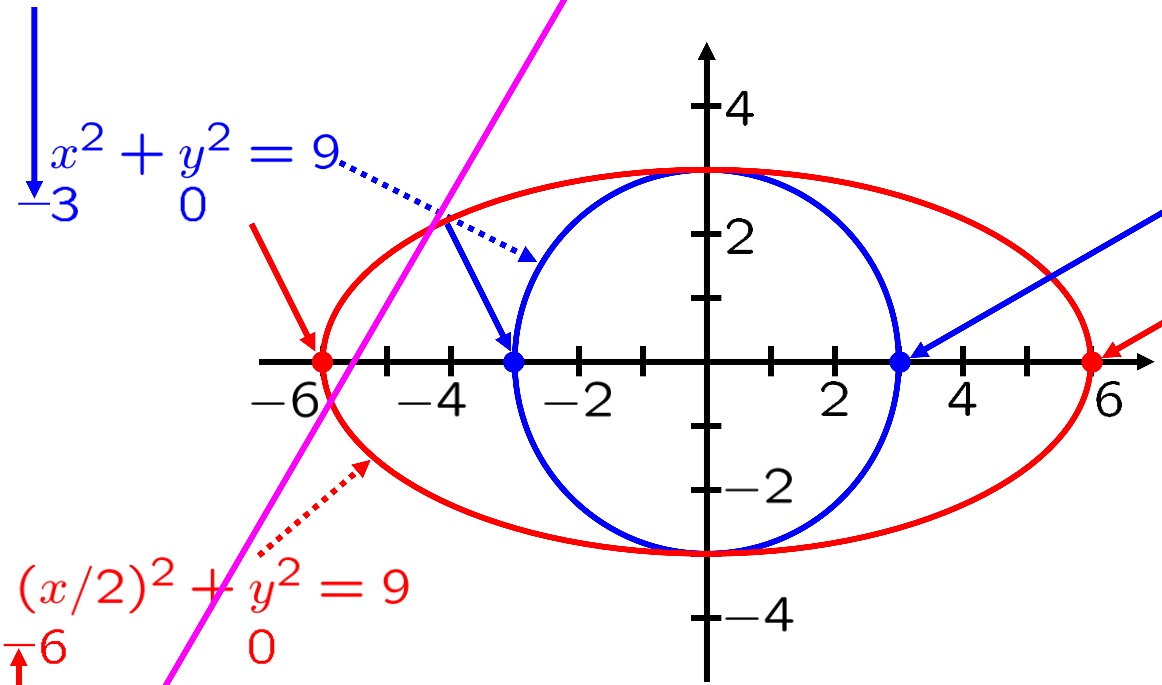


Shift old graph a units upward
to get new graph.

Dilation

General problem:

Graph some equation in x and y.
Given a number $a > 0$.
Replace x by x/a in the equation.
Graph the new equation.



Stretch old graph by a factor of a in x-direction to get new graph.

Dilation

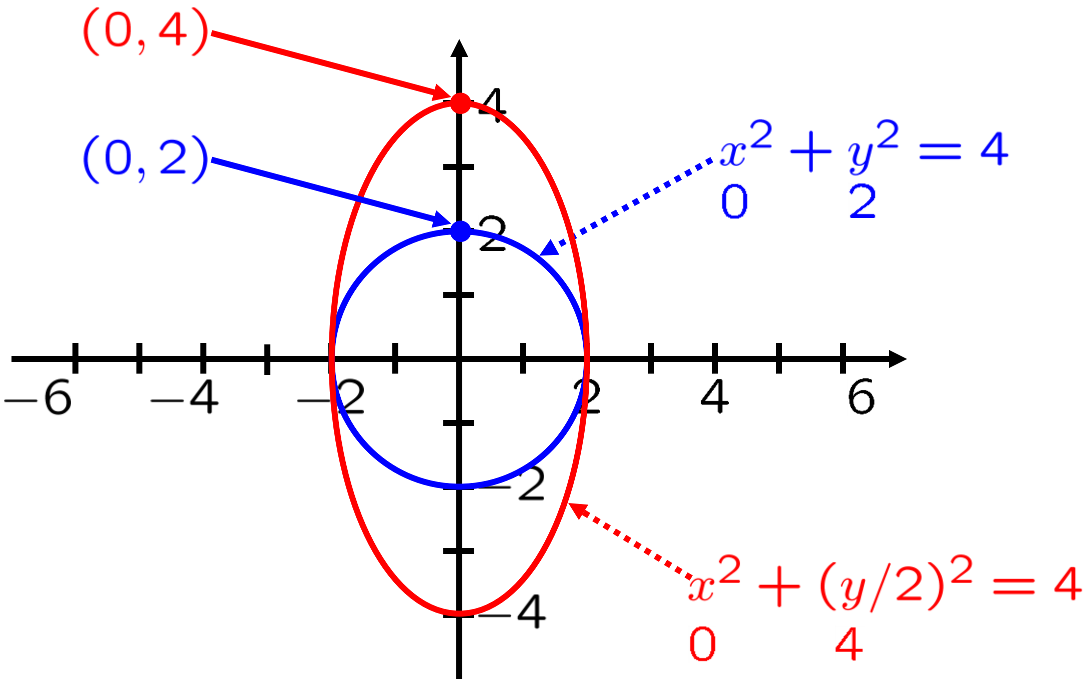
General problem:

Graph some equation in x and y.

Given a number $a > 0$.

Replace $\boxed{y}$ by $\boxed{y/a}$ in the equation.

Graph the new equation.



Stretch old graph by a factor of a in y-direction to get new graph.

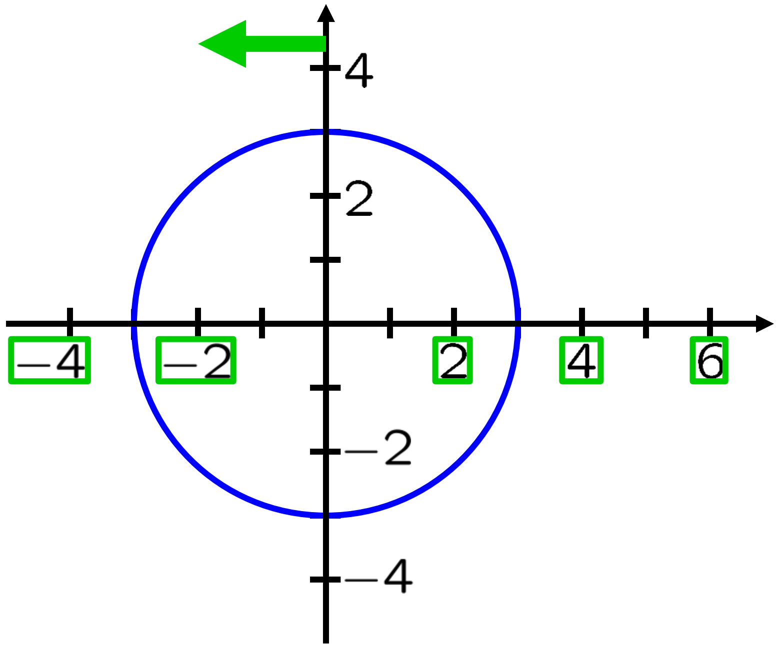
Alternate approach: Change the axes

General problem:

Graph $x^2 = 9 - y^2$.

Replace x by $x - 2$.

Graph $(x-2)^2 = 9 - y^2$.



Shift y-axis 2 units to left
and add 2 to markings on the x-axis.

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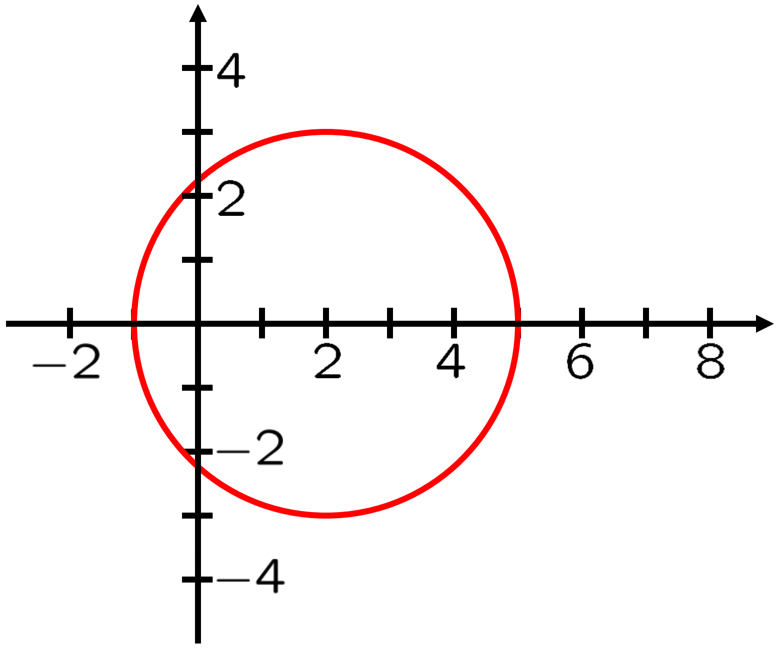
Graph $(x - 2)^2 = 9 - y^2$.



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Alternate approach: Change the axes

General problem:

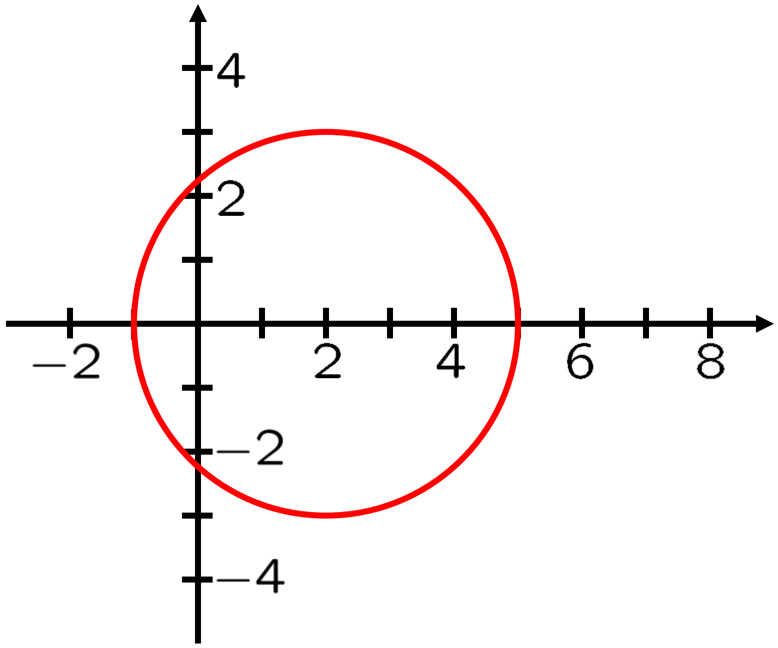
Graph some equation in x and y.

Given a number a.

Replace x by $x - a$ in the equation.

Graph the new equation.

e.g.:



Shift y-axis a units to left
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Alternate approach: Change the axes

General problem:

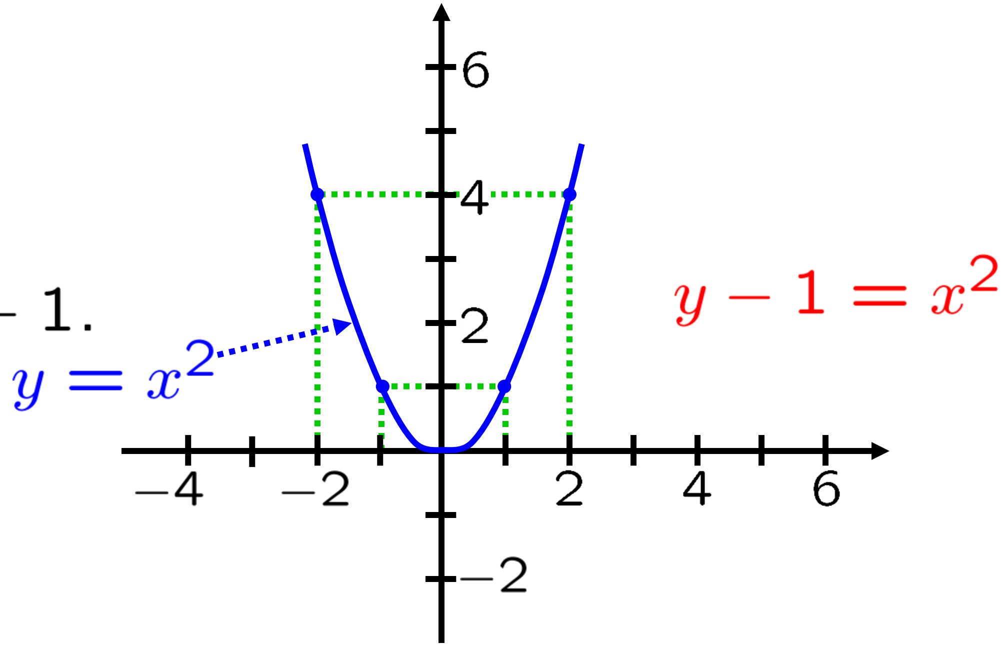
Graph some equation in x and y.

Given a number a.

Replace y by $y - a$ in the equation.

Graph the new equation.

Replace y by $y - 1$.



Shift x-axis a units down and add a to markings on the y-axis.

Alternate approach: Change the axes

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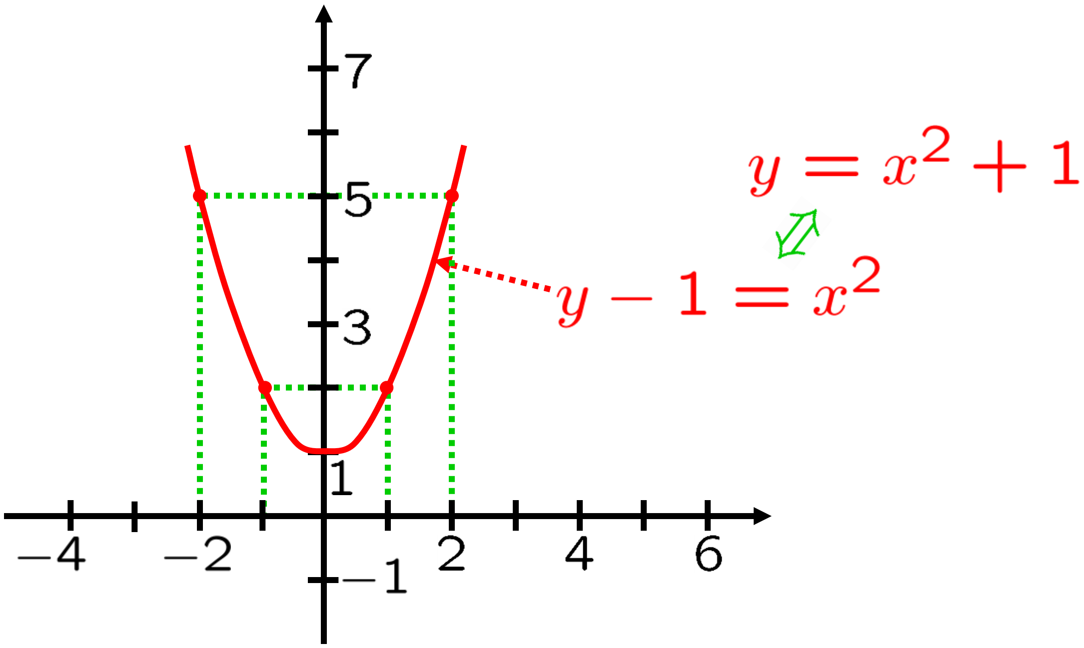
Graph some equation in x and y.

Given a number a.

Replace y by $y-a$ in the equation.

Graph the new equation.



Shift x-axis a units down

and add a to markings on the y-axis.

Alternate approach: Change the axes

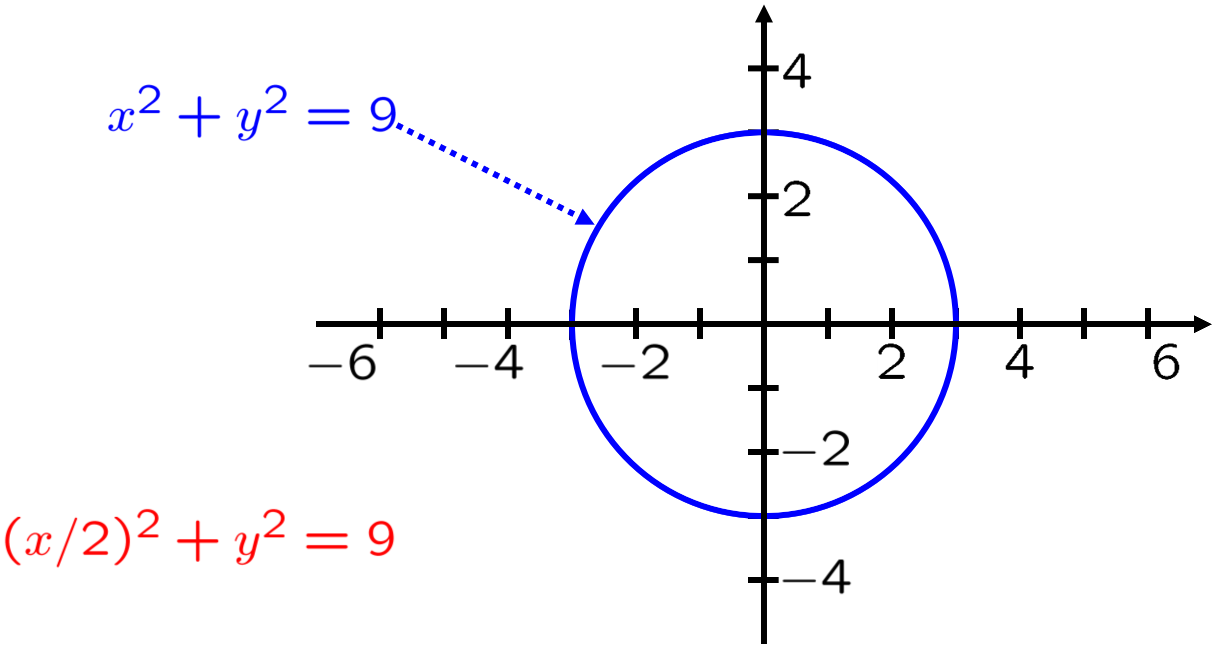
General problem:

Graph some equation in x and y.

Given a number $a > 0$.

Replace x by x/a in the equation.

Graph the new equation.



Multiply markings on the x-axis by a.

Alternate approach: Change the axes

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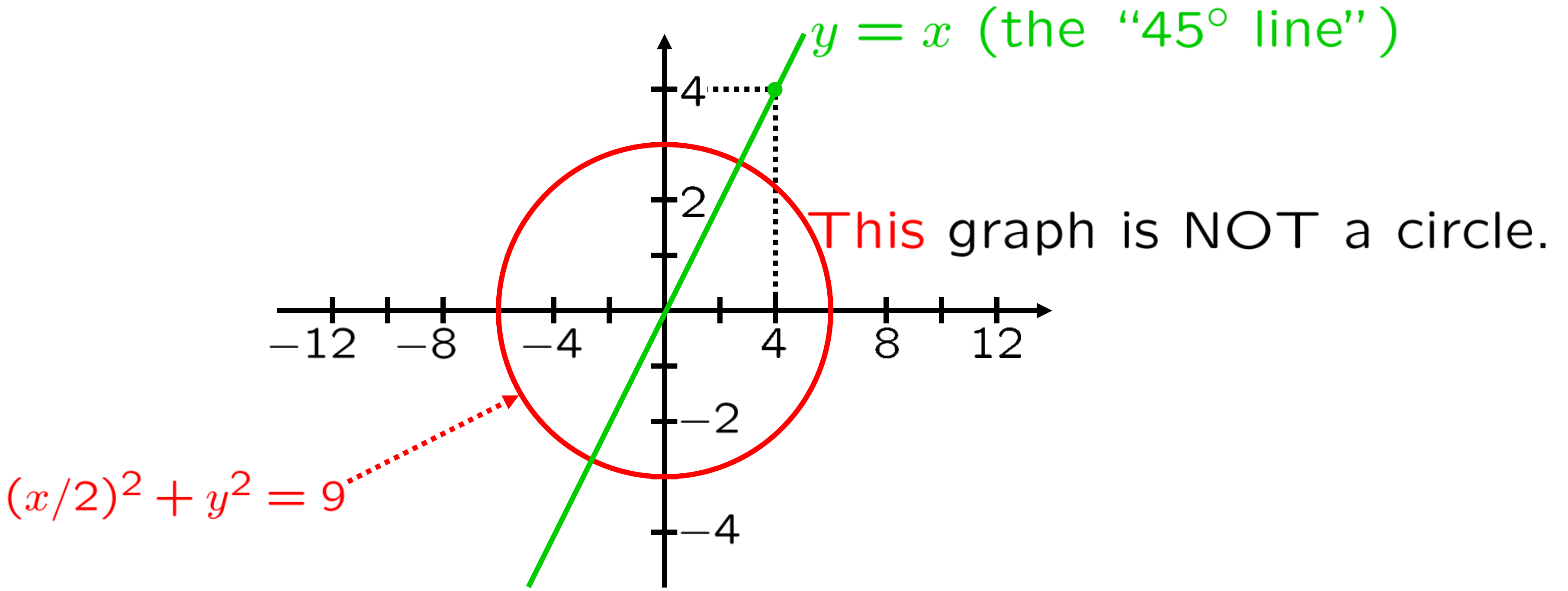

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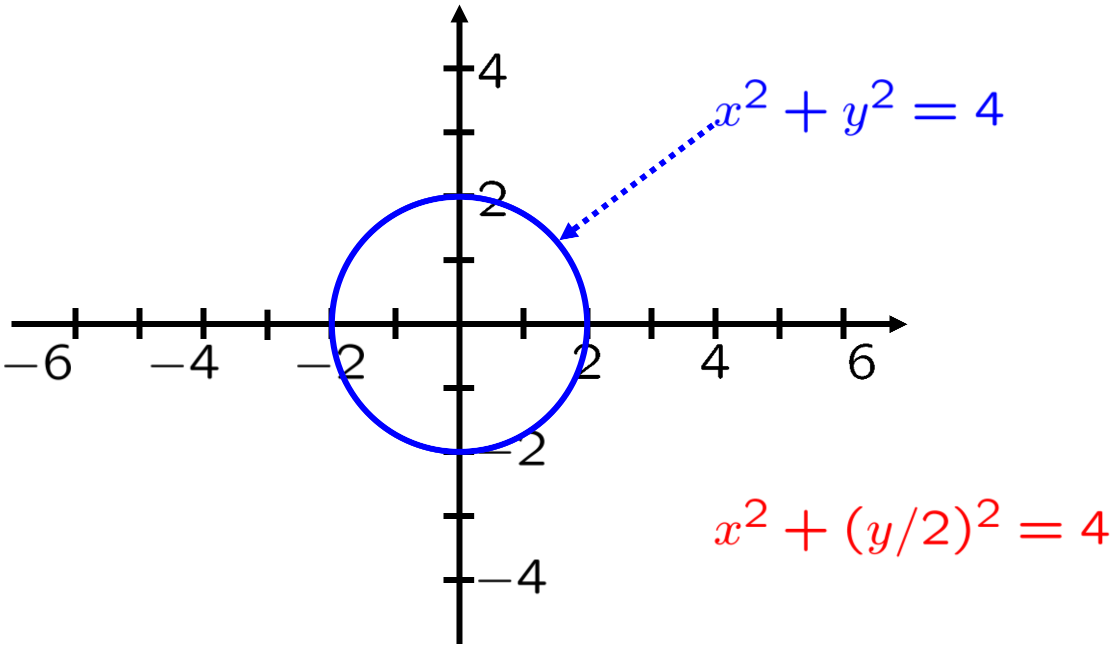

Multiply markings on the y-axis by a.

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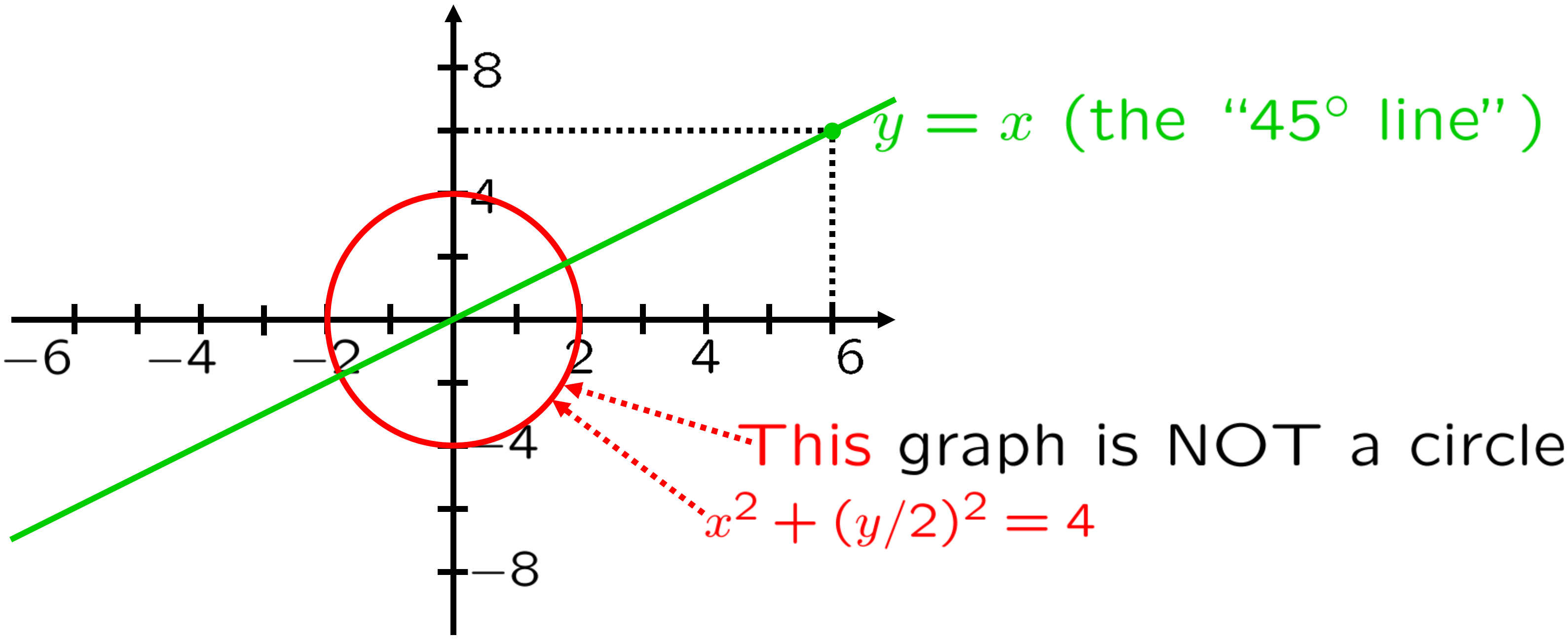


Multiply markings on the y-axis by a.

Lines

Given two points on a line, find its slope.

SKILL
pts to slope

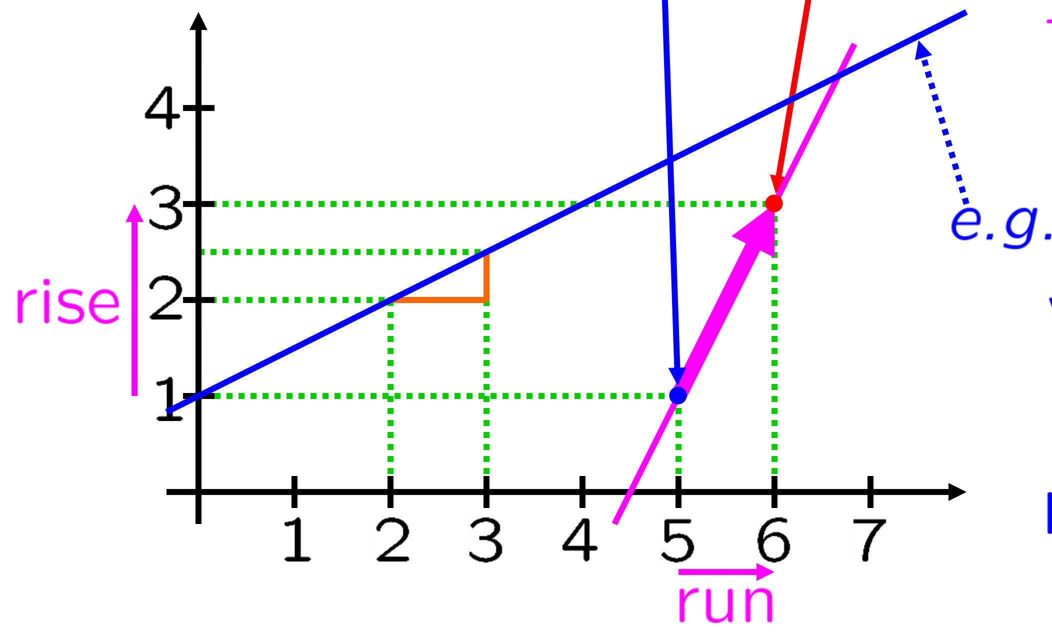
e.g.: $(5,1),(6,3)$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{3-1}{6-5} = \frac{2}{1} = 2 \quad \blacksquare$$

difference quotient

Explanation:

2 units rise
per unit run
fairly steep



slope measures
steepness

What if we had
0.5 units rise
per unit run?
less steep

Lines

Given two points on a line, find its slope.

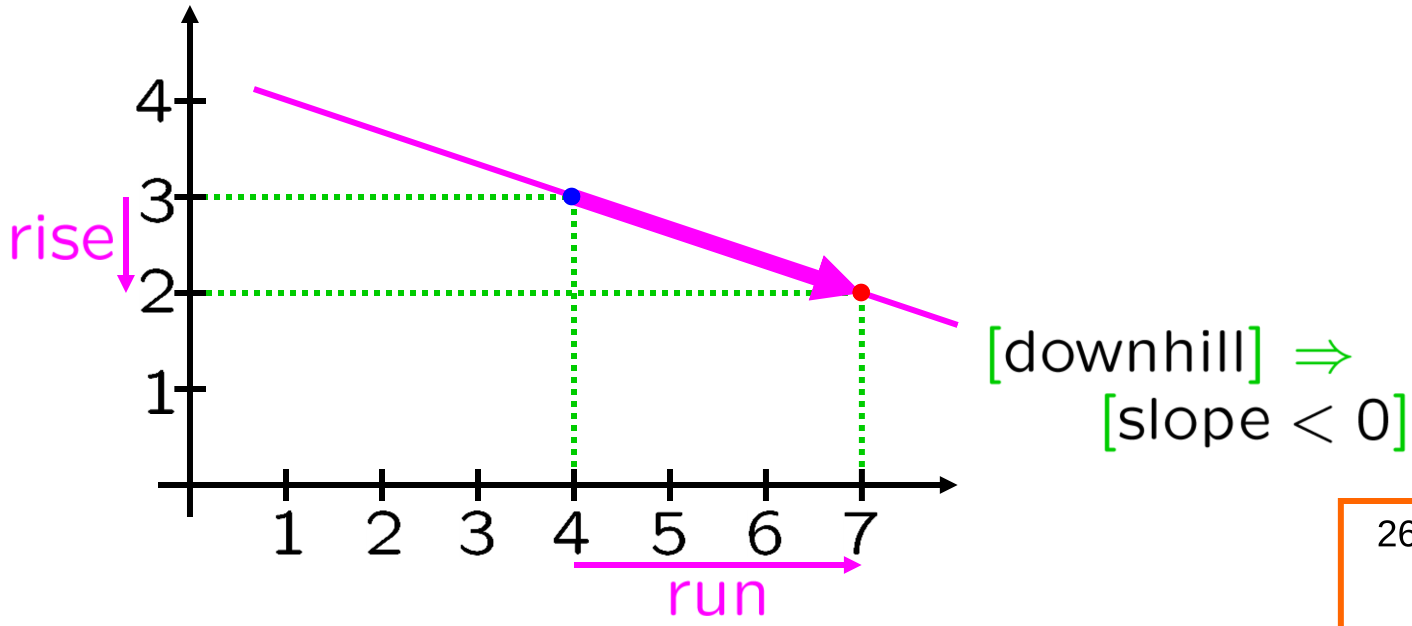
SKILL
pts to slope

e.g.: $(4,3),(7,2)$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \boxed{\frac{2-3}{7-4}} = -\frac{1}{3} \quad \blacksquare$$

difference quotient

Explanation:



Lines

Given two points on a line, find its slope.

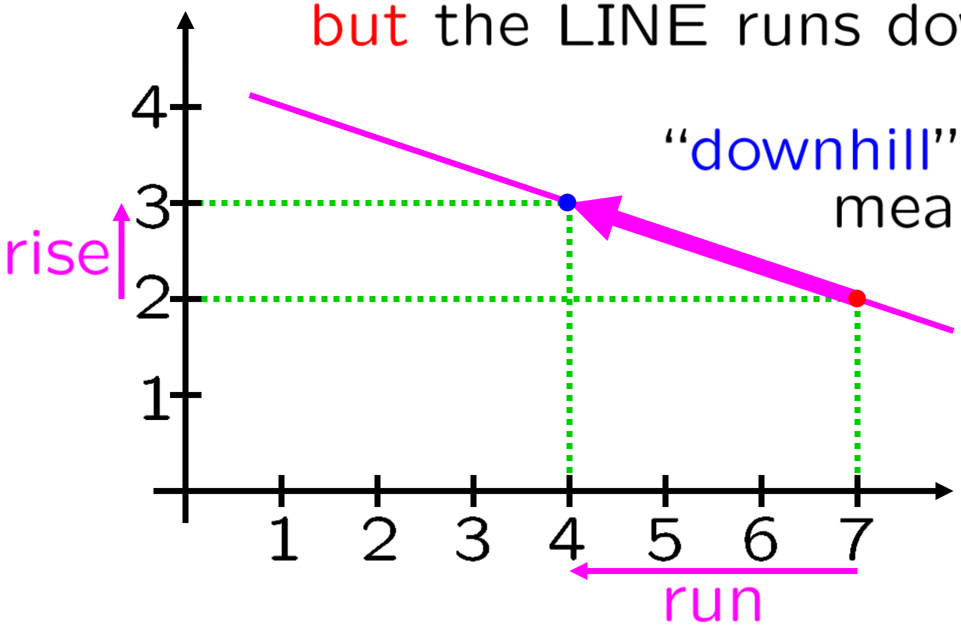
SKILL
pts to slope

e.g.: $(7,2)$,$(7,2)$

$-\frac{1}{3} =$ slope $= -\frac{1}{3}$

Explanation:

The ARROW runs uphill,
but the LINE runs downhill.



"downhill" for a LINE
means: left-to-right
downhill

[downhill] $\Rightarrow$
[slope $<$ 0]

Lines

Given two points on a line, find its slope.

SKILL
pts to slope

e.g.: $(7,2),(4,3)$

$$-\frac{1}{3} = \text{slope} = \frac{\text{rise}}{\text{run}} = \boxed{\frac{3-2}{4-7}} = -\frac{1}{3} \quad \blacksquare$$

difference quotient

Explanation:

The ARROW runs uphill,
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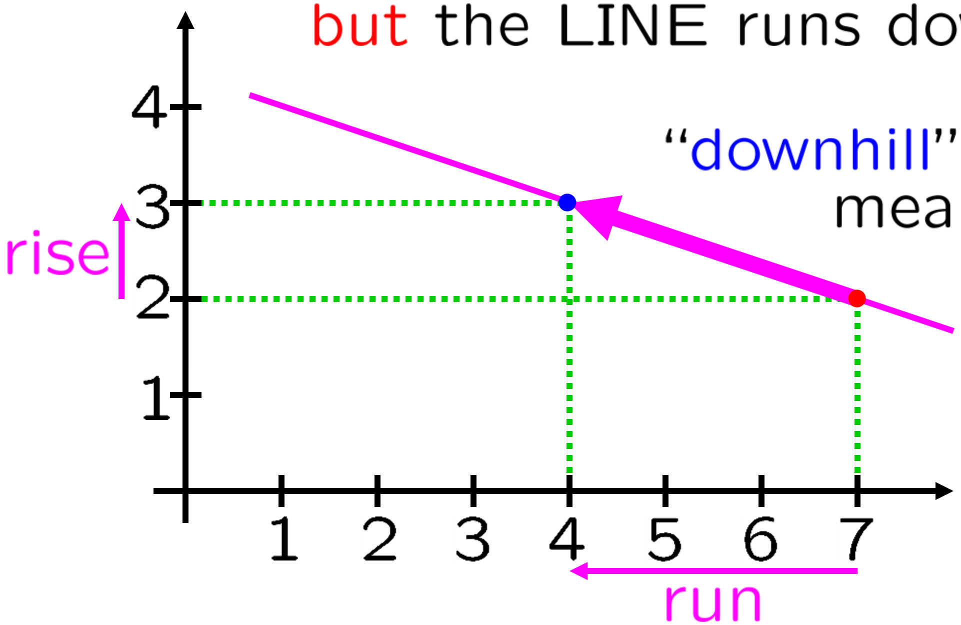


"downhill" for a LINE
means: left-to-right
downhill

[downhill] $\Rightarrow$
[slope < 0]

Lines

Given a pt and slope, find an eq'n for the line.

e.g.: pt $= (5, 1)$, slope $= 2$

$$y - 1 = 2(x - 5)$$

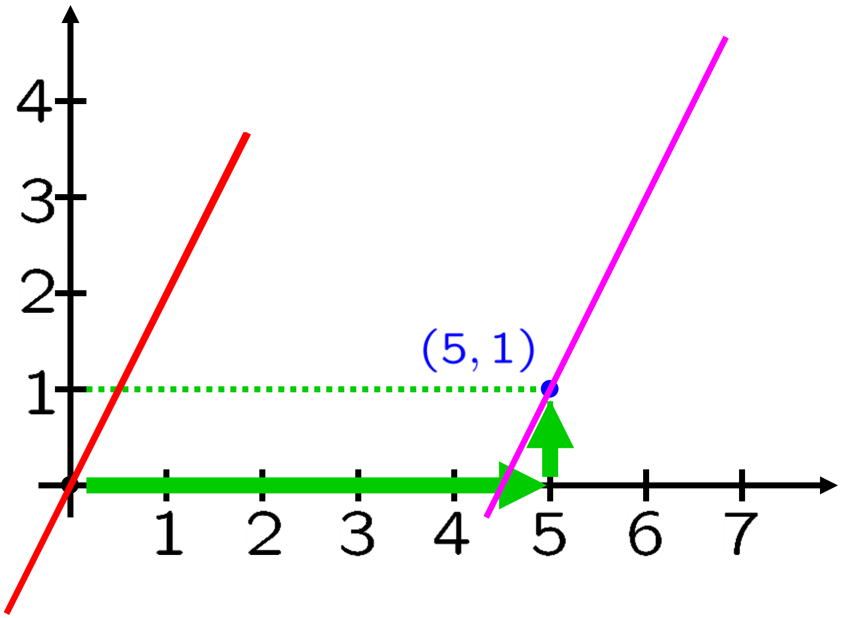
SKILL
pt/slope to eqn

simpler *e.g.*:

pt $= (0, 0)$, slope $= 2$

$$y = 2x$$

$x :\to x - 5$
$y :\to y - 1$



Lines

Given a pt and slope, find an eq'n for the line.

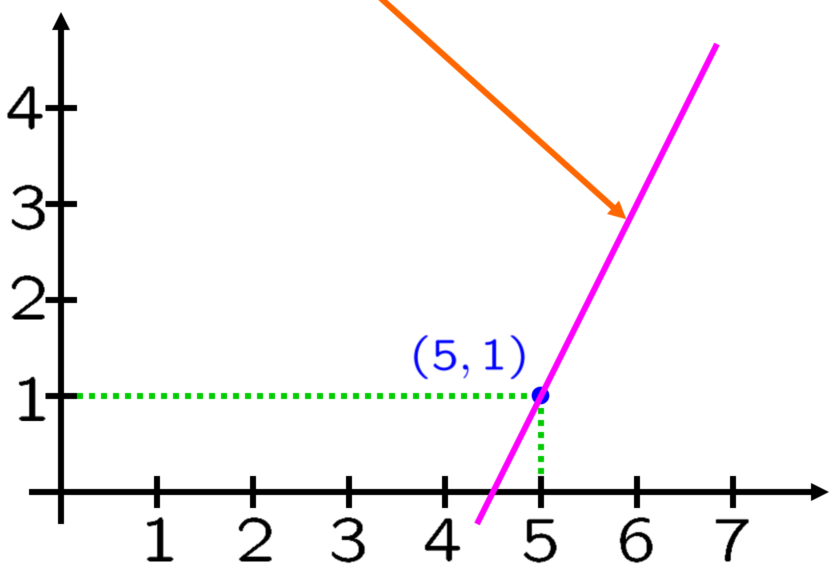
e.g.: pt $= (5,1)$, slope $= 2$

$$y - 1 = 2(x - 5)$$

$$L(x) = 2(x - 5)?$$

SKILL
equation of line
to linear fn

Find the linear $L(x)$ s.t.
this line is the graph of $y = L(x)$.



Lines

Given a pt and slope, find an eq'n for the line.

e.g.: pt = (5, 1), slope = 2

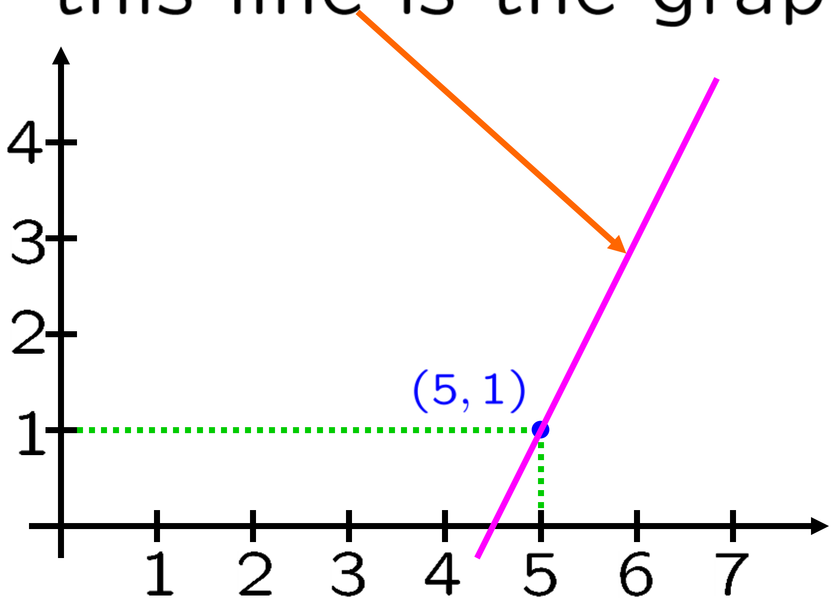
$$y - 1 = 2(x - 5)$$

$$y = 1 + 2(x - 5)$$

$$L(x) = 1 + 2(x - 5)$$

SKILL
equation of line
to linear fn

Find the linear $L(x)$ s.t.
this line is the graph of $y = L(x)$.



Lines

Given two points, find a linear $L(x)$ s.t.
$y = L(x)$ is the line through them.

SKILL
pts to linear fn

e.g.: $(3, 4)$ and $(9, 16)$

One approach:

$$\text{slope} = \frac{16 - 4}{9 - 3} = 2$$

$$y - 4 = 2(x - 3)$$
$$y = 4 + 2(x - 3)$$
$$L(x) = 4 + 2(x - 3)$$

Lines

Given two points, find a linear $L(x)$ s.t.
$y = L(x)$ is the line through them.

SKILL
pts to linear fn

e.g.: $(3,4)$ and $(9,16)$

Another approach:

$L(x)$ is a linear combination of $x-3$ and $x-9$,
with carefully chosen coefficients . . .

$$L(x) = \left[\frac{16}{6}\right][x-3] + \left[\frac{4}{-6}\right][x-9]$$

16 = ; 4 = ; 9 ; 3 ; 9 ; 3 ; 9 ; 3

$$L(x) = 4 + 2(x-3)$$

Lines

Given two points, find a linear $L(x)$ s.t.
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Exercise: Show that these are the same.

$$L(x) = 4 + 2(x-3)$$

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SKILL
pts to linear fn

e.g.: $(3, 4)$ and $(9, 16)$

Another approach:

slightly easier variant:
use $9 - x$

$L(x)$ is a linear combination of $x - 3$ and $x - 9$,
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$$L(x) = \left[\frac{16}{6}\right][x-3] + \left[\frac{4}{-6}\right][x-9]$$

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Lines

Given two points, find a linear $L(x)$ s.t. $y = L(x)$ is the line through them.

SKILL
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$$\underset{4}{\overset{16}{}} = L\underset{3}{\overset{9}{(x)}} = \left[\frac{16}{6}\right]\underset{3}{\overset{9}{[x-3]}} + \left[\frac{4}{6}\right]\underset{3}{\overset{9}{[9-x]}}$$

$$L(x) = 4 + 2(x-3)$$

Lines

Given two points, find a linear $L(x)$ s.t.
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SKILL
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Exercise: Show that these are the same.

$$L(x) = \boxed{4 + 2(x-3)}$$

SKILL

two pts to $y = mx + b$

Whitman problems
§1.1, p. 5, #1-3

SKILL

chg eq'n to $y = mx + b$
and find intercepts

Whitman problems
§1.1, p. 5, #4-8

SKILL

ck whether lines parallel

Whitman problems
§1.1, p. 5, #9

SKILL

find lines along
edges of a triangle

Whitman problems
§1.1, p. 5, #10

SKILL

words to eq'n of line

Whitman problems
§1.1, p. 5-7, #11-18

SKILL

translate/dilate graph

Whitman problems
§1.4, p. 16-17, #1-19

SKILL

eq'ns of circles

Whitman problems
§1.2, p. 8, #1,3-6

SKILL

dist & slope
circle eq'n & line eq'n

Whitman problems
§1.2, p. 8, #2a-f

STOP