CALCULUS
Summation

$$1^2 + \cdots + n^2 \quad = \quad ??$$

Solution: For any sequence $a_0, a_1, a_2, \ldots$,

$\boxed{\triangle a_0, \triangle a_1, \triangle a_2, \ldots}$ is the sequence

$$a_1 - a_0, \quad a_2 - a_1, \quad a_3 - a_2, \quad \ldots$$

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E.g.: Let $x_n := 2n$, so $x_0, x_1, x_2, \ldots$ is

$$0, \quad 2, \quad 4, \quad 6, \quad \ldots$$

Then $\triangle x_0, \triangle x_1, \triangle x_2, \ldots$ is

$$2-0, \quad 4-2, \quad 6-4, \quad 8-6, \quad \ldots$$

More simply, $\triangle(2n) = (2(n+1)) - (2n) = 2.$

Generally, $\boxed{\triangle a_n} = ([a_n]_{n:\to n+1}) - (a_n) = a_{n+1} - a_n.$

Note: $[(\triangle a_n = 0), \forall n] \quad \Rightarrow \quad [a_0 = a_1 = a_2 = \cdots,$
i.e., a_n is constant]

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

$$\triangle n^4 = ((n+1)^4) - (n^4)$$
$$= (\cancel{1n^4} + 4n^3 + 6n^2 + 4n + 1) - (\cancel{n^4})$$
$$= 4n^3 + 6n^2 + 4n + 1$$

7.1

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n^4 is quartic (degree = 4)

$$\triangle n^4 = ((n+1)^4) - (n^4)$$

$$= (1n^4 + 4n^3 + 6n^2 + 4n + 1) - (n^4)$$

$$= 4n^3 + 6n^2 + 4n + 1$$

cubic (degree = 3)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

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Then $\triangle s_n = (n+1)^2$

$$s_{n+1} = 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2$$
$$s_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

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$\frac{1}{3} \times \left[\triangle n^3 = 3n^2+3n+1 \right]$

$\triangle n^2 = 2n+1$

$\triangle n = 1$

$\triangle \left(\frac{1}{3}n^3\right) = n^2 + \quad n + \frac{1}{3}$

$\triangle$ is LINEAR!!

Linear operations: addition, scalar multiplication

$\triangle$ is additive: $\triangle(a_n+b_n) = (\triangle a_n) + (\triangle b_n)$ — $\triangle$ distributes over addition.

$\triangle$ is commutes with scalar mult.: $\triangle(ca_n) = c(\triangle a_n)$

"commutes" refers to traveling $\triangle(\hat{c}a_n)$

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Product rule: $\triangle(a_n b_n) =$ $a_n(b_{n+1} - b_n)$

$$a_{n+1}b_{n+1} - a_n b_n \;=\; (a_{n+1} - a_n)b_{n+1} + \underbrace{a_n b_{n+1} - a_n b_n}$$

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a_n is the "first part" b_n is the "second part"

"differencing by parts"

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$$\tfrac{1}{2} \times \left[\triangle n^2 = 2n + 1 \right]$$

$$\triangle n = 1$$

△ is LINEAR!!

△ is ADDITIVE!!

ADD

$$\triangle\left(\tfrac{1}{3}n^3\right) = n^2 + n + \tfrac{1}{3}$$

$$\triangle\left(\tfrac{1}{2}n^2\right) = n + \tfrac{1}{2}$$

$$\tfrac{1}{3} + \tfrac{1}{2} = \tfrac{2}{6} + \tfrac{3}{6} = \tfrac{5}{6}$$

$$\triangle\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2\right) = n^2 + 2n + \tfrac{5}{6}$$

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$1^2 + \cdots + n^2 \quad = \quad$?? cubic (degree = 3)

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$$\tfrac{1}{6} \times \left[\triangle n = 1 \right]$$

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$$\triangle\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2\right) = n^2 + 2n + \tfrac{5}{6}$$

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$$\triangle \left(\tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n - s_n \right) = 0$$

$\tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n - s_n$ is a constant sequence.

$$\left[\tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n - s_n \right]_{n:\to 0} = 0 + 0 + 0 - 0 = 0$$

$$\tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n - s_n = 0$$

$$\triangle \left(\tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n \right) = n^2 + 2n + 1 = \triangle s_n$$

$$1^2 + \cdots + n^2 \quad = \quad \frac{2n^3 + 3n^2 + n}{6}$$

cubic (degree = 3)

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$$\frac{2n^3 + 3n^2 + n}{6}$$

$$\|$$

$$\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n = s_n = 1^2 + 2^2 + \cdots + n^2$$

$$\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n \boxed{- s_n} = 0$$

$$1^2 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

EXERCISE: $$1 + \cdots + n = \frac{n^2 + n}{2}$$

$$1^2 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

EXERCISE: $$1^3 + \cdots + n^3 = \frac{n^4 + 2n^3 + n^2}{4}$$

$$\left(1+\cdots+n\right)^2 = \left(\frac{n^2+n}{2}\right)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

$$1^2+\cdots+n^2 = \frac{2n^3+3n^2+n}{6} = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+\cdots+n^3 = \frac{n^4+2n^3+n^2}{4} = \frac{n^2(n+1)^2}{4}$$

$$(1+\cdots+n)^2 = \left(\frac{n^2+n}{2}\right)^2 = \underbrace{\left(\frac{n(n+1)}{2}\right)^2}$$

EQUAL

$$1^3+\cdots+n^3 = \frac{n^4+2n^3+n^2}{4} = \overbrace{\frac{n^2(n+1)^2}{4}}$$

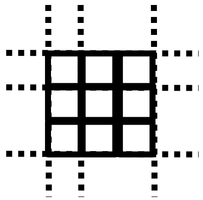
$$= (1+\cdots+n)^2$$

$1^3 + \cdots + n^3 =$

$$(1 + \cdots + n)^2 = 1^3 + \cdots + n^3$$

$(1$ WHY?? $n)^2$

$$(1)^2 = 1^3$$



$1 + 2$ by $1 + 2$

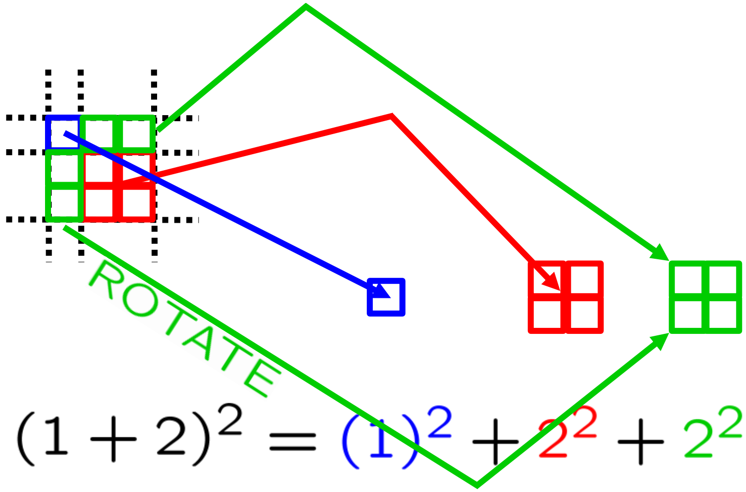
$$(1 + 2)^2$$

Split along dashed lines . . .

$$(1 + \cdots + n)^2 \quad = \quad 1^3 + \cdots + n^3$$

WHY??

$(1)^2 = 1^3$



$(1+2)^2 = (1)^2 + 2^2 + 2^2$

Split along dashed lines . . .

Two squares (blue and red)

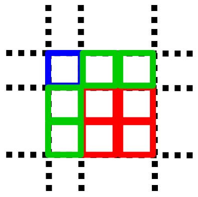
and two rectangles (both green).

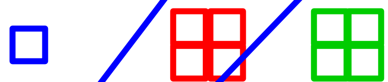
Rectangles pair up to form a square . . .

$$(1 + \cdots + n)^2 \quad = \quad 1^3 + \cdots + n^3$$

WHY??

$$(1)^2 = 1^3$$





$$(1+2)^2 = (1)^2 + 2^2 + 2^2$$

$$= 1^3 + \boxed{2} \cdot 2^2$$

$$= 1^3 + 2^3$$

$$(1 + \cdots + n)^2 \quad = \quad 1^3 + \cdots + n^3$$

WHY??

$$(1)^2 = 1^3$$
$$(1+2)^2 = 1^3 + 2^3$$

$$(1+2)^2$$

$$(1+2+3)^2$$
$$= 1^3 + 2^3$$

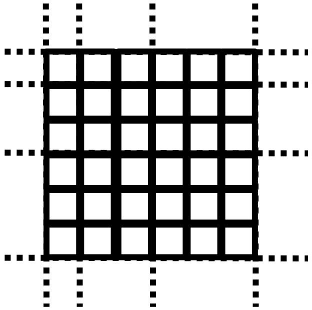
$$(1+\cdots+n)^2 \quad = \quad 1^3+\cdots+n^3$$

WHY??

$$(1)^2 = 1^3$$

Make the old square blue . . .

$$(1+2)^2 = 1^3 + 2^3$$



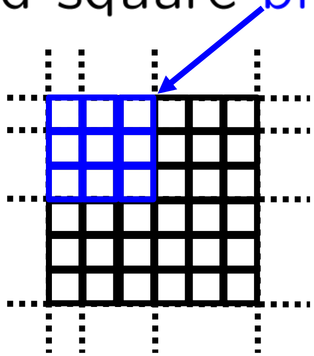
$1+2+3$ by $1+2+3$

$$(1+2+3)^2$$

$$(1)^2 = 1^3$$

Make the old square blue . . .

$$(1+2)^2 = 1^3 + 2^3$$



$$(1+2+3)^2$$

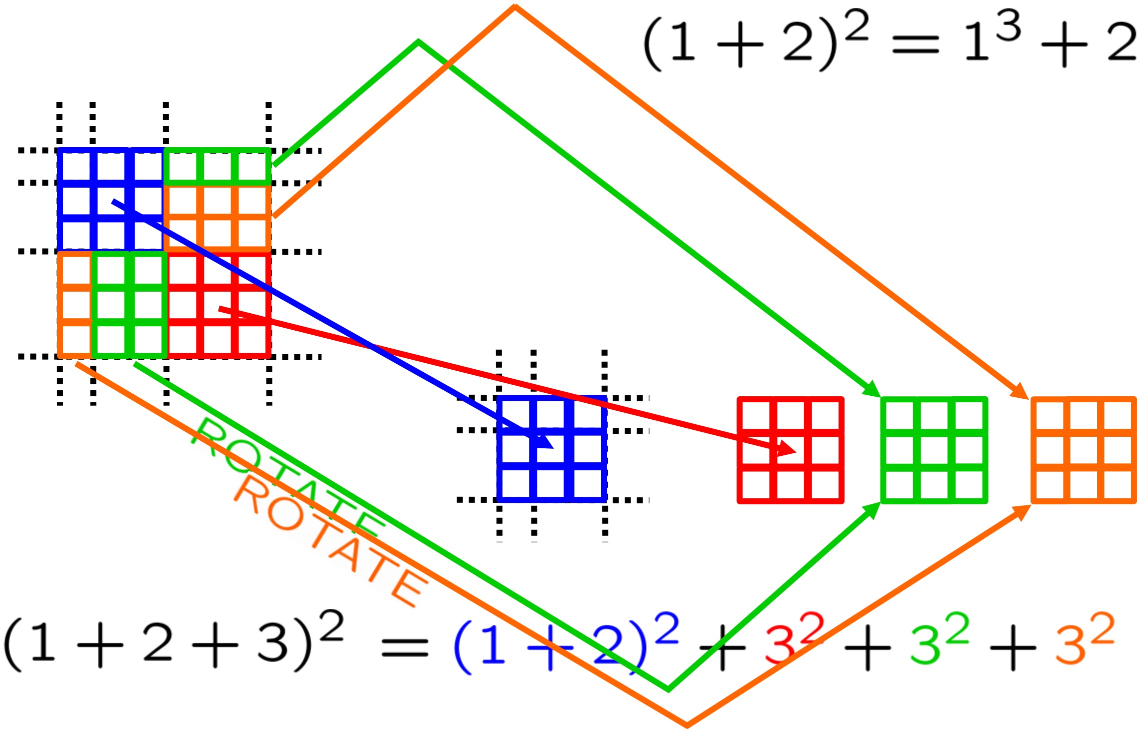
Split remainder along dashed lines . . .

$$(1+\cdots+n)^2 \quad = \quad 1^3+\cdots+n^3$$

WHY??

$$(1)^2 = 1^3$$

$$(1+2)^2 = 1^3 + 2^3$$



$$(1+2+3)^2 = (1+2)^2 + 3^2 + 3^2 + 3^2$$

Split remainder along dashed lines . . .

Two squares (blue and red)

and four rectangles (two green and two orange).

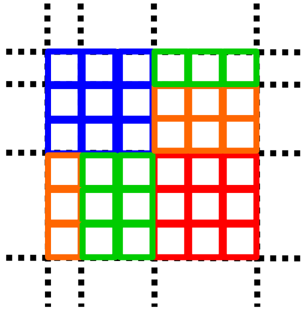
Rectangles pair up to form squares . . .

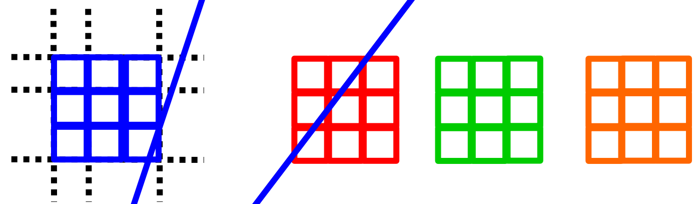
$$(1+\cdots+n)^2 \quad = \quad 1^3+\cdots+n^3$$

WHY??

$$(1)^2 = 1^3$$

$$(1+2)^2 = 1^3 + 2^3$$





$$\begin{aligned}(1+2+3)^2 &= (1+2)^2 + 3^2 + 3^2 + 3^2 \\ &= 1^3 + 2^3 + \boxed{3} \cdot 3^2 \\ &= 1^3 + 2^3 + 3^3\end{aligned}$$

$$(1+\cdots+n)^2 \quad = \quad 1^3+\cdots+n^3$$

WHY??

$$(1)^2 = 1^3$$

$$(1+2)^2 = 1^3 + 2^3$$

$$(1+2+3)^2 = 1^3 + 2^3 + 3^3$$

$$(1+2+3)^2$$

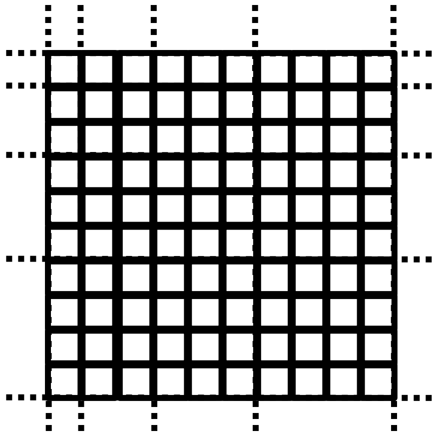
$$(1+2+3+4)^2 \; 1^3 + 2^3 + 3^3$$

$$(1+\cdots+n)^2 \quad = \quad 1^3+\cdots+n^3$$

WHY??

Make the old square blue . . .

$$(1)^2 = 1^3$$
$$(1+2)^2 = 1^3 + 2^3$$
$$(1+2+3)^2 = 1^3 + 2^3 + 3^3$$



$1+2+3+4$ by $1+2+3+4$

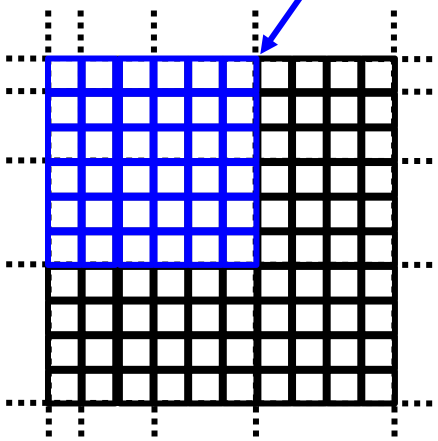
$$(1+2+3+4)^2$$

$$(1+\cdots+n)^2 \quad = \quad 1^3+\cdots+n^3$$

WHY??

Make the old square blue . . .

$$(1)^2 = 1^3$$
$$(1+2)^2 = 1^3 + 2^3$$
$$(1+2+3)^2 = 1^3 + 2^3 + 3^3$$



$1+2+3+4$ by $1+2+3+4$

$$(1+2+3+4)^2$$

Split remainder along dashed lines . . .

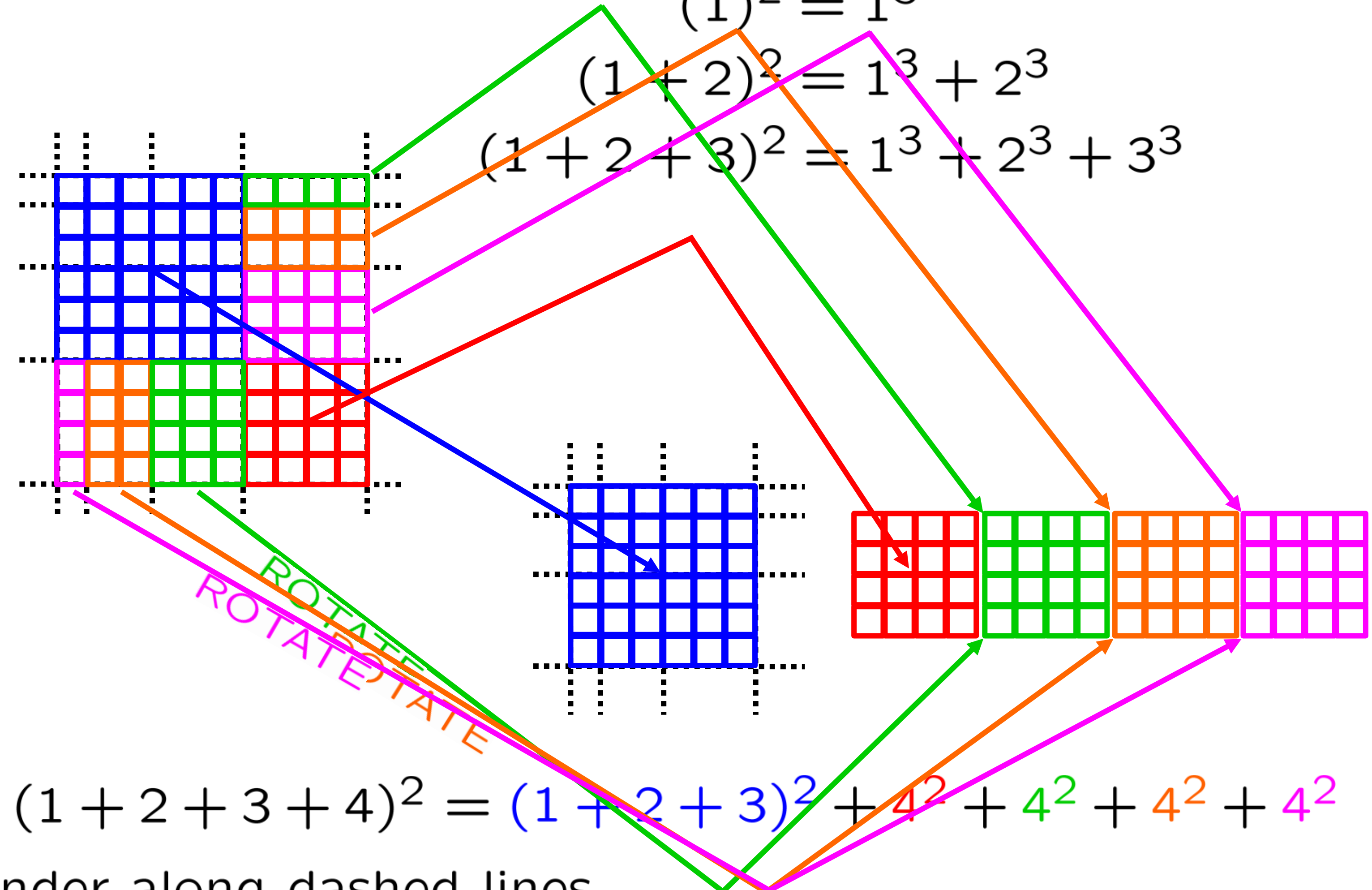
$$(1+\cdots+n)^2 \quad = \quad 1^3+\cdots+n^3$$

WHY??

$$(1)^2 = 1^3$$

$$(1+2)^2 = 1^3 + 2^3$$

$$(1+2+3)^2 = 1^3 + 2^3 + 3^3$$



$$(1+2+3+4)^2 = (1+2+3)^2 + 4^2 + 4^2 + 4^2 + 4^2$$

Split remainder along dashed lines . . .

Two squares (blue and red)

and six rectangles (two green, two orange and two purple).

Rectangles pair up to form squares . . .

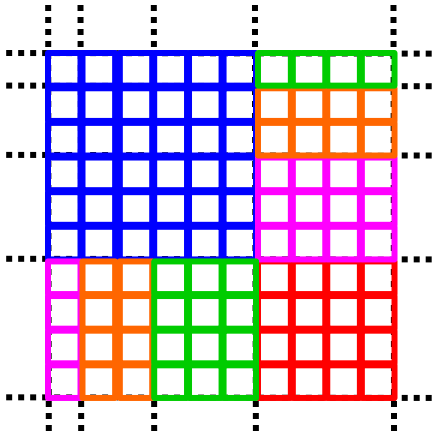
$$(1 + \cdots + n)^2 = 1^3 + \cdots + n^3$$

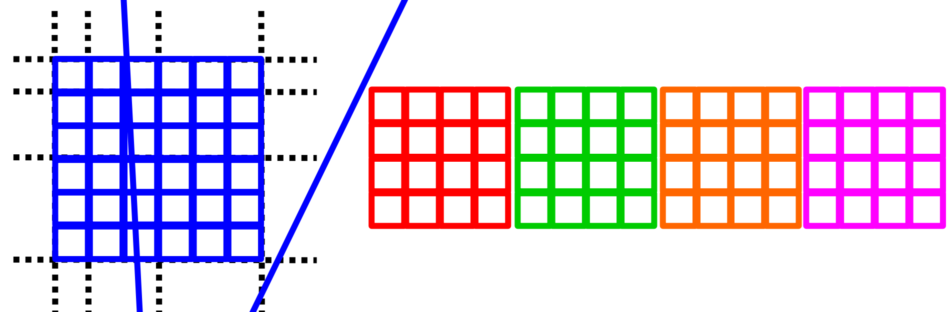
WHY??

$$(1)^2 = 1^3$$

$$(1+2)^2 = 1^3 + 2^3$$

$$(1+2+3)^2 = 1^3 + 2^3 + 3^3$$





$$(1+2+3+4)^2 = (1+2+3)^2 + 4^2 + 4^2 + 4^2 + 4^2$$

$$= 1^3 + 2^3 + 3^3 + \boxed{4} \cdot 4^2$$

$$= 1^3 + 2^3 + 3^3 + 4^3$$

$$(1 + \cdots + n)^2 \quad = \quad 1^3 + \cdots + n^3$$

WHY??

$$(1)^2 = 1^3$$
$$(1+2)^2 = 1^3 + 2^3$$
$$(1+2+3)^2 = 1^3 + 2^3 + 3^3$$
$$(1+2+3+4)^2 = 1^3 + 2^3 + 3^3 + 4^3$$
etc.

$$(1+2+3+4)^2$$

$$= 1^3 + 2^3 + 3^3 + 4^3$$

$$(1 + \cdots + n)^2 \quad = \quad 1^3 + \cdots + n^3$$

WHY??

$$(1)^2 = 1^3$$

$$(1+2)^2 = 1^3 + 2^3$$

$$(1+2+3)^2 = 1^3 + 2^3 + 3^3$$

$$(1+2+3+4)^2 = 1^3 + 2^3 + 3^3 + 4^3$$

etc.

STOP

$$(1+\cdots+n)^2 \quad = \quad 1^3+\cdots+n^3$$

WHY??