CALCULUS
Inverse functions

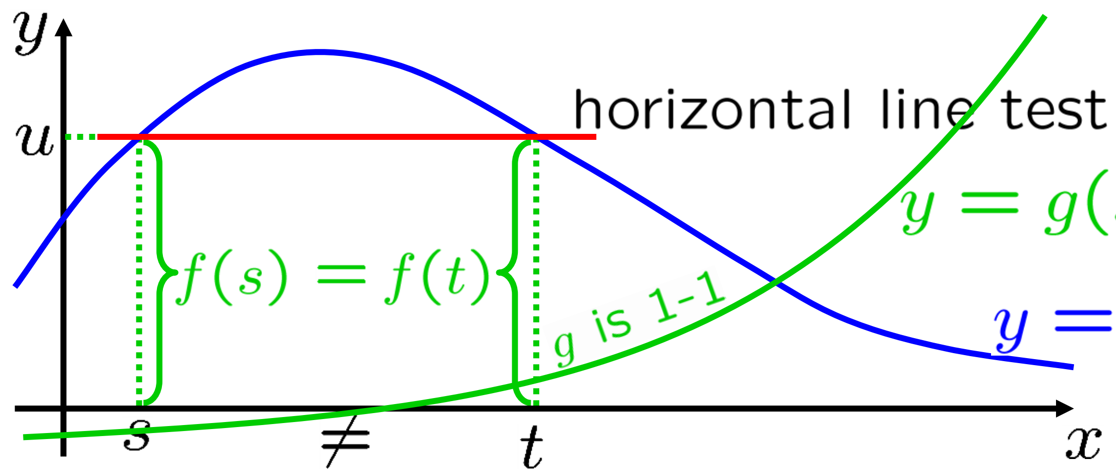
DEF'N: A function f is **one-to-one** (or **1-1**) (or **injective**) if

$$\forall s,t \in \text{dom}[f], \qquad s \neq t \quad \Rightarrow \quad f(s) \neq f(t).$$

HORIZONTAL LINE TEST:

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

SKILL
recognize 1-1 gph



two-to-one (s and t map to u)
NOT one-to-one

DEF'N: A function f is **one-to-one** (or **1-1**) if

$\forall s, t \in \text{dom}[f], \qquad s \neq t \quad \Rightarrow \quad f(s) \neq f(t).$

DEF'N: Let $f : A \to C$ be a function.

The **image of** f is $\{f(a) \,|\, a \in A\}$,

and is denoted by either $\boxed{\text{im}[f]}$ or $\boxed{f(A)}$.

DEF'N: We say $f : A \to C$ is **onto** B if $f(A) = B$.

DEF'N: Let $f : A \to C$ be both one-to-one and onto B.

The **inverse of** f is the function

$f^{-1} : B \to A$ defined by

$f^{-1}(y) \quad := \quad [\text{ the unique } x \in A \text{ s.t. } f(x) = y \,] \quad .$

$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$

Spp

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To have an inverse, one-to-one is needed.

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To have an inverse,

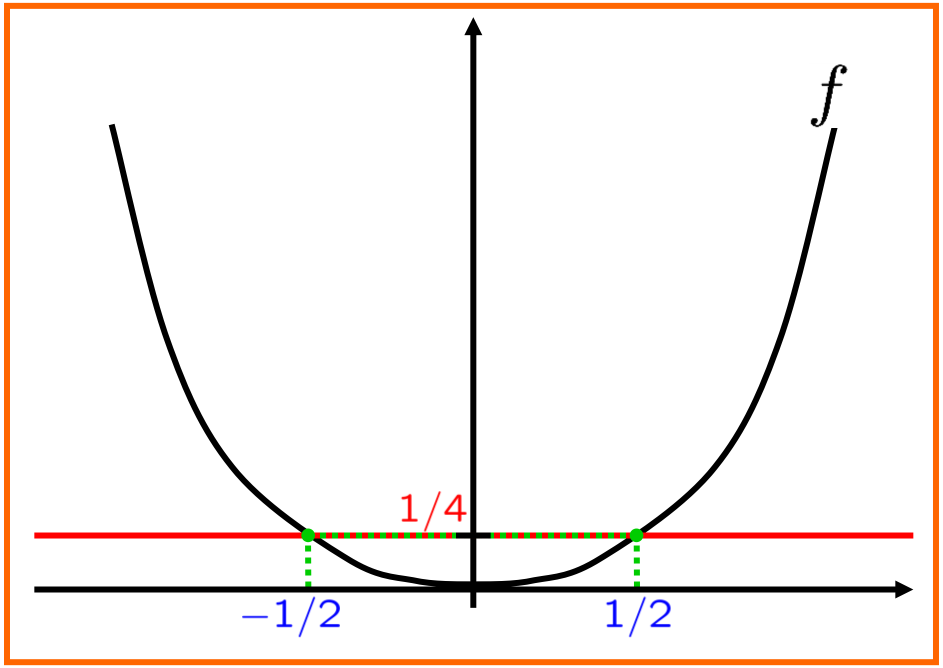
one-to-one is needed.

$f^n(x) := [f(x)]^n$, provided $n \neq -1$.

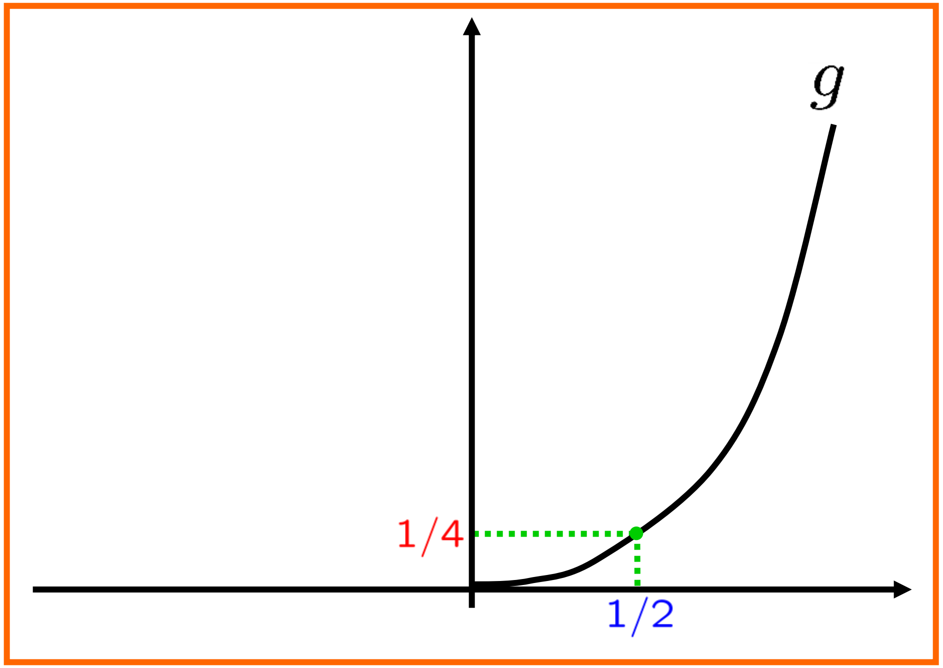
$\forall c \neq 0, c^{-1} = \dfrac{1}{c}$

Typically, $f^{-1}(x) \neq [f(x)]^{-1} = \dfrac{1}{f(x)}$.

$f(x) = x^2$



$g := f|[0, \infty)$



$f(1/2) = 1/4$
$f(-1/2) = 1/4$

f is not one-to-one.

No good way to define $f^{-1}(1/4)$ as a number.

Is it $1/2$ or $-1/2$?

$g(1/2) = 1/4$
$g(-1/2)$ is undefined!

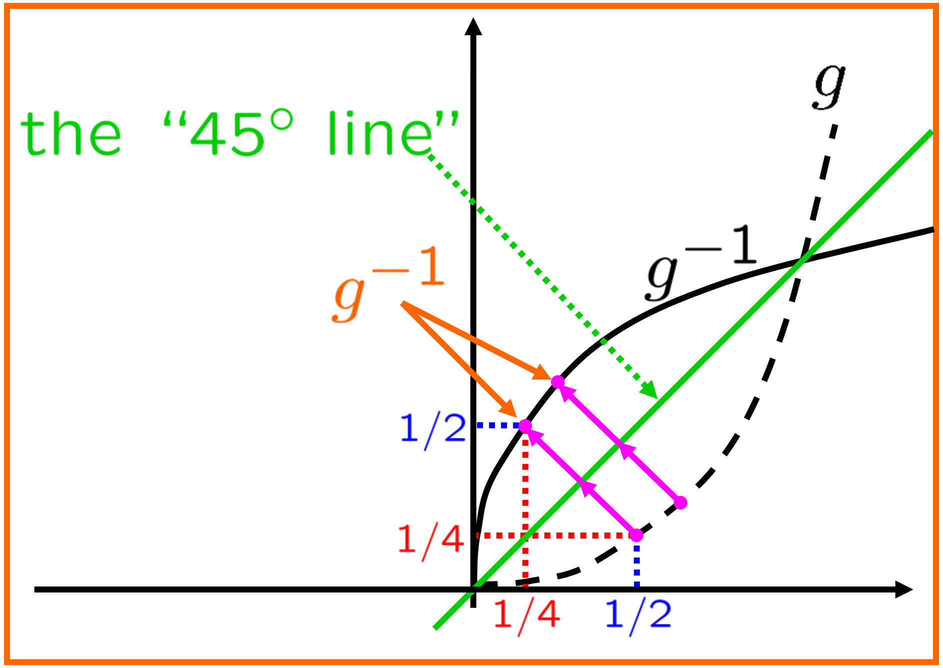
g is one-to-one.

$g : [0, \infty) \to [0, \infty)$ is one-to-one.

Goal: Graph $g^{-1} = \sqrt{\bullet}$.

Spp

$g^{-1} = \sqrt{\bullet}$



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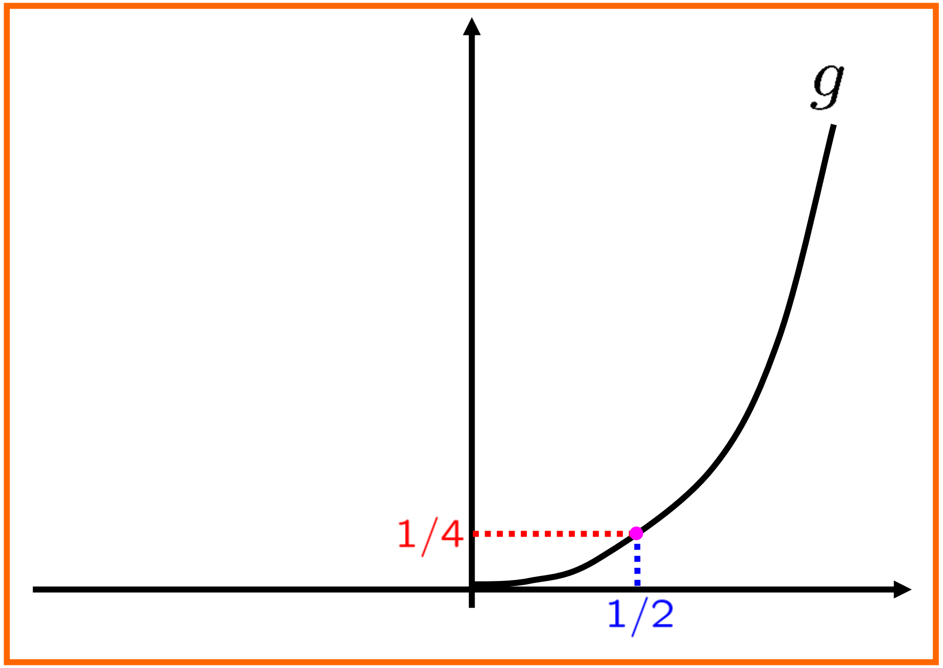


$g^{-1}(1/4) = 1/2$

$g(1/2) = 1/4$

$g(-1/2)$ is undefined!

THE MORAL:
To get the gph of the inverse,
reflect the graph of the fn
through the 45° line

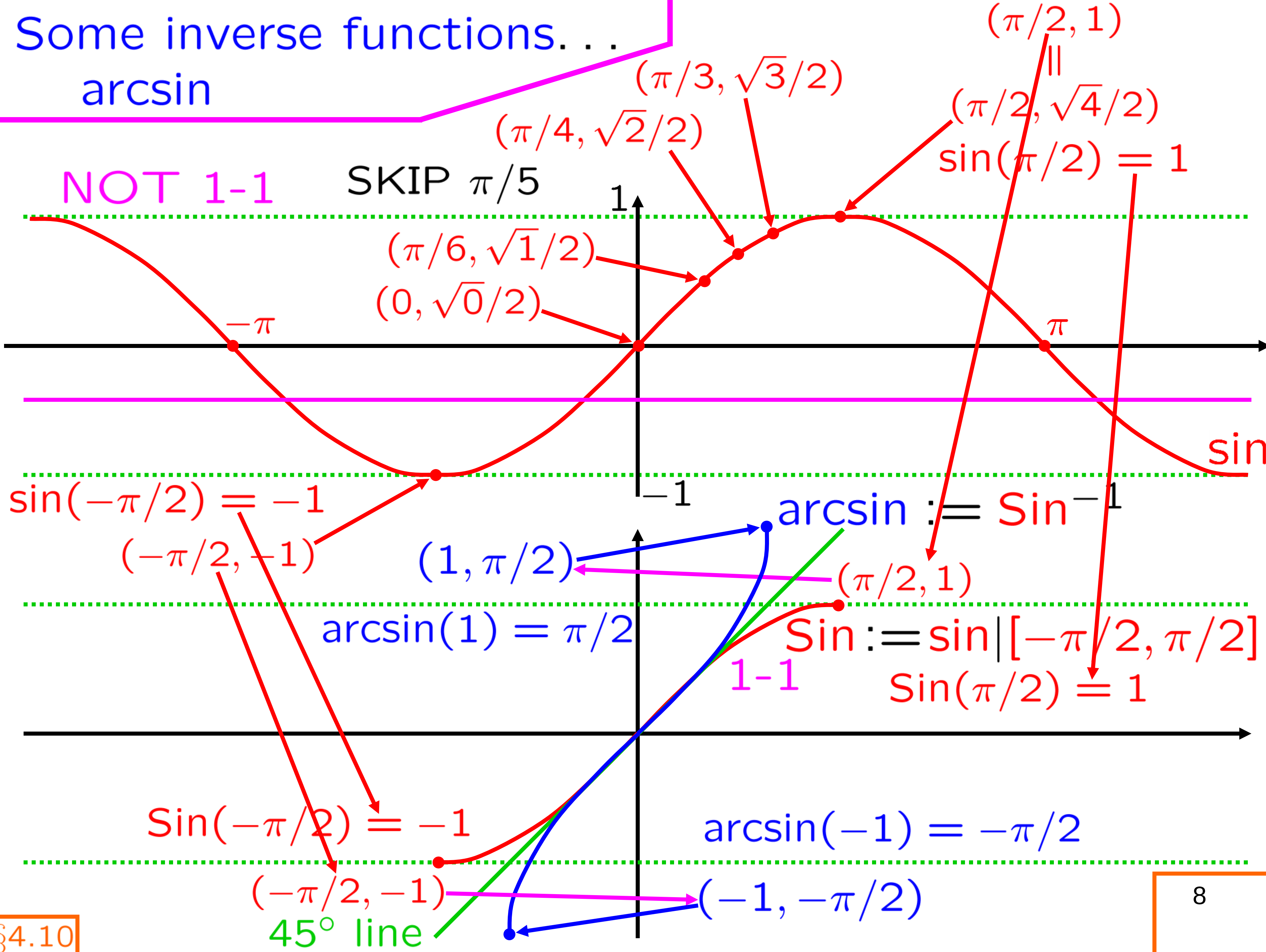
g is one-to-one.

$g : [0, \infty) \to [0, \infty)$ is one-to-one.

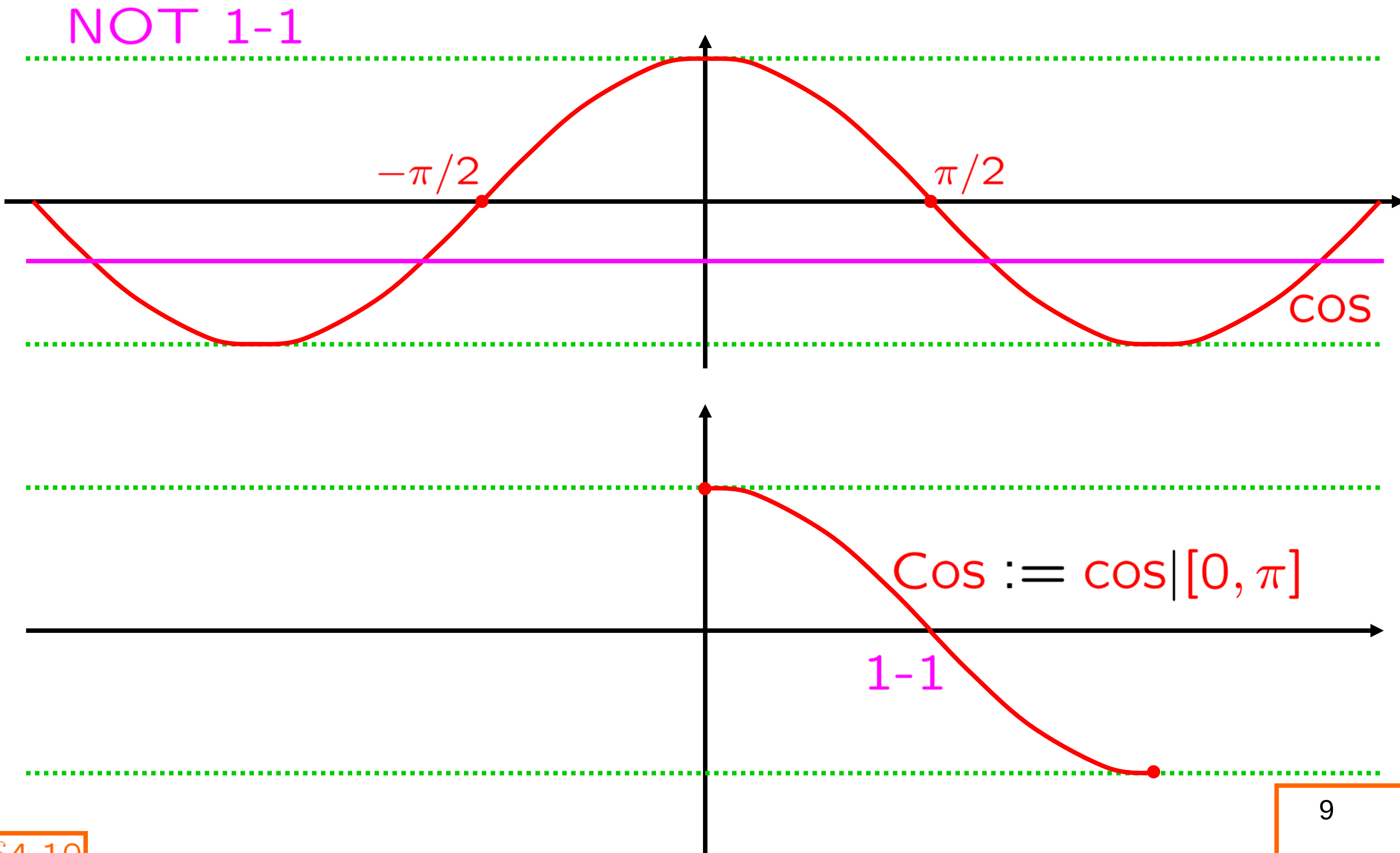
Goal: Graph $g^{-1} = \sqrt{\bullet}$.

Spp

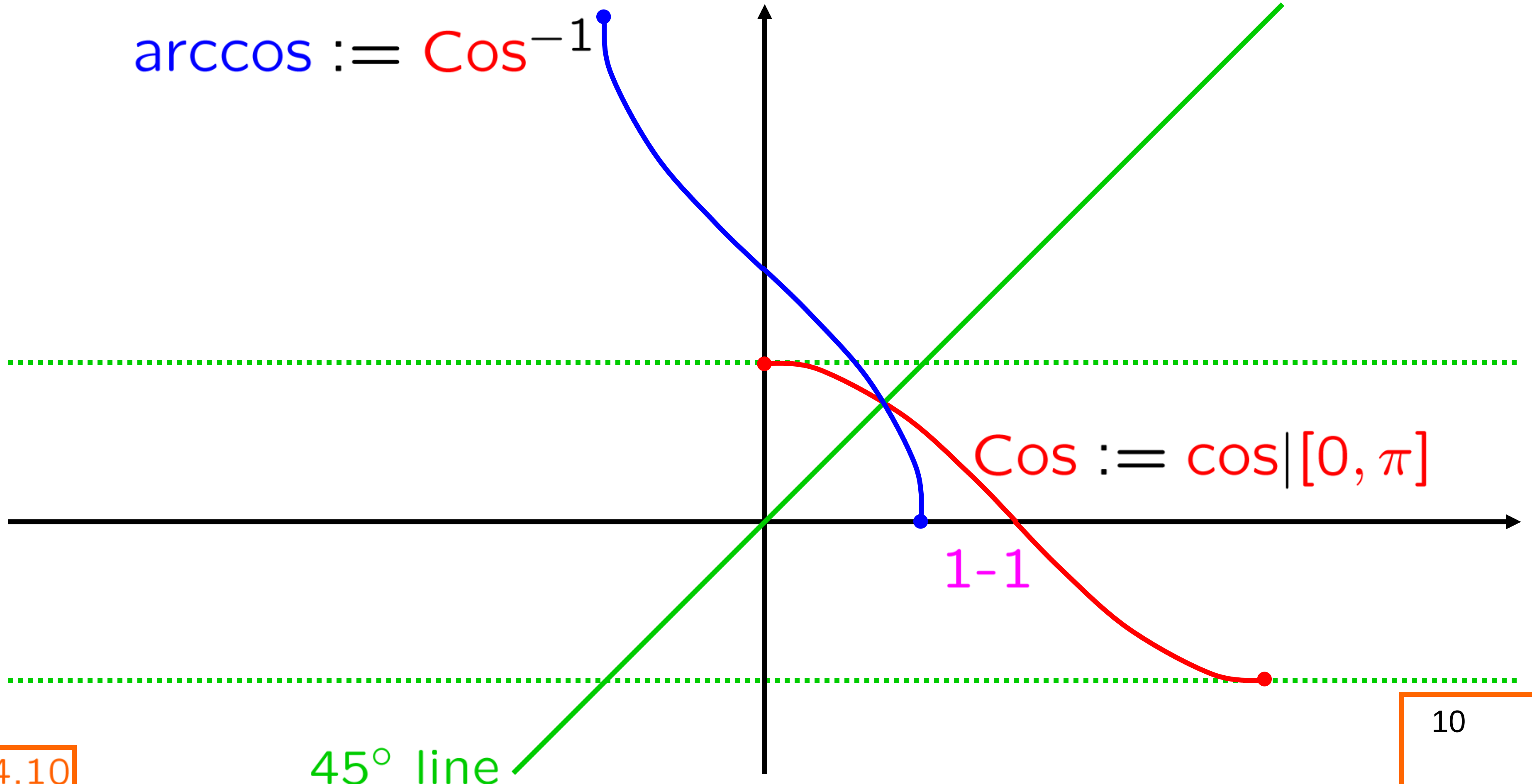
Some inverse functions. . .
arcsin



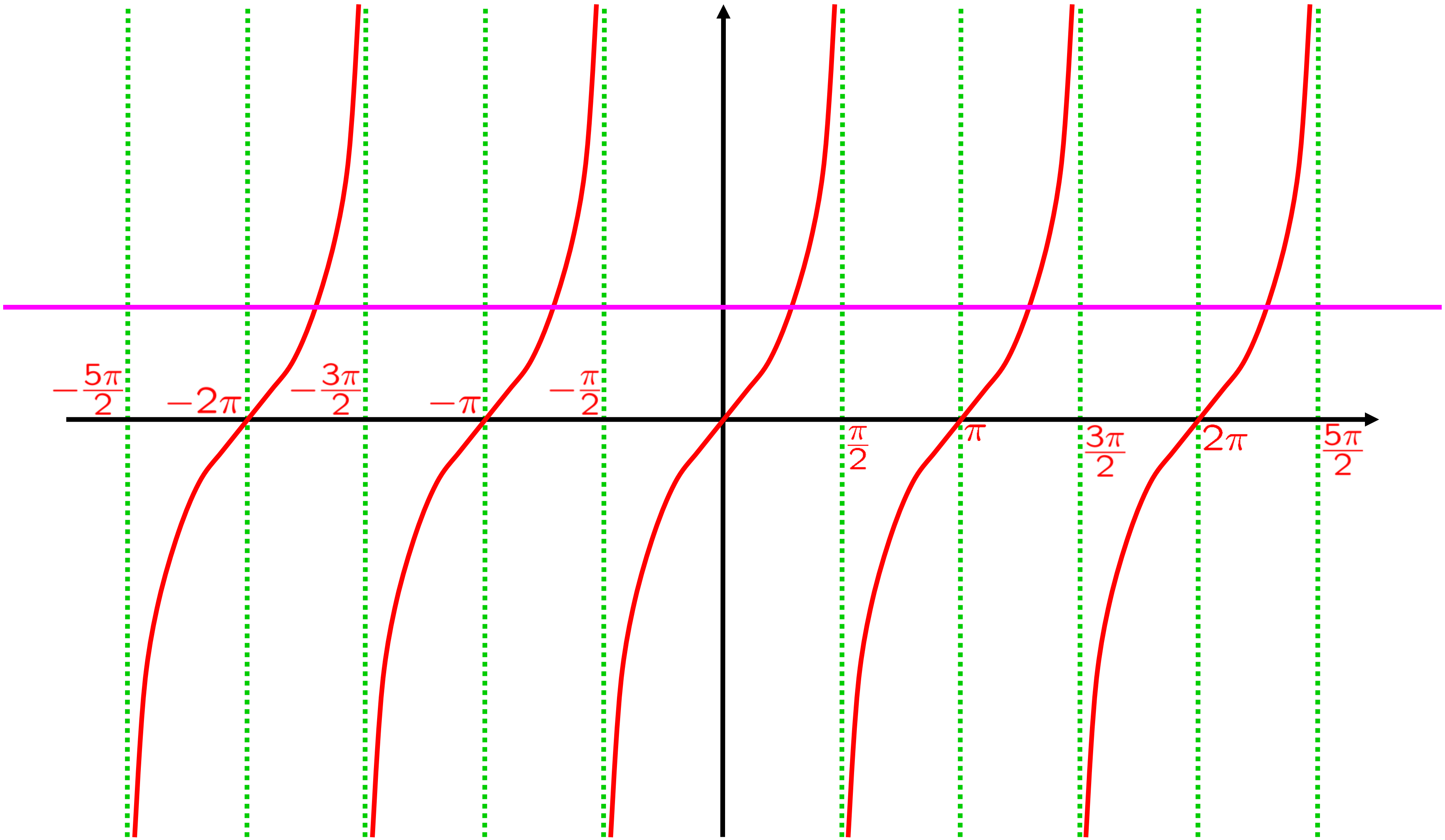
Some inverse functions. . .
arcsin



Some inverse functions. . .
arcsin, arccos



Some inverse functions...
arcsin, arccos

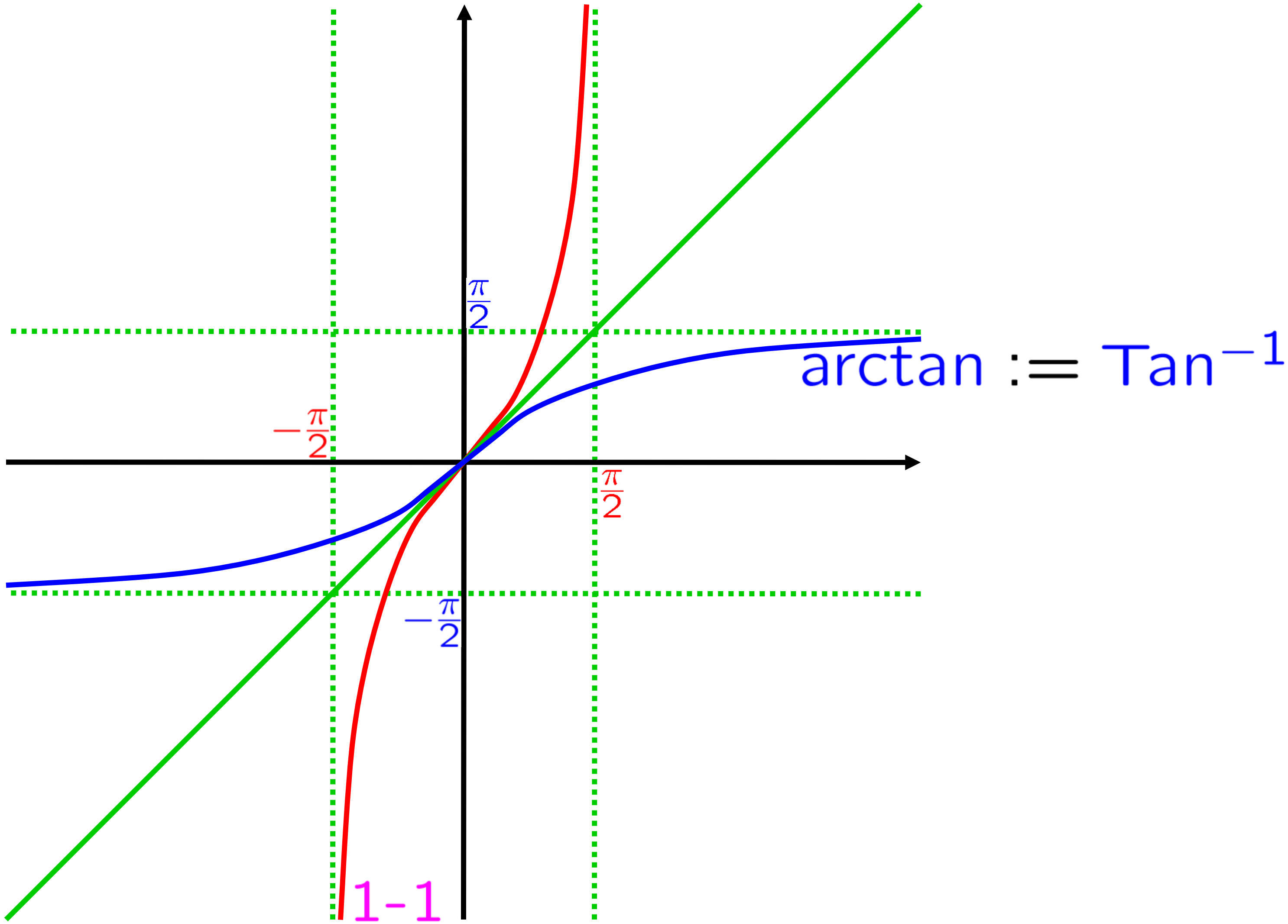


NOT 1-1

tan

$$\text{Tan} := \text{tan}|(-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

Some inverse functions...
arcsin, arccos, arctan



$\mathrm{Tan} := \tan|(-\frac{\pi}{2},\frac{\pi}{2})$

$\mathrm{Tan} := \tan|(-\frac{\pi}{2},\frac{\pi}{2})$

Some inverse functions. . .
arcsin, arccos, arctan

NOT 1-1

cot

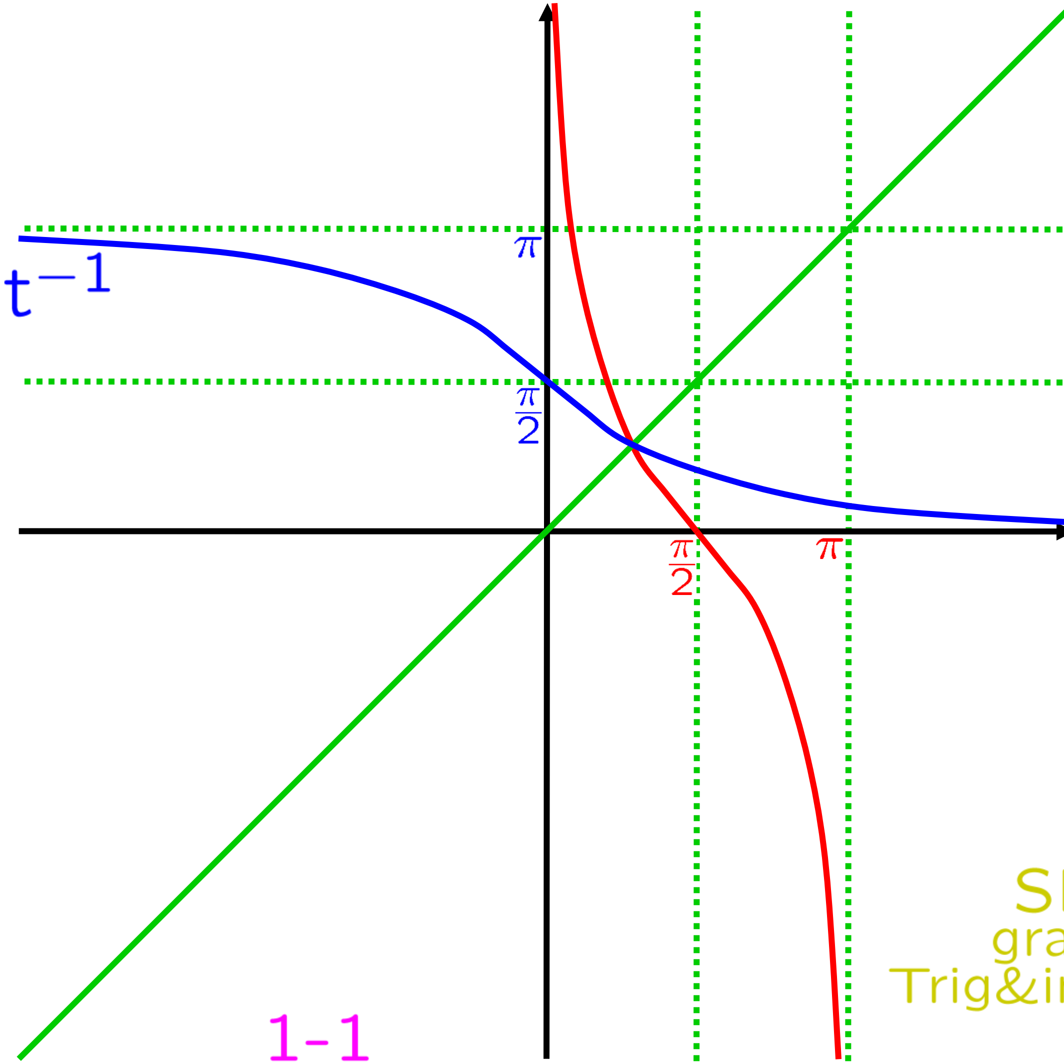
$\mathrm{Cot} := \mathrm{cot}|(0, \pi)$

§4.10

Some inverse functions. . .
arcsin, arccos, arctan, arccot

We do **NOT** try to define arcsec or arccsc

$\text{arccot} := \text{Cot}^{-1}$



45° line

1-1

$\text{Cot} := \cot|(0,\pi)$

$\text{Cot} := \cot|(0,\pi)$

SKILL
graphs of
Trig&inverse Trig

$\sin : (-\infty, \infty) \to [-1, 1]$

$\text{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$\text{Sin}^{-1} = \arcsin$

$\text{Sin}^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

x	$\sin x$	$\text{Sin}\, x$
0	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{0}}{2}$
$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{1}}{2}$
SKIP $\pi/5$		
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{4}}{2}$
π	0	undefined
2π	0	undefined

$\notin$

x	$\text{Sin}^{-1} x = \arcsin x$
$\frac{\sqrt{0}}{2}$	0
$-\frac{\sqrt{1}}{2}$	$-\frac{\pi}{6}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$
$-\frac{\sqrt{4}}{2}$	$-\frac{\pi}{2}$

SKILL
comp inv Trig

$\sin(-x) = -(\sin x)$

$\text{Sin}(-x) = -(\text{Sin}\, x)$

$\arcsin(-x) = -(\arcsin x)$

$\sin : (-\infty, \infty) \to [-1, 1]$

$\text{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$\text{Sin}^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, where $\text{Sin}^{-1} = \arcsin x$

x	$\sin x$	$\text{Sin}\, x$
0	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{0}}{2}$
$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{1}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{4}}{2}$
π	0	undefined
2π	0	undefined

x	$\text{Sin}^{-1} x = \arcsin x$
$\frac{\sqrt{0}}{2}$	0
$-\frac{\sqrt{1}}{2}$	$-\frac{\pi}{6}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$
$-\frac{\sqrt{4}}{2}$	$-\frac{\pi}{2}$

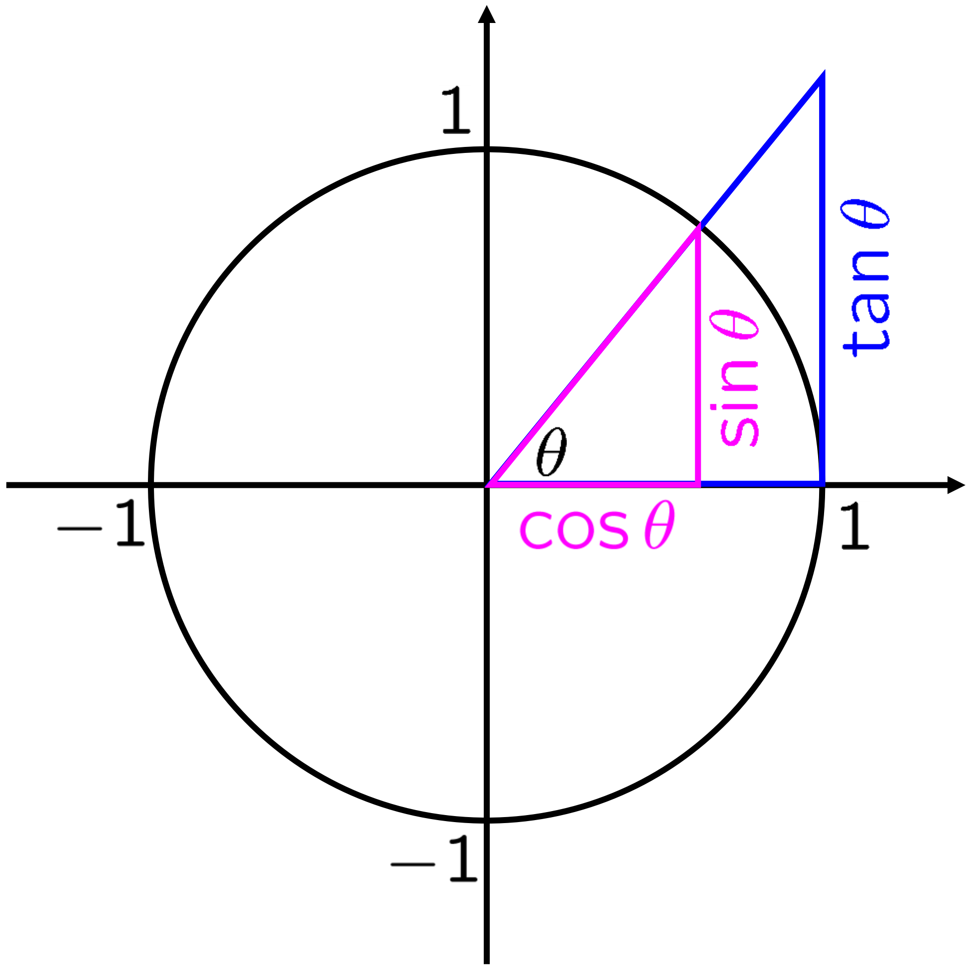
Next subtopic:

Why "arc"?

SKILL
comp inv Trig

EXERCISE:
Make similar tables for arccos, arctan, arccot.

Recall. . .

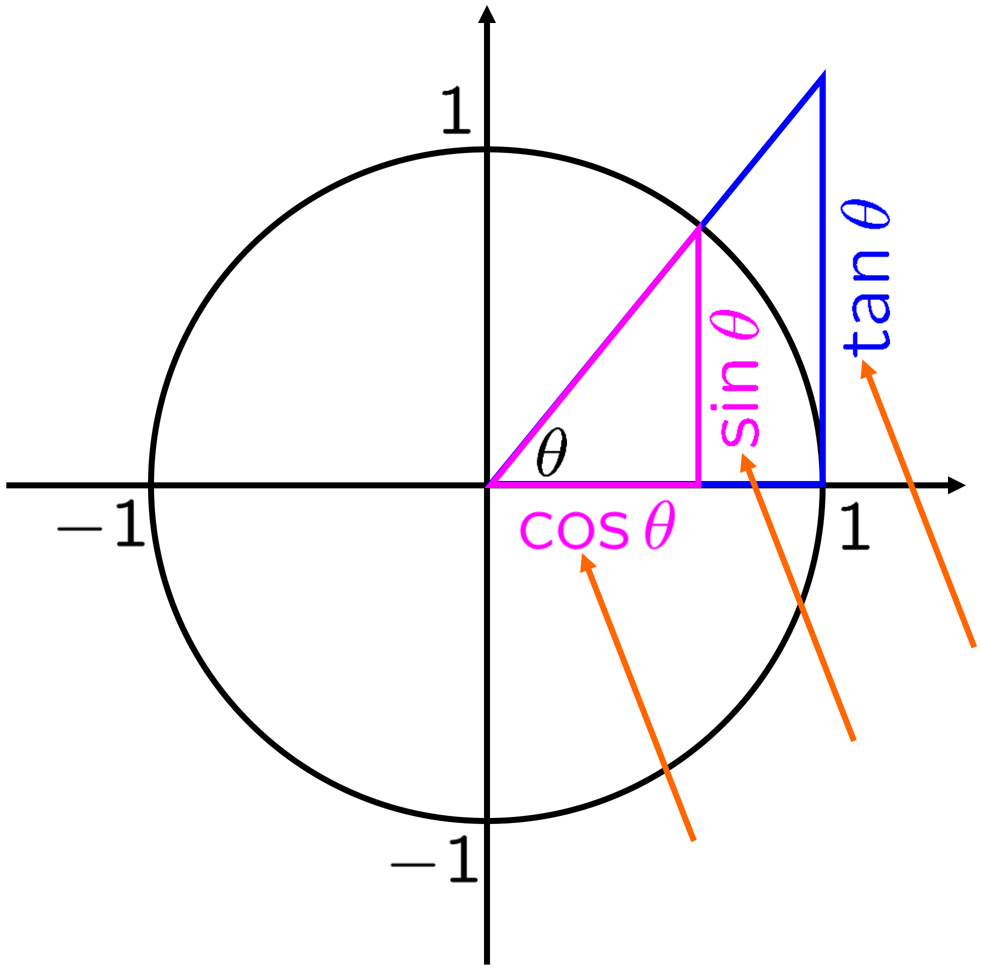


$0 < \theta < \frac{\pi}{2}$, so

$$\mathrm{Sin}\,\theta = \sin\theta,$$
$$\mathrm{Cos}\,\theta = \cos\theta,$$
$$\mathrm{Tan}\,\theta = \tan\theta$$

and $\mathrm{Cot}\,\theta = \cot\theta$

$$\mathrm{Sin} : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to [-1, 1]$$
$$\mathrm{Cos} : [0, \pi] \to [-1, 1]$$
$$\mathrm{Tan} : (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \to \mathbb{R}$$
$$\mathrm{Cot} : (0, \pi) \to \mathbb{R}$$



$0 < \theta < \frac{\pi}{2}$, so

$\text{Sin}\,\theta = \sin\theta$,

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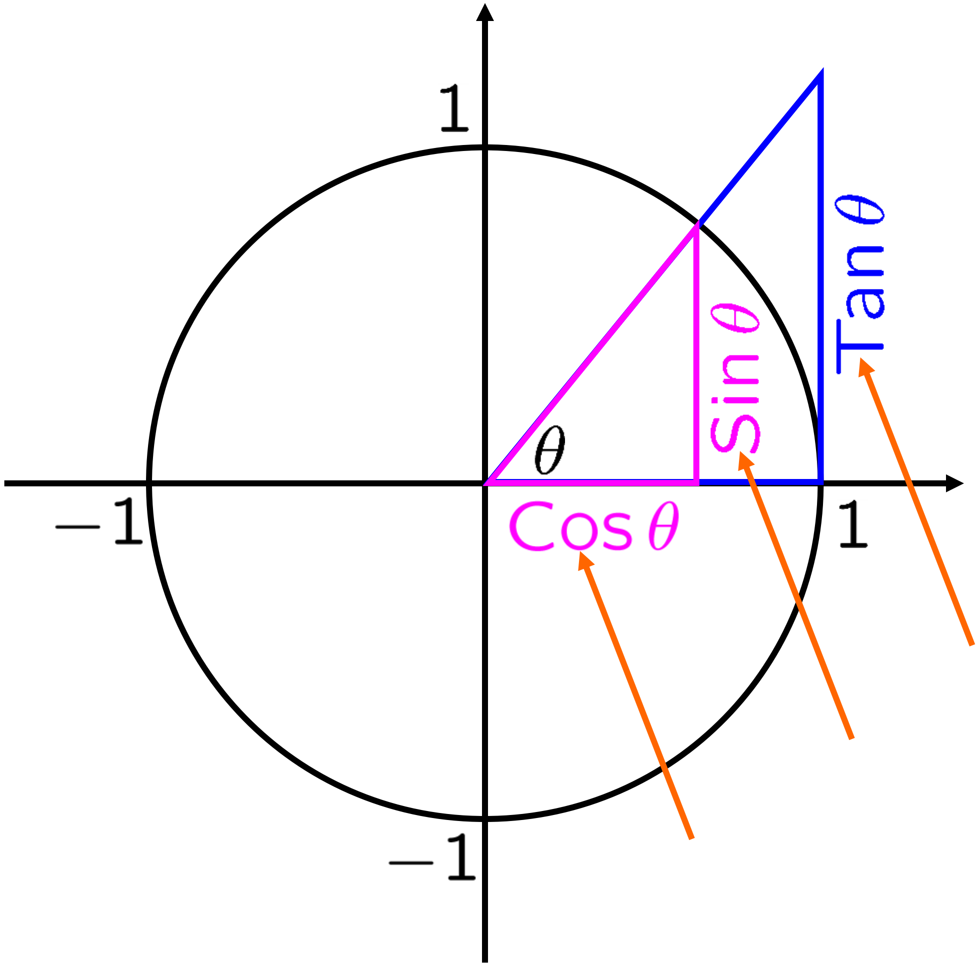
and $\text{Cot}\,\theta = \cot\theta$

$\text{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$\text{Cos} : [0, \pi] \to [-1, 1]$

$\text{Tan} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

$\text{Cot} : (0, \pi) \to \mathbb{R}$



Let $S := \text{Sin}\,\theta$,
let $C := \text{Cos}\,\theta$
and let $T := \text{Tan}\,\theta$.

$0 < \theta < \frac{\pi}{2}$, so

$\mathrm{Sin}\,\theta = \sin\theta$,

$\mathrm{Cos}\,\theta = \cos\theta$,

$\mathrm{Tan}\,\theta = \tan\theta$

and $\mathrm{Cot}\,\theta = \cot\theta$

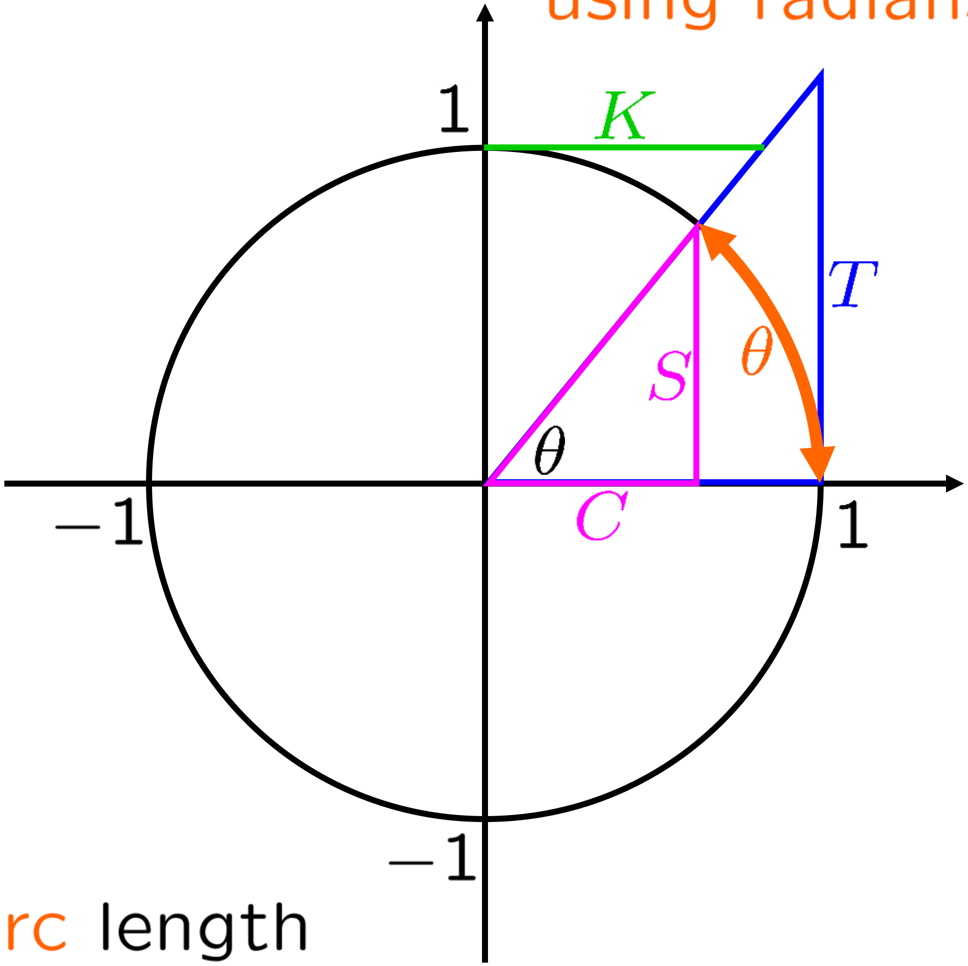
because we're using radians...

$\mathrm{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow[\text{1-1}]{\text{onto}} [-1, 1]$

$\mathrm{Cos} : [0, \pi] \xrightarrow[\text{1-1}]{\text{onto}} [-1, 1]$

$\mathrm{Tan} : (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow[\text{1-1}]{\text{onto}} \mathbb{R}$

$\mathrm{Cot} : (0, \pi) \xrightarrow[\text{1-1}]{\text{onto}} \mathbb{R}$



θ measures arc length

$\theta = \mathrm{arc}\sin S$

$\theta = \mathrm{arc}\cos C$

$\theta = \mathrm{arc}\tan T$

$\theta = \mathrm{arc}\cot K$

Let $S := \mathrm{Sin}\,\theta$,

let $C := \mathrm{Cos}\,\theta$

and let $T := \mathrm{Tan}\,\theta$.

Let $K := \mathrm{Cot}\,\theta$.

SKILL
dom&image of
Trig&inverse Trig

LEARN:

$\text{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow[\text{1-1}]{\text{onto}} [-1, 1]$

$\text{Cos} : [0, \pi] \xrightarrow[\text{1-1}]{\text{onto}} [-1, 1]$

$\text{Tan} : (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow[\text{1-1}]{\text{onto}} \mathbb{R}$

$\text{Cot} : (0, \pi) \xrightarrow[\text{1-1}]{\text{onto}} \mathbb{R}$

LEARN:

$\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arccos : [-1, 1] \to [0, \pi]$

$\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$\text{arccot} : \mathbb{R} \to (0, \pi)$

Next subtopic:

Inverses of complementary Trig fns yield complementary angles

$\text{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ $\qquad$ $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\text{Cos} : [0, \pi] \to [-1, 1]$ $\qquad$ $\arccos : [-1, 1] \to [0, \pi]$

Fact: $\forall x \in [-1, 1]$,

$$(\arcsin x) + (\arccos x) = \frac{\pi}{2}$$

$\text{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ $\qquad$ $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

Fact: $\forall x \in \mathbb{R}$,

$\text{Cos} : [0, \pi] \to [-1, 1]$

$$(\arctan x) + (\text{arccot}\, x) = \frac{\pi}{2}$$

$\text{Tan} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ $\qquad$ $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$\text{Tan} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ $\qquad$ $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$\text{Cot} : (0, \pi) \to \mathbb{R}$ $\qquad$ $\text{arccot} : \mathbb{R} \to (0, \pi)$

$\text{Cot} : (0, \pi) \to \mathbb{R}$ $\qquad$ $\text{arccot} : \mathbb{R} \to (0, \pi)$

Next subtopic:

Inverses of complementary Trig fns yield complementary angles

$\text{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ $\qquad$ $\text{arcsin} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\text{Cos} : [0, \pi] \to [-1, 1]$ $\qquad$ $\text{arccos} : [-1, 1] \to [0, \pi]$

Fact: $\forall x \in [-1, 1]$,

$$(\text{arcsin}\, x) + (\text{arccos}\, x) = \frac{\pi}{2}$$

Proof: Let $\theta := \text{arcsin}\, x := \text{Sin}^{-1} x$.

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \qquad x = \text{Sin}\, \theta = \sin \theta$$

$$\frac{\pi}{2} - \frac{\pi}{2} \le \frac{\pi}{2} - \theta \le \frac{\pi}{2} + \frac{\pi}{2} \qquad \|$$

$$0 \le \frac{\pi}{2} - \theta \le \pi \qquad \text{Cos}(\frac{\pi}{2} - \theta) = \cos(\frac{\pi}{2} - \theta)$$

Cos^{-1}

$$\text{arccos}\, x = \text{Cos}^{-1} x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - (\text{arcsin}\, x)$$

QED

$\text{Sin} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ arcsin : $[-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\text{Cos} : [0, \pi] \to [-1, 1]$ arccos : $[-1, 1] \to [0, \pi]$

Fact: $\forall x \in [-1, 1]$,

We can also see this by looking at graphs . . .

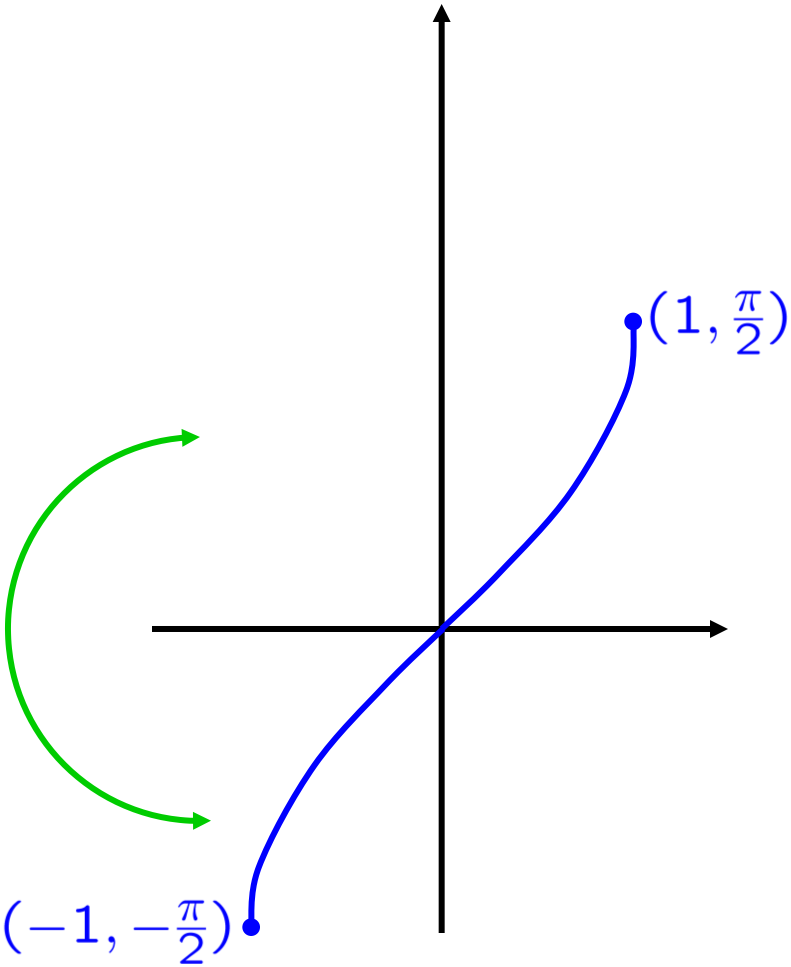
$$(\arcsin x) + (\arccos x) = \frac{\pi}{2}$$

Fact: $\forall x \in \mathbb{R}$,

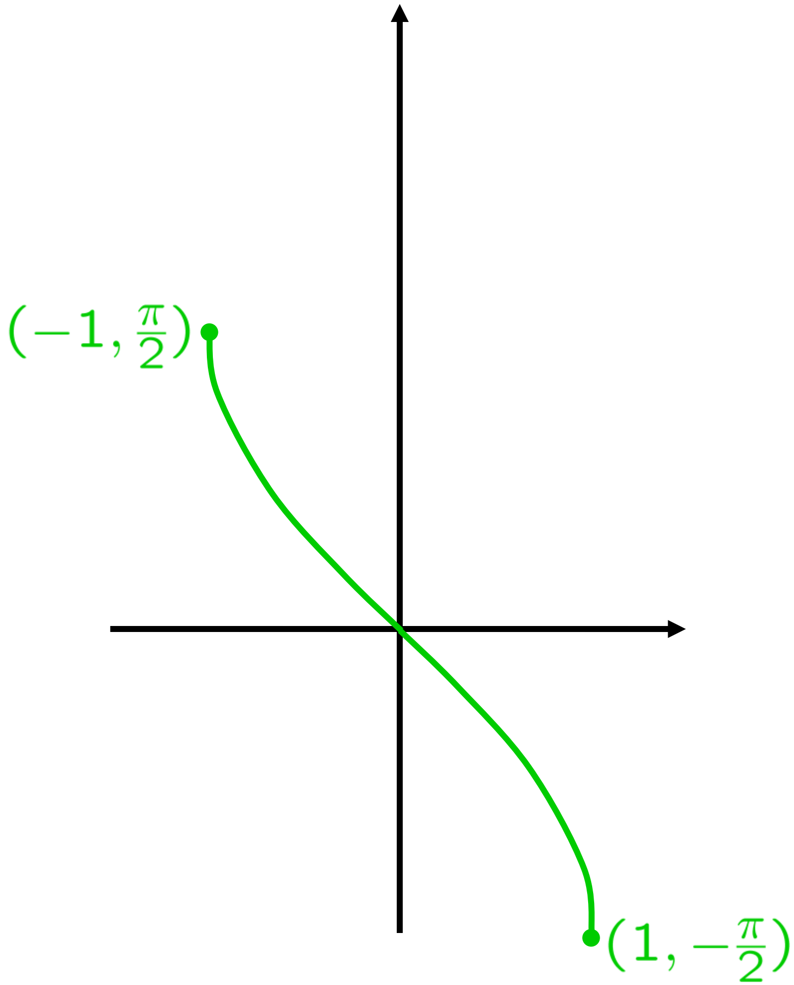
$$(\arctan x) + (\operatorname{arccot} x) = \frac{\pi}{2}$$

Proof: Similar. QED

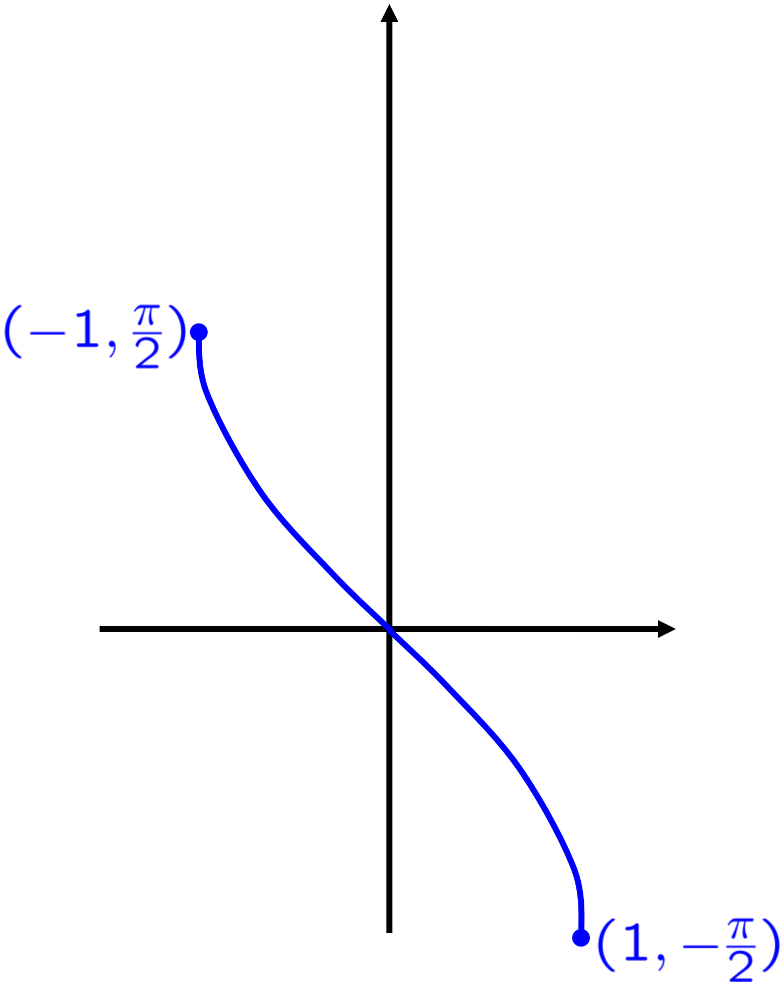
See Exercise 7, §4.10, p. 92.



$$y = \text{arcsin}\, x$$
$$y :\rightarrow -y$$

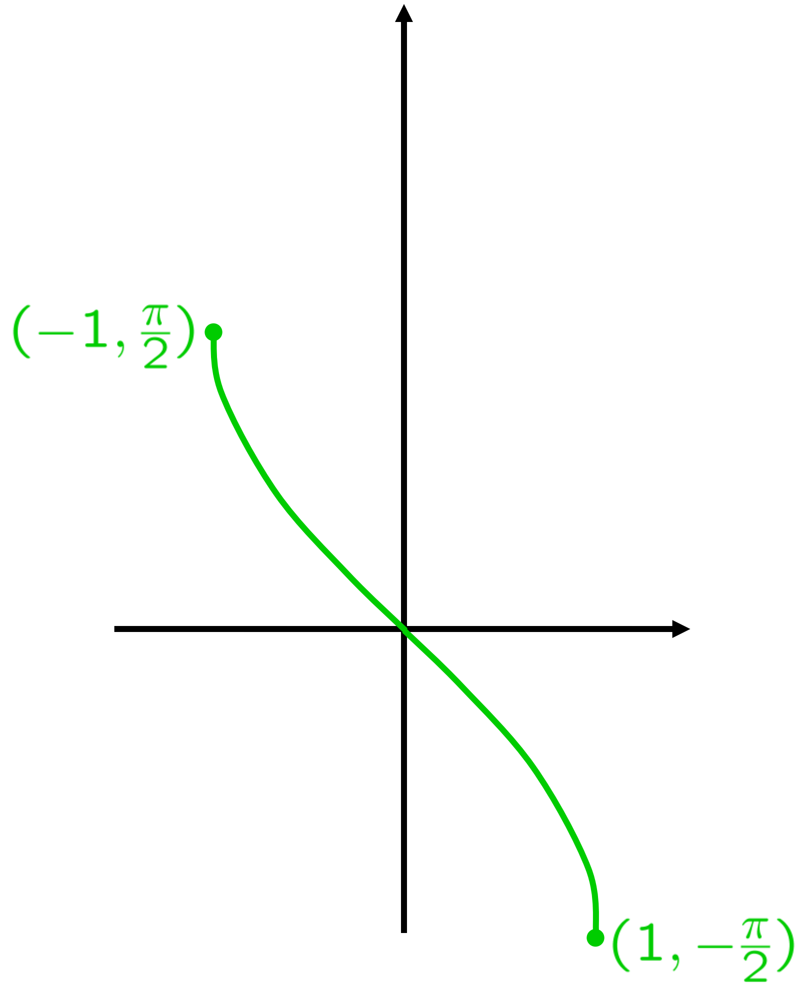


$$-y = \text{arcsin}\, x$$
$$\Updownarrow$$
$$y = -\,\text{arcsin}\, x$$



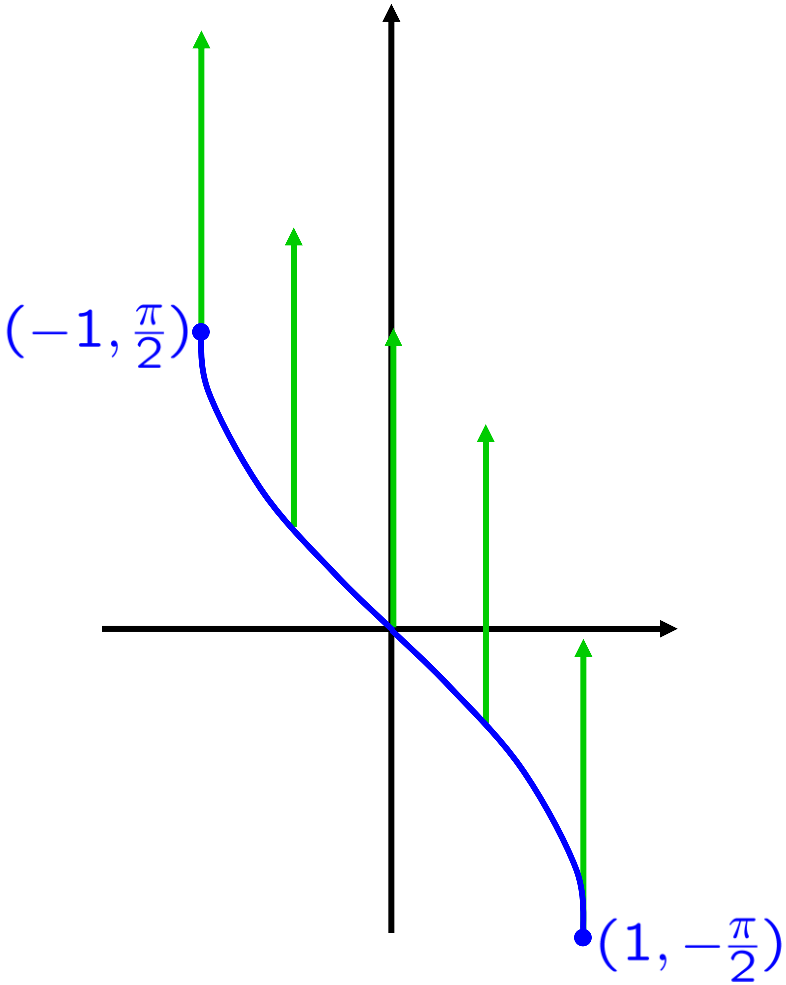
$$y = -\operatorname{arcsin} x$$

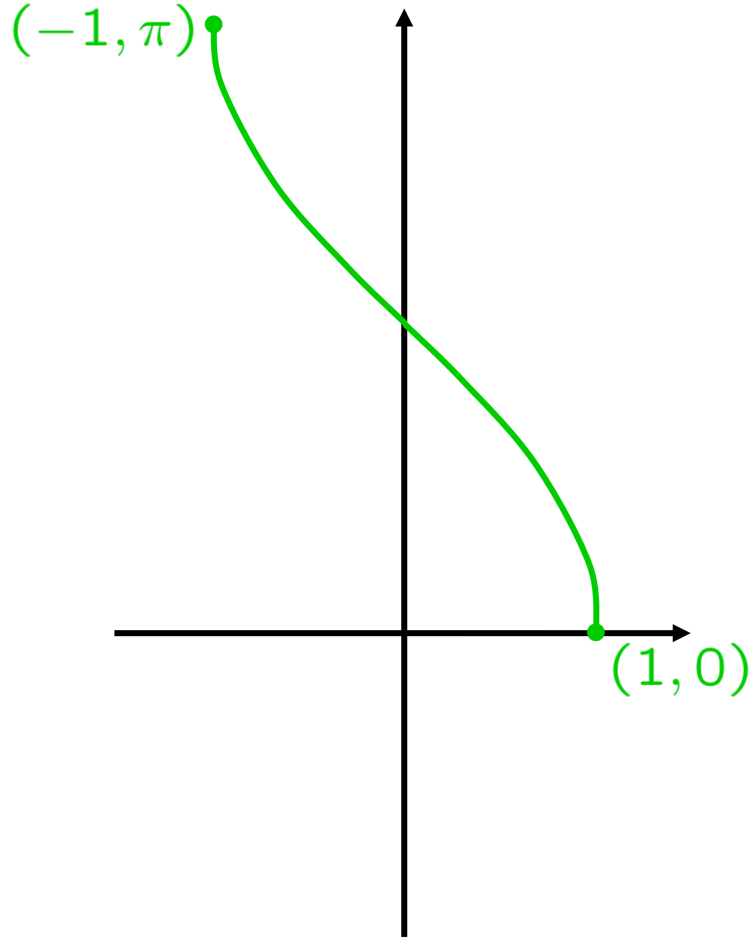
$$y :\longrightarrow y - \frac{\pi}{2}$$



$$y - \frac{\pi}{2} = -\operatorname{arcsin} x$$

$$y = -\operatorname{arcsin} x$$





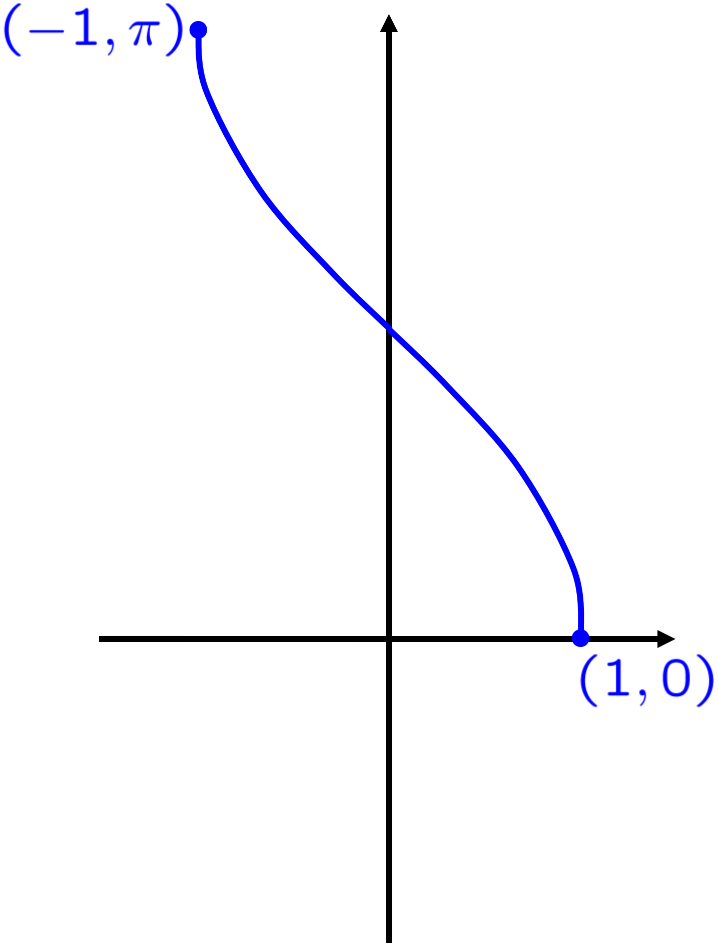
$$y = -\arcsin x$$

$$y :\longrightarrow y - \frac{\pi}{2}$$

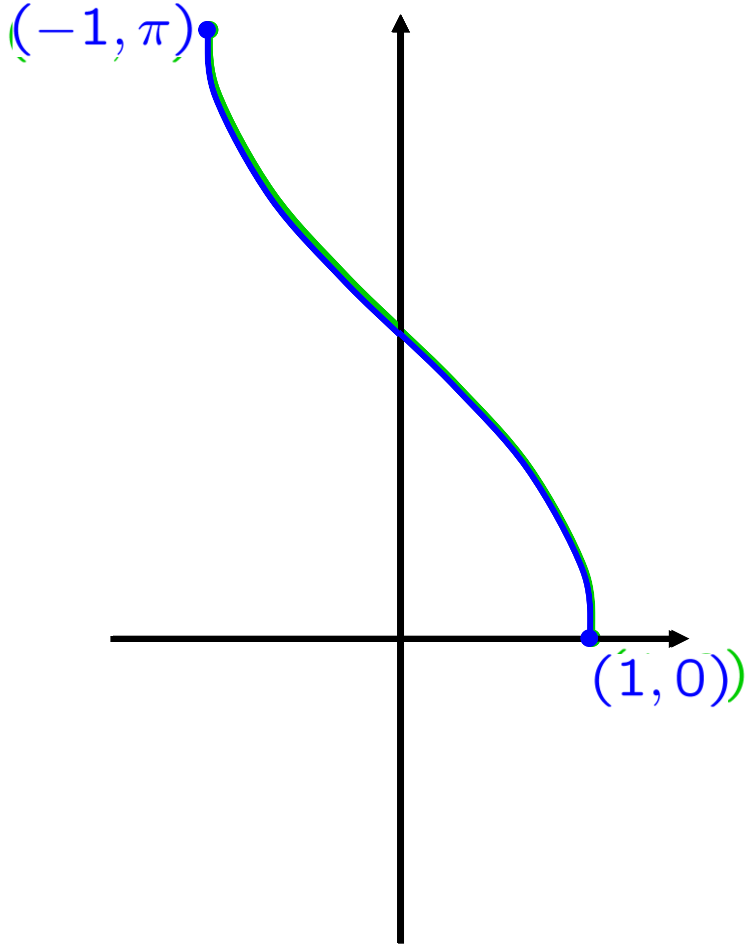
$$y - \frac{\pi}{2} = -\arcsin x$$

$$\Updownarrow$$

$$y = \frac{\pi}{2} - \arcsin x$$



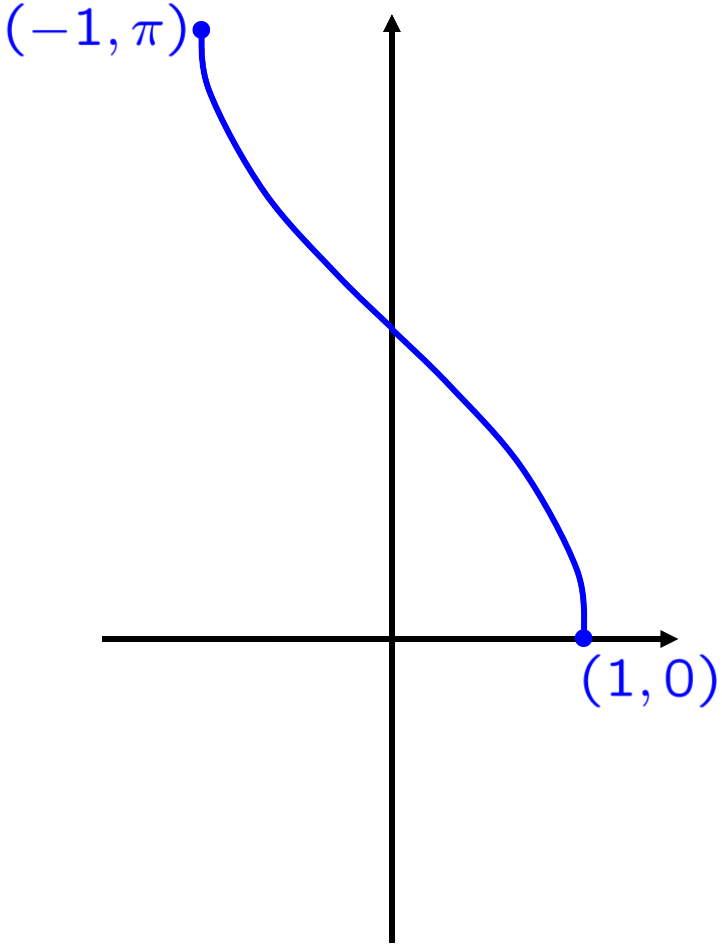
$y = \frac{\pi}{2} - \arcsin x$

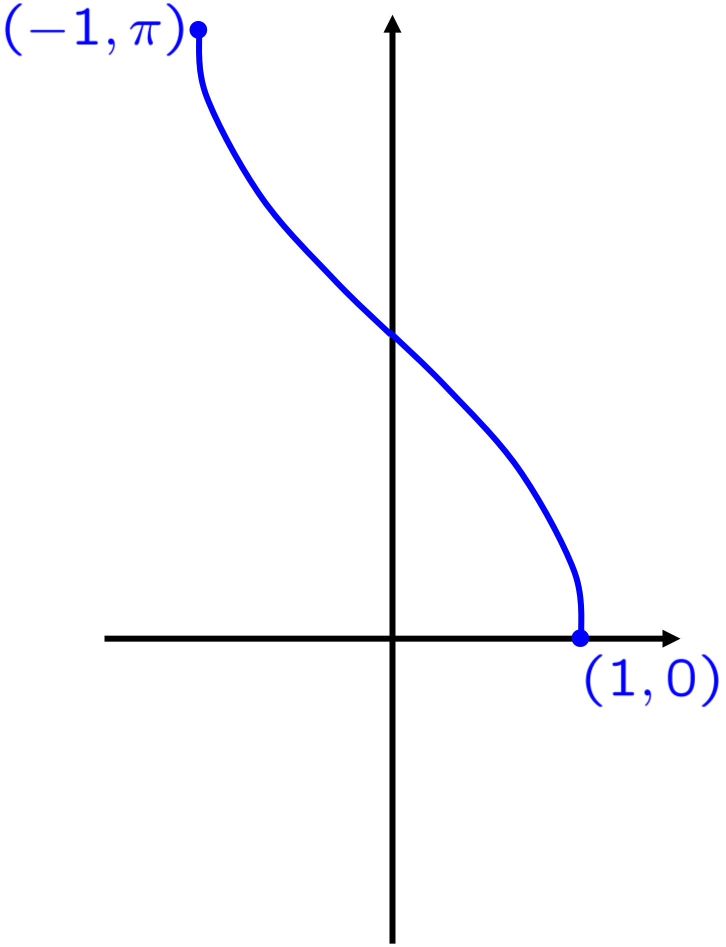


$y = \arccos x$

$y = \frac{\pi}{2} - \arcsin x$

Spp





$y = \frac{\pi}{2} - \arcsin x$

$y = \arccos x$

$$\frac{\pi}{2} - \arcsin x = \arccos x$$

$$\frac{\pi}{2} = \arcsin x + \arccos x$$

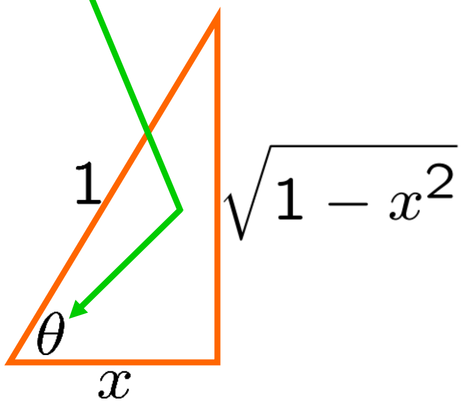
Next subtopic:
Inverse trig,
then trig . . .

Spp

INVERSE TRIG FN THEN TRIG FN

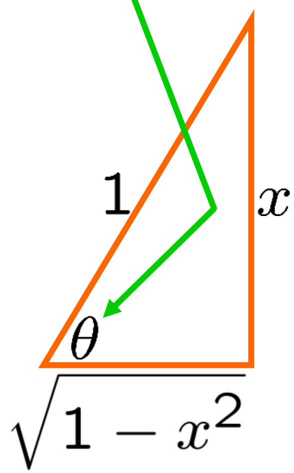
$$\tan(\arccos x) = \tan(\theta) = \frac{\sqrt{1-x^2}}{x}$$

Want: $\cos\theta = x$



$$\cos(\arcsin x) = \cos(\theta) = \sqrt{1-x^2}$$

Want: $\sin\theta = x$

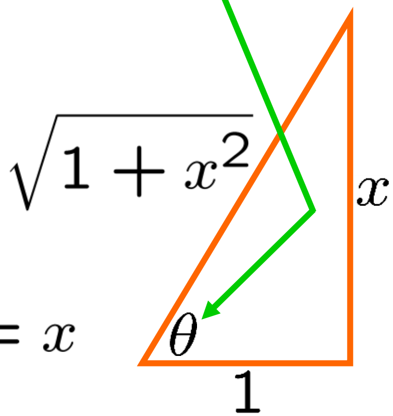


Spp

INVERSE TRIG FN THEN TRIG FN

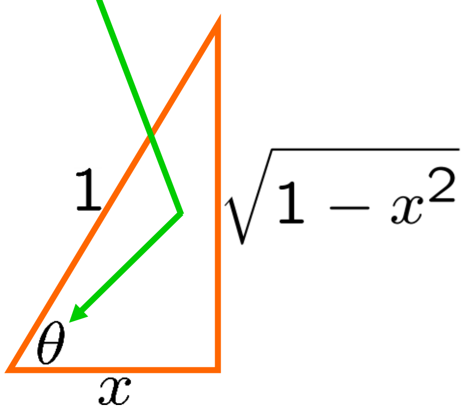
$$\cos(\arctan x) = \cos(\theta) = \frac{1}{\sqrt{1+x^2}}$$

Want: $\tan\theta = x$



$$\sec(\arccos x) = \sec(\theta) = \frac{1}{x}$$

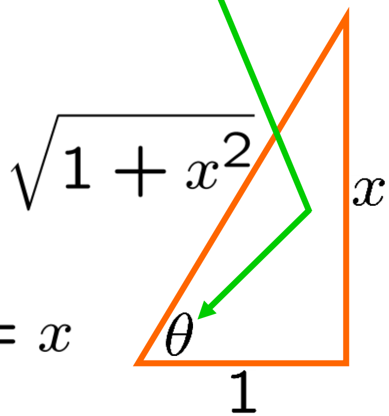
Want: $\cos\theta = x$



INVERSE TRIG FN THEN TRIG FN

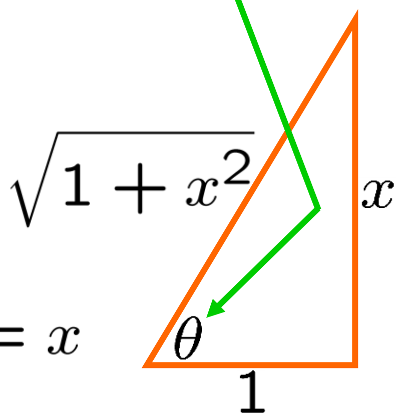
$$\sin(\arctan x) = \sin(\theta) = \frac{x}{\sqrt{1+x^2}}$$

Want: $\tan\theta = x$



$$\csc(\arctan x) = \csc(\theta) = \frac{\sqrt{1+x^2}}{x}$$

Want: $\tan\theta = x$



SKILL
write inv. trig then trig
as a “no-trig” function