CALCULUS
Additivity of limit

Fact: x^2 and $\cos x$ are both continuous at $x = \pi/4$,

i.e., $$\lim_{x\to\pi/4} x^2 = (\pi/4)^2 = \frac{\pi^2}{16}$$

and $$\lim_{x\to\pi/4} \cos x = \cos(\pi/4) = \frac{\sqrt{2}}{2}$$

Def'n:
$f(x)$ is **continuous at** $x = a$
if $\lim_{x\to a} f(x) = f(a)$.

Def'n 2.18, p. 42:
f is **continuous at** a
if $\lim_{x\to a} f(x) = f(a)$.

meaning?

$$\lim_{x\to\pi/4} x^2 = \frac{\pi^2}{1} \lim_{x\to\pi/4} x^2 \qquad \lim_{x\to\pi/4} {}^{\pi^2}\cos x = \frac{\sqrt{2}}{2}$$

$$\lim_{x\to\pi/4} \cos x \qquad = \frac{\sqrt{2}}{2}$$

meaning?

$$\lim_{x\to\pi/4} x^2 = \frac{\pi^2}{16} \qquad \lim_{x\to\pi/4} \cos x = \frac{\sqrt{2}}{2}$$

The letters ε and δ are traditional, but can be changed, if convenient, *e.g.*, $\delta :\to \alpha$

For every $\varepsilon > 0$, there is a $\delta > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \delta \quad \Rightarrow \quad \left| x^2 - \frac{\pi^2}{16} \right| < \varepsilon.$$

meaning?

$$\lim_{x\to\pi/4} x^2 = \frac{\pi^2}{16} \qquad \lim_{x\to\pi/4} \cos x = \frac{\sqrt{2}}{2}$$

For every $\varepsilon > 0$,
there is an $\boxed{\alpha} > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \boxed{\alpha} \quad \Rightarrow \quad \left| x^2 - \frac{\pi^2}{16} \right| < \varepsilon.$$

Example: $\varepsilon = 0.1$

Alice: $\alpha = 0.061271$ works

There is an $\alpha > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \alpha \quad \Rightarrow \quad \left| x^2 - \frac{\pi^2}{16} \right| < 0.1.$$

meaning?

$$\lim_{x \to \pi/4} x^2 = \frac{\pi^2}{16} \qquad \lim_{x \to \pi/4} \cos x = \frac{\sqrt{2}}{2}$$

The letters ε and δ are traditional,
but can be changed, if convenient,
e.g., $\delta :\to \beta$

For every $\varepsilon > 0$,
there is a $\boxed{\delta} > 0$ s.t.

$$0 < \left|x - \frac{\pi}{4}\right| < \boxed{\delta} \quad \Rightarrow \quad \left|\cos x - \frac{\sqrt{2}}{2}\right| < \varepsilon.$$

$$\lim_{x \to \pi/4} x^2 = \frac{\pi^2}{16} \qquad\qquad \lim_{x \to \pi/4} \cos x = \frac{\sqrt{2}}{2}$$

For every $\varepsilon > 0$,
there is a $\boxed{\beta} > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \boxed{\beta} \quad \Rightarrow \quad \left|\cos x - \frac{\sqrt{2}}{2}\right| < \varepsilon.$$

Example: $\varepsilon = 0.07$

Ben: $\beta = 0.094659$ works

There is a $\beta > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \beta \quad \Rightarrow \quad \left|\cos x - \frac{\sqrt{2}}{2}\right| < 0.07.$$

$$\lim_{x \to \pi/4} x^2 = \frac{\pi^2}{16} \qquad \lim_{x \to \pi/4} \cos x = \frac{\sqrt{2}}{2}$$

Goal: Show that $\displaystyle\lim_{x \to \pi/4} (x^2 + \cos x) = \frac{\pi^2}{16} + \frac{\sqrt{2}}{2}.$

Example: $\varepsilon = 0.07$

There is a $\beta > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \beta \quad \Rightarrow \quad \left|\cos x - \frac{\sqrt{2}}{2}\right| < 0.07.$$

$$\lim_{x\to\pi/4} x^2 = \frac{\pi^2}{16} \qquad\qquad \lim_{x\to\pi/4} \cos x = \frac{\sqrt{2}}{2}$$

Goal: Show that $\lim_{x\to\pi/4}(x^2+\cos x) = \frac{\pi^2}{16} + \frac{\sqrt{2}}{2}$.

Goal: For every $\varepsilon > 0$,
find a $\delta > 0$ s.t.

$$0 < \left|x - \tfrac{\pi}{4}\right| < \delta \quad \Rightarrow$$

$$\left|(x^2+\cos x) - \left(\frac{\pi^2}{16} + \frac{\sqrt{2}}{2}\right)\right| < \varepsilon.$$

Example: $\varepsilon = 0.008$

$$\lim_{x\to\pi/4} x^2 = \frac{\pi^2}{16} \qquad\qquad \lim_{x\to\pi/4} \cos x = \frac{\sqrt{2}}{2}$$

Goal: Find $\delta > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \delta \quad \Rightarrow$$

$$\left|(x^2 + \cos x) - \left(\frac{\pi^2}{16} + \frac{\sqrt{2}}{2}\right)\right| < 0.008.$$

Example: $\varepsilon = 0.008$

$$\lim_{x\to\pi/4} x^2 = \frac{\pi^2}{16} \qquad \lim_{x\to\pi/4} \cos x = \frac{\sqrt{2}}{2}$$

Goal: Find $\delta > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \delta \quad \Rightarrow$$

$$\left|(x^2 + \cos x) - \left(\frac{\pi^2}{16} + \frac{\sqrt{2}}{2}\right)\right| < 0.008.$$

HALF

Choose $\alpha > 0$ s.t. $0 < |x - \frac{\pi}{4}| < \alpha \Rightarrow \left|x^2 - \frac{\pi^2}{16}\right| < 0.004.$

Goal: Find $\delta > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \delta \quad \Rightarrow$$

$$\left|(x^2 + \cos x) - \left(\frac{\pi^2}{16} + \frac{\sqrt{2}}{2}\right)\right| < 0.008.$$

$$\lim_{x\to\pi/4} x^2 = \frac{\pi^2}{16} \qquad \lim_{x\to\pi/4} \cos x = \frac{\sqrt{2}}{2}$$

Goal: Find $\delta > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \delta \Rightarrow \left|(x^2 + \cos x) - \left(\frac{\pi^2}{16} + \frac{\sqrt{2}}{2}\right)\right| < 0.008.$$

Alice: $\alpha = 0.002542$

Choose $\alpha > 0$ s.t. $0 < |x - \frac{\pi}{4}| < \alpha \Rightarrow \left|x^2 - \frac{\pi^2}{16}\right| < 0.004.$

HALF

Ben: $\beta = 0.005640$

Choose $\beta > 0$ s.t. $0 < |x - \frac{\pi}{4}| < \beta \Rightarrow \left|\cos x - \frac{\sqrt{2}}{2}\right| < 0.004.$

$\delta = 0.002542$

Let $\delta := \min\{\alpha, \beta\}$.

Let x satisfy $0 < |x - \frac{\pi}{4}| < \delta$.

Want: $\left|(x^2+\cos x) - \left(\frac{\pi^2}{16}+\frac{\sqrt{2}}{2}\right)\right| < 0.008$

Goal: Find $\delta > 0$ s.t.

$$0 < |x - \tfrac{\pi}{4}| < \delta \quad \Rightarrow$$

$$\left|(x^2+\cos x) - \left(\frac{\pi^2}{16}+\frac{\sqrt{2}}{2}\right)\right| < 0.008.$$

Alice: $\alpha = 0.002542$

Choose $\alpha > 0$ s.t. $0 < |x - \frac{\pi}{4}| < \alpha \Rightarrow \left|x^2 - \frac{\pi^2}{16}\right| < 0.004.$

Ben: $\beta = 0.005640$

Choose $\beta > 0$ s.t. $0 < |x - \frac{\pi}{4}| < \beta \Rightarrow \left|\cos x - \frac{\sqrt{2}}{2}\right| < 0.004.$

$\delta = 0.002542$

Let $\delta := \min\{\alpha, \beta\}$.

Let x satisfy $0 < |x - \frac{\pi}{4}| < \delta$.

Want: $\left|(x^2 + \cos x) - \left(\frac{\pi^2}{16} + \frac{\sqrt{2}}{2}\right)\right| < 0.008$

$0 < |x - \frac{\pi}{4}| < \alpha$, so $\left|x^2 - \frac{\pi^2}{16}\right| < 0.004$

Choose $\alpha > 0$ s.t. $0 < |x - \frac{\pi}{4}| < \alpha \Rightarrow \left|x^2 - \frac{\pi^2}{16}\right| < 0.004.$

Choose $\beta > 0$ s.t. $0 < |x - \frac{\pi}{4}| < \beta \Rightarrow \left|\cos x - \frac{\sqrt{2}}{2}\right| < 0.004.$

Let $\delta := \min\{\alpha, \beta\}$. $\leq \alpha$

Let x satisfy $0 < |x - \frac{\pi}{4}| < \delta$. $\leq \alpha$

Want: $\left|(x^2+\cos x)-\left(\frac{\pi^2}{16}+\frac{\sqrt{2}}{2}\right)\right| < 0.008$

$0 < |x-\frac{\pi}{4}| < \alpha$, so $\left|x^2-\frac{\pi^2}{16}\right| < 0.004$

$0 < |x-\frac{\pi}{4}| < \beta$, so $\left|\cos x-\frac{\sqrt{2}}{2}\right| < 0.004$

Choose $\alpha > 0$ s.t. $0 < |x-\frac{\pi}{4}| < \alpha \Rightarrow \left|x^2-\frac{\pi^2}{16}\right| < 0.004.$

Choose $\beta > 0$ s.t. $0 < |x-\frac{\pi}{4}| < \beta \Rightarrow \left|\cos x-\frac{\sqrt{2}}{2}\right| < 0.004.$

Let $\delta := \min\{\alpha,\beta\}$. $\leq \beta$

Let x satisfy $0 < |x-\frac{\pi}{4}| < \delta$. $\leq \beta$

Want: $\left|(x^2+\cos x)-\left(\frac{\pi^2}{16}+\frac{\sqrt{2}}{2}\right)\right|<0.008$

$0<|x-\frac{\pi}{4}|<\alpha$, so $\left|x^2-\frac{\pi^2}{16}\right|<0.004$

$0<|x-\frac{\pi}{4}|<\beta$, so $\left|\cos x-\frac{\sqrt{2}}{2}\right|<0.004$

By additivity of error,

$$\left|(x^2+\cos x)-\left(\frac{\pi^2}{16}+\frac{\sqrt{2}}{2}\right)\right|<0.004+0.004$$

Exercise: Using algebra and the triangle inquality, prove this from these.

Want: $\left|(x^2 + \cos x) - \left(\frac{\pi^2}{16} + \frac{\sqrt{2}}{2}\right)\right| < 0.008$

$0 < |x - \frac{\pi}{4}| < \alpha$, so $\left|x^2 - \frac{\pi^2}{16}\right| < 0.004$

$0 < |x - \frac{\pi}{4}| < \beta$, so $\left|\cos x - \frac{\sqrt{2}}{2}\right| < 0.004$

By additivity of error,

$$\left|(x^2 + \cos x) - \left(\frac{\pi^2}{16} + \frac{\sqrt{2}}{2}\right)\right| < 0.004 + 0.004 = 0.008$$

QED

cf. §2.3, p. 32, THEOREM 2.7:

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$

$$\Rightarrow \quad \lim_{x \to c}(f(x)) + (g(x)) = L + M$$

Pf: Given $\varepsilon > 0$.

Want: $\exists\delta > 0$ s.t. $\left(0 < |x - c| < \delta \quad \Rightarrow \quad |[(f(x)) + (g(x))] - [L + M]| < \varepsilon \right)$

HALF

Choose $\alpha > 0$ s.t. $0 < |x - c| < \alpha \Rightarrow |(f(x)) - L| < \varepsilon/2$.

Choose $\beta > 0$ s.t. $0 < |x - c| < \beta \Rightarrow |(g(x)) - M| < \varepsilon/2$.

$\leq \alpha$

Let $\delta := \min\{\alpha, \beta\}$. Let x satisfy $0 < |x - c| < \delta$.

Want: $|[(f(x)) + (g(x))] - [L + M]| < \varepsilon$

$0 < |x - c| < \delta \leq \alpha$, so $|(f(x)) - L| < \varepsilon/2$

cf. §2.3, p. 32, THEOREM 2.7:

$\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

$\Rightarrow \lim_{x\to c}(f(x)) + (g(x)) = L + M$

Pf: Given $\varepsilon > 0$.

Want: $\exists \delta > 0$ s.t. $\left(\begin{array}{l} 0 < |x - c| < \delta \quad \Rightarrow \\ \quad |[(f(x)) + (g(x))] - [L + M]| < \varepsilon \end{array} \right)$

Choose $\alpha > 0$ s.t. $0 < |x - c| < \alpha \Rightarrow |(f(x)) - L| < \varepsilon/2$.

Choose $\beta > 0$ s.t. $0 < |x - c| < \beta \Rightarrow |(g(x)) - M| < \varepsilon/2$.

Let $\delta := \min\{\alpha, \beta\}$. $\leq \beta$ Let x satisfy $0 < |x - c| < \delta$.

Want: $|[(f(x)) + (g(x))] - [L + M]| < \varepsilon$

$0 < |x - c| < \delta \leq \alpha$, so $|(f(x)) - L| < \varepsilon/2$

$0 < |x - c| < \delta \leq \beta$, so $|(g(x)) - M| < \varepsilon/2$

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cf. §2.3, p. 32, THEOREM 2.7:

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$

$\Rightarrow \lim_{x \to c} (f(x)) + (g(x)) = L + M$

Pf: Given $\varepsilon > 0$.

Want: $\exists \delta > 0$ s.t. $\left(0 < |x - c| < \delta \quad \Rightarrow \quad |[(f(x)) + (g(x))] - [L + M]| < \varepsilon \right)$

Choose $\alpha > 0$ s.t. $0 < |x - c| < \alpha \Rightarrow |(f(x)) - L| < \varepsilon/2$.

Choose $\beta > 0$ s.t. $0 < |x - c| < \beta \Rightarrow |(g(x)) - M| < \varepsilon/2$.

Let $\delta := \min\{\alpha, \beta\}$. Let x satisfy $0 < |x - c| < \delta$.

Want: $|[(f(x)) + (g(x))] - [L + M]| < \varepsilon$

$0 < |x - c| < \delta \leq \alpha$, so $|(f(x)) - L| < \varepsilon/2$

$0 < |x - c| < \delta \leq \beta$, so $|(g(x)) - M| < \varepsilon/2$

By additivity of error,

$$|[(f(x)) + (g(x))] - [L + M]| < (\varepsilon/2) + (\varepsilon/2)$$

$$= \varepsilon. \qquad \text{QED}$$