CALCULUS
Limit laws

cf. §2.3, p. 32, THEOREM 2.7:

Works for $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a}$, $\lim\limits_{x\to \infty}$, $\lim\limits_{x\to -\infty}$.

Need to assume that the limits on the RHS exist.
Can conclude that the limit on the LHS exists.

lim distributes over addition

lim is additive

$$\lim[(f(x)) + (g(x))] = [\lim f(x)] + [\lim g(x)]$$

$$\lim[c(f(x))] = c[\lim(f(x))]$$

lim commutes with scalar multiplication

"commutes" refers to traveling –
watch c and lim travel:

$$\lim[c(f(x))]$$

real

"**scalar**" means number, *e.g.*, 12, 7, $-\pi$, *etc.*

"**scalar multiplication**" means
multiplication by a (real) number

§2.3 c is a (real) number, not a function or expression cf. §2.3, p. 32, THEOREM 2.7:

Works for $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to a}$, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$.

Need to assume that the limits on the RHS exist.

Can conclude that the limit on the LHS exists.

lim is linear

lim distributes over addition

lim is additive

i.e.: both

$$\lim[(f(x))+(g(x))] = [\lim f(x)]+[\lim g(x)]$$

and

$$\lim[c(f(x))] = c[\lim(f(x))]$$

lim commutes with scalar multiplication

lim distributes over multiplication

lim is multiplicative

i.e.: $$\lim[(f(x))(g(x))] = [\lim f(x)][\lim g(x)]$$

lim distributes over division

i.e.: $$\lim\left[\frac{f(x)}{g(x)}\right] = \frac{\lim f(x)}{\lim g(x)}$$

provided this denominator is not zero

§2.3

lim is linear

lim distributes over multiplication

lim distributes over division

Works for $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a}$, $\lim\limits_{x\to\infty}$, $\lim\limits_{x\to -\infty}$.

Works for $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a}$, $\lim\limits_{x\to\infty}$, $\lim\limits_{x\to -\infty}$.

CONCLUSIONS:

lim respects linear combination

lim is linear

i.e.: $\lim[a(f(x))+b(g(x))] = a[\lim f(x)]+b[\lim g(x)]$

lim distributes over multiplication

lim distributes over division

lim is linear
lim distributes over multiplication
lim distributes over division

Works for
$\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a}$,
$\lim\limits_{x\to\infty}$, $\lim\limits_{x\to-\infty}$.

CONCLUSIONS:
lim respects linear combination

i.e.: $\text{lim}[a(f(x))+b(g(x))] = a[\text{lim } f(x)]+b[\text{lim } g(x)]$

lim is additive \\ // lim commutes with scalar mult

$$[\text{lim}[a(f(x))]] + [\text{lim}[b(g(x))]]$$

QED

lim is linear
lim distributes over multiplication
lim distributes over division

Works for
$\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a}$,
$\lim\limits_{x\to\infty}$, $\lim\limits_{x\to-\infty}$.

CONCLUSIONS:
lim respects linear combination

i.e.: $\lim[a(f(x))+b(g(x))] = a[\lim f(x)]+b[\lim g(x)]$

e.g.: $\lim[5(f(x))-9(g(x))] = 5[\lim f(x)]-9[\lim g(x)]$

coefficients

EXAMPLE 1:

$$\lim_{x\to-3} f(x) = 4, \qquad \lim_{x\to-3} g(x) = -2$$

Evaluate $\lim\limits_{x\to-3} [(f(x)) + 5(g(x))]$.

$$= 4 + 5(-2) = -6$$

SKILL
use linearity of lim

lim is linear
lim distributes over multiplication
lim distributes over division

Works for $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to a}$, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$.

CONCLUSIONS:

lim respects linear combination

i.e.: $\lim[a(f(x))+b(g(x))] = a[\lim f(x)]+b[\lim g(x)]$

e.g.: $\lim[5(f(x))-9(g(x))] = 5[\lim f(x)]-9[\lim g(x)]$

coeffs 1 and -1

lim distributes over subtraction (cf. TH'M 2.7, p. 32)

i.e.: $\lim[(f(x)) - (g(x))] = [\lim f(x)] - [\lim g(x)]$

lim comm. with pos. int. powers

i.e.: $\lim[(f(x))^n] = [\lim f(x)]^n$

$$\lim[(f(x))(f(x))^{n-1}] = [\lim f(x)][\lim(f(x))^{n-1}]$$
$$= \cdots = [\lim f(x)] \cdots [\lim f(x)]$$
$$= [\lim f(x)]^n$$

equal QED

lim is linear
lim distributes over multiplication
lim distributes over division

Works for $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to a}$, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$.

CONCLUSIONS:

lim respects linear combination

i.e.: $\lim[a(f(x))+b(g(x))] = a[\lim f(x)]+b[\lim g(x)]$

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lim distributes over subtraction (cf. TH'M 2.7, p. 32)

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lim comm. with pos. int. powers

i.e.: $\lim[(f(x))^n] = [\lim f(x)]^n$

Def'n 2.18, p. 42: f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$.

MORE FACTS:

constants are continuous

i.e.: $\lim_{x\to a} c = c$ at every real

$f :\to$ const fn with value c

the identity is continuous

i.e.: $\lim_{x\to a} \iota(x) = \iota(a)$

x a

$f :\to \iota$

Def'n: The **identity** is the function $\iota : \mathbb{R} \to \mathbb{R}$ def'd by $\iota(x) = x$.

§2.3

lim is linear
lim distributes over multiplication
lim distributes over division

Works for $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to a}$, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$.

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e.g.: $\lim[5(f(x))-9(g(x))] = 5[\lim f(x)]-9[\lim g(x)]$

lim distributes over subtraction (cf. TH'M 2.7, p. 32)

i.e.: $\lim[(f(x)) - (g(x))] = [\lim f(x)] - [\lim g(x)]$

lim comm. with pos. int. powers

i.e.: $\lim[(f(x))^n] = [\lim f(x)]^n$

Def'n 2.18, p. 42: f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$.

MORE FACTS:

constants are continuous at every real

$\lim c = c$

$\lim_{x\to\infty} x = \infty$, $\lim_{x\to-\infty} x = -\infty$

the identity is continuous at every real

i.e.: $\lim_{x\to a} x = a$

Def'n: The **identity** is the function $\iota : \mathbb{R} \to \mathbb{R}$ def'd by $\iota(x) = x$.

lim is linear
lim distributes over multiplication
lim distributes over division
lim respects linear combination
lim distributes over subtraction
lim respects linear combination
lim comm. with pos. int. powers
constants are continuous
the identity is continuous

Def'n 2.18, p. 42:
f is **continuous at** a
if $\lim\limits_{x \longrightarrow a} f(x) = f(a)$.

pos. int. powers of contin. functions are again contin.

lim distributes over subtraction

lim comm. with pos. int. powers

Def'n 2.18, p. 42:
f is **continuous at** a
if $\lim\limits_{x \longrightarrow a} f(x) = f(a)$.

constants are continuous

the identity is continuous

lim is linear
lim distributes over multiplication
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lim comm. with pos. int. powers
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Def'n 2.18, p. 42:
f is **continuous at** a
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pos. int. powers of contin. functions are again contin.

Proof:

Know: $\lim_{x \to a} f(x) = f(a)$

Want: $\lim_{x \to a} [f(x)]^n = [f(a)]^n$

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = [f(a)]^n$$

QED

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Def'n 2.18, p. 42:
f is **continuous at** a if $\lim_{x \to a} f(x) = f(a)$.

MORE FACTS:

pos. int. powers of contin. functions are again contin.

pos. int. powers are continuous at every real

i.e.: $\lim_{x \to a} x^n = a^n$

pos. int. roots are continuous at every real (odd roots), at every pos. real (even roots)

i.e.: $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (If n is even, we assume that: $a > 0$.)

$\Uparrow$

Pf: Want: $\left(\lim_{x \to a} \sqrt[n]{x}\right)^n = \left(\sqrt[n]{a}\right)^n$ QED

$\left(\sqrt[n]{x}\right)^n = x$

$x :\to a$

$\left(\sqrt[n]{a}\right)^n = a$

$\lim_{x \to a} \left(\sqrt[n]{x}\right)^n = \lim_{x \to a} x = a$

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Def'n 2.18, p. 42:

f is **continuous at** a if $\lim\limits_{x\to a} f(x) = f(a)$.

pos. int. powers of contin. functions are again contin.

MORE FACTS:

pos. int. powers are continuous

i.e.: $\lim\limits_{x\to a} x^n = a^n$

sin is continuous at every real

pos. int. roots are continuous at every real (odd roots) at every pos. real (even roots)

i.e.: $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (If n is even, we assume that: $a > 0$.)

e.g.: $\lim\limits_{x\to 7} \left[\sin\left(x^2\right)\right] = \sin\left(\lim\limits_{x\to 7} x^2\right) = \sin(49)$ ■

THEOREM (fn↔lim):

an expression inside a function

...es with fns at pts of continuity

f continuous at $\lim g(x) \Rightarrow \lim f(g(x)) = f(\lim g(x))$.

lim is linear
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Def'n 2.18, p. 42:
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i.e.: $\lim\limits_{x\to a} x^n = a^n$

pos. int. roots are continuous at every real (odd roots) at every pos. real (even roots)

i.e.: $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (If n is even, we assume that: $a > 0$.)

$\lim \sqrt[n]{g(x)} = \sqrt[n]{\lim g(x)}$ lim commutes with roots
(If n is even, we assume that: $\lim g(x) > 0$.)

lim commutes with fns at pts of continuity

THEOREM (fn↔lim):
f continuous at $\lim g(x) \Rightarrow \lim f(g(x)) = f(\lim g(x))$.

lim is linear
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lim distributes over division
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lim distributes over subtraction
lim comm. with pos. int. powers
constants are continuous
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pos. int. powers are continuous

pos. int. powers are continuous

Def'n 2.18, p. 42:
f is **continuous at** a
if $\lim_{x \to a} f(x) = f(a)$.

Def: A **polynomial in** x
is a finite linear comb.
of $1, x, x^2, \ldots$

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Def'n 2.18, p. 42:
f is **continuous at** a
if $\lim_{x \longrightarrow a} f(x) = f(a)$.

Def: A **polynomial in** x
is a finite linear comb.
of 1, x, x^2, . . .

e.g.: $5 + 3x - 2x^2$
$4 + x^{1,000,000}$
$3 + 5x^2 - 19x^5$

Defs: **polynomial in** t
polynomial in u
polynomial in q
etc. **polynomial**

e.g.: $5 + 3\bullet - 2\bullet^2$
$4 + \bullet^{1,000,000}$
$3 + 5\bullet^2 - 19\bullet^5$

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f is **continuous at** a if $\lim\limits_{x \longrightarrow a} f(x) = f(a)$.

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Defs: **polynomial in** t
polynomial in u
polynomial in q
etc. **polynomial**

Def: A **rational fn** is a quotient of two polys.

polynomial in q
etc. **polynomial**

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Defs: **polynomial in** t
polynomial in u
polynomial in q
etc. **polynomial**

e.g.: $\dfrac{5 + 3x - 2x^2}{4 + x^{1,000,000}}$

$$\frac{7 + 2t^9 + 4t^{10}}{(1 - t)(4 + t)}$$

$$\left[\frac{4}{w} - \frac{2w^2 - 7w}{w^7}\right] \Big/ \left[\frac{7}{w^3} - \frac{(1 + w)(2w^2 - 7w)}{w^7}\right]$$

Def: A **rational fn** is a
quotient of two polys.

Defs: **rat'l expr. of** t
rat'l expr. of v
rat'l expr. of b
rat'l expr. of x
etc.

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any polynomial is continuous
(at all real numbers)
any rational fn is continuous
(at numbers in its domain)

FACT: (Polys and rat'l fns are contin)
If f is a polynomial or rational function,
and if $a \in \mathrm{dom}[f]$,
then $\lim\limits_{x \to a} f(x) = f(a)$.

Def'n 2.18, p. 42:
f is **continuous at** a
if $\lim\limits_{x \to a} f(x) = f(a)$.

Def: A **polynomial in** x is a finite linear comb. of 1, x, x^2, . . .

Defs: **polynomial in** t
polynomial in u
polynomial in q
etc. **polynomial**

Def: A **rational fn** is a quotient of two polys.

Defs: **rat'l expr. of** t
rat'l expr. of v
rat'l expr. of b
rat'l expr. of x
etc.

Next: Determinate and indeterminate forms. . .

$$\frac{1}{10^6} \approx 0$$

The form determines the answer.

Form of answer: "$\frac{1}{\infty} = 0$"

determinate form

Problem: Evaluate $\lim_{x\to\infty} \frac{1}{x}$.

$$\lim_{x\to\infty} x = \infty$$

Solution: $\lim_{x\to\infty} \frac{1}{x} = 0$ ■

"$1/(-\infty) = 0$"

$$\frac{1}{-10^6} \approx 0$$

$$\lim f(x) = -\infty \quad \Rightarrow \quad \lim \frac{1}{f(x)} = 0$$

Works for $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to a}$, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$.

$$\lim f(x) = \infty \quad \Rightarrow \quad \lim \frac{1}{f(x)} = 0$$

"$1/(0^+) = \infty$"   "$1/(0^-) = -\infty$"

WARNING: "1/0 is NOT always ∞"

"1/0 is (slightly) indeterminate"

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x\to 0} \frac{1}{x} \text{ DNE}$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim f(x) = -\infty \quad \Rightarrow \quad \lim \frac{1}{f(x)} = 0$$

Works for $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a}$, $\lim\limits_{x\to\infty}$, $\lim\limits_{x\to-\infty}$.

$$\lim f(x) = \infty \quad \Rightarrow \quad \lim \frac{1}{f(x)} = 0$$

Spp

“$1/(0^+) = \infty$” “$1/(0^-) = -\infty$”

$$\lim f(x) = 0^+ \Rightarrow \lim \frac{1}{f(x)} = \infty$$

$$\lim f(x) = 0^- \Rightarrow \lim \frac{1}{f(x)} = -\infty$$

e.g.

MEANING?

$$\lim_{x \to a^+} f(x) = L^-$$ means, intuitively:

if x is close to a, but greater than a,
then $f(x)$ is close to L, but less than L,

$$\lim f(x) = -\infty \quad \Rightarrow \quad \lim \frac{1}{f(x)} = 0$$

Works for $\lim\limits_{x \to a^+}$, $\lim\limits_{x \to a^-}$, $\lim\limits_{x \to a}$, $\lim\limits_{x \to \infty}$, $\lim\limits_{x \to -\infty}$.

$$\lim f(x) = \infty \quad \Rightarrow \quad \lim \frac{1}{f(x)} = 0$$

"$1/(0^+) = \infty$" "$1/(0^-) = -\infty$"

$$\lim f(x) = 0^+ \Rightarrow \lim \frac{1}{f(x)} = \infty$$

$$\lim f(x) = 0^- \Rightarrow \lim \frac{1}{f(x)} = -\infty$$

$\lim_{x\to 0} x^2 = 0^+$, so $\lim_{x\to 0} \frac{1}{x^2} = \infty$

MEANING? ☺

nonstandard

$\lim_{x\to a^+} f(x) = L^-$ means, intuitively:

if x is close to a, but greater than a,

then $f(x)$ is close to L, but less than L,

and means, rigorously:

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $a < x < a + \delta \Rightarrow L - \varepsilon < f(x) < L$.

"$\infty \cdot \infty = \infty$" $(10^6)(10^6)$ is very positive

"$\infty \cdot (-\infty) = -\infty$" $(10^6)(-10^6)$

"$(-\infty) \cdot \infty = -\infty$" $(-10^6)(10^6)$

"$(-\infty) \cdot (-\infty) = \infty$" $(-10^6)(-10^6)$

MORE DETERMINATE AND INDETERMINATE FORMS

Spp

"$1/(0^+) = \infty$" "$1/(0^-) = -\infty$"

$$\lim f(x) = 0^+ \Rightarrow \lim \frac{1}{f(x)} = \infty$$

$$\lim f(x) = 0^- \Rightarrow \lim \frac{1}{f(x)} = -\infty$$

$\lim\limits_{x\to 0} x^2 = 0^+$, so $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$

$\lim\limits_{x\to a^+} f(x) = L^-$ (nonstandard) means, intuitively:

if x is close to a, but greater than a,

then $f(x)$ is close to L, but less than L,

and means, rigorously:

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $a < x < a + \delta \Rightarrow L - \varepsilon < f(x) < L$.

"$\infty \cdot \infty = \infty$"

"$\infty \cdot (-\infty) = -\infty$"

"$(-\infty) \cdot \infty = -\infty$"

"$(-\infty) \cdot (-\infty) = \infty$"

"$\infty + 5 = \infty$"

"$\infty - 7 = \infty$"

"$\infty + c = \infty$"

"$(-\infty) + c = -\infty$"

MORE DETERMINATE AND INDETERMINATE FORMS

“$1/(0^+) = \infty$”

“$1/(0^-) = -\infty$”

“$6 \cdot \infty = \infty$”

“$\sqrt{\infty} = \infty$”

“$3 \cdot \infty = \infty$”

“$(-3) \cdot \infty = -\infty$”

“$c > 0 \Rightarrow c \cdot \infty = \infty$”

“$c < 0 \Rightarrow c \cdot \infty = -\infty$”

“$\infty \cdot \infty = \infty$”

“$\infty \cdot (-\infty) = -\infty$”

“$(-\infty) \cdot \infty = -\infty$”

“$(-\infty) \cdot (-\infty) = \infty$”

“$\infty + c = \infty$”

“$(-\infty) + c = -\infty$”

MORE DETERMINATE AND INDETERMINATE FORMS

Spp

"$1/(0^+) = \infty$" "$1/(0^-) = -\infty$"

"$\sqrt{\infty} = \infty$"

"$c > 0 \Rightarrow c \cdot (-\infty) = -\infty$"

"$c < 0 \Rightarrow c \cdot (-\infty) = \infty$"

"$c > 0 \Rightarrow c \cdot \infty = \infty$"

"$c < 0 \Rightarrow c \cdot \infty = -\infty$"

$$\lim_{x\to\infty} (1/x^2)(x) = 0$$

$$\lim_{x\to\infty} (1/x^2)(x^2) = 1$$

$$\lim_{x\to\infty} (1/x^2)(x^3) = \infty$$

"$0 \cdot \infty$ and $0 \cdot (-\infty)$ are indeterminate"

$$\lim_{x\to\infty} (-1/x^2)(-x) = 0$$

$$\lim_{x\to\infty} (-1/x^2)(-x^2) = 1$$

$$\lim_{x\to\infty} (-1/x^2)(-x^3) = \infty$$

"$\infty \cdot \infty = \infty$"

"$\infty \cdot (-\infty) = -\infty$"

"$(-\infty) \cdot \infty = -\infty$"

"$(-\infty) \cdot (-\infty) = \infty$"

"$\infty + c = \infty$"

"$(-\infty) + c = -\infty$"

MORE DETERMINATE AND INDETERMINATE FORMS

Spp

"$1/(0^+) = \infty$" "$1/(0^-) = -\infty$"

"$e^{\infty} = \infty$" "$e^{-\infty} = 0$"

"$\ln(\infty) = \infty$"

"$\sqrt{\infty} = \infty$"

"$(0^+)^{\infty} = 0$"

"$1^{\pm\infty}$ is indet."

"$(0^+)^0$, ∞^0 indet."

"$c > 0 \Rightarrow c \cdot (-\infty) = -\infty$"

"$c < 0 \Rightarrow c \cdot (-\infty) = \infty$"

"$c > 0 \Rightarrow c \cdot \infty = \infty$"

"$c < 0 \Rightarrow c \cdot \infty = -\infty$"

"$0 \cdot \infty$ and $0 \cdot (-\infty)$ are indeterminate"

"$0/0$, ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, $(-\infty)/(-\infty)$ all indeterminate"

"l'Hôpital indeterminate forms"

"$\infty \cdot \infty = \infty$"

"$\infty \cdot (-\infty) = -\infty$"

"$(-\infty) \cdot \infty = -\infty$"

"$(-\infty) \cdot (-\infty) = \infty$"

"$\infty + c = \infty$"

"$(-\infty) + c = -\infty$"

MORE DETERMINATE AND INDETERMINATE FORMS

Spp

"$1/(0^+) = \infty$" "$1/(0^-) = -\infty$"

"$(-\infty) + (-\infty) = -\infty$"

"$e^{\infty} = \infty$" "$e^{-\infty} = 0$" "$\infty + \infty = \infty$"

"$\ln(\infty) = \infty$" "$\infty - \infty$ is indeterminate"

"$\sqrt{\infty} = \infty$" "$c > 0 \Rightarrow c \cdot (-\infty) = -\infty$"

"$(0^+)^{\infty} = 0$" "$c < 0 \Rightarrow c \cdot (-\infty) = \infty$"

"$1^{\pm\infty}$ is indet." "$c > 0 \Rightarrow c \cdot \infty = \infty$"

"$(0^+)^0$, ∞^0 indet." "$c < 0 \Rightarrow c \cdot \infty = -\infty$"

"$0 \cdot \infty$ and $0 \cdot (-\infty)$ are indeterminate"

"$0/0$, ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, $(-\infty)/(-\infty)$ all indeterminate"

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"$(-\infty) \cdot \infty = -\infty$" "$\infty + c = \infty$"

"$(-\infty) \cdot (-\infty) = \infty$" "$(-\infty) + c = -\infty$"

MORE DETERMINATE AND INDETERMINATE FORMS

Spp

“$1/(0^+) = \infty$”

“$(-\infty) + \infty$ i“$1/(0^-) = -\infty$”e”

“$1/(0^-) = -\infty$”

“$(-\infty) + (-\infty) = -\infty$”

“$e^{\infty} = \infty$” “$e^{-\infty} = 0$”

“$\infty + \infty = \infty$”

“$\ln(\infty) = \infty$”

“$\infty - \infty$ is indeterminate”

“$\sqrt{\infty} = \infty$”

“$c > 0 \Rightarrow c \cdot (-\infty) = -\infty$”

“$(0^+)^{\infty} = 0$”

“$c < 0 \Rightarrow c \cdot (-\infty) = \infty$”

“$1^{\pm\infty}$ is indet.”

“$c > 0 \Rightarrow c \cdot \infty = \infty$”

“$(0^+)^0$, ∞^0 indet.”

“$c < 0 \Rightarrow c \cdot \infty = -\infty$”

Next: Monotonicity and squeeze th’m. . .

“$0 \cdot \infty$ and $0 \cdot (-\infty)$ are indeterminate”

“$0/0$, ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, $(-\infty)/(-\infty)$ all indeterminate”

“l’Hôpital indeterminate forms”

“$\infty \cdot \infty = \infty$”

“$\infty \cdot (-\infty) = -\infty$”

“$(-\infty) \cdot \infty = -\infty$”

“$(-\infty) \cdot (-\infty) = \infty$”

“$\infty + c = \infty$”

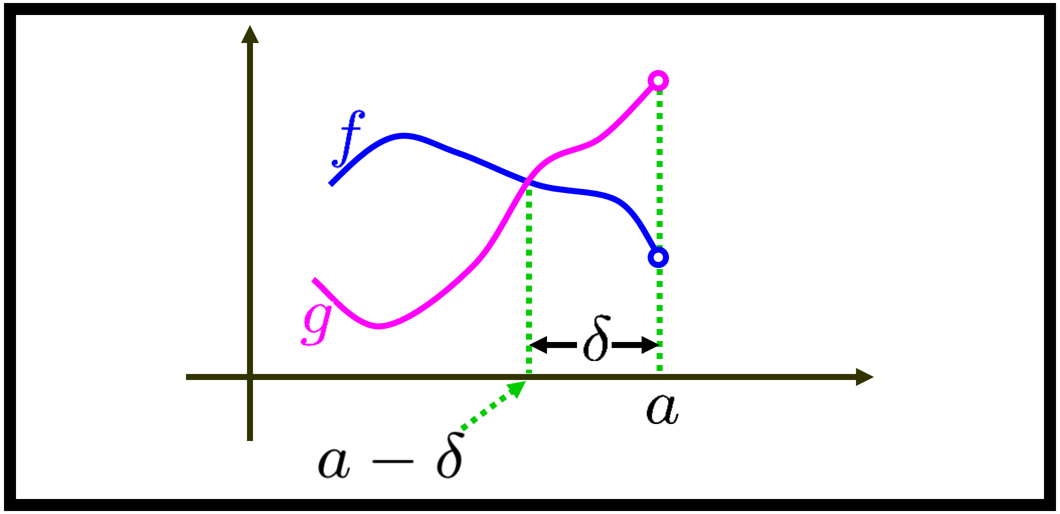
“$(-\infty) + c = -\infty$”

MORE DETERMINATE AND INDETERMINATE FORMS

TH'M (left montonicity of limit):

If, for some $\delta > 0$,

$$f(x) \leq g(x) \quad \text{on} \quad a - \delta < x < a,$$



Any smaller positive number would work for δ, too.

TH'M (left montonicity of limit):

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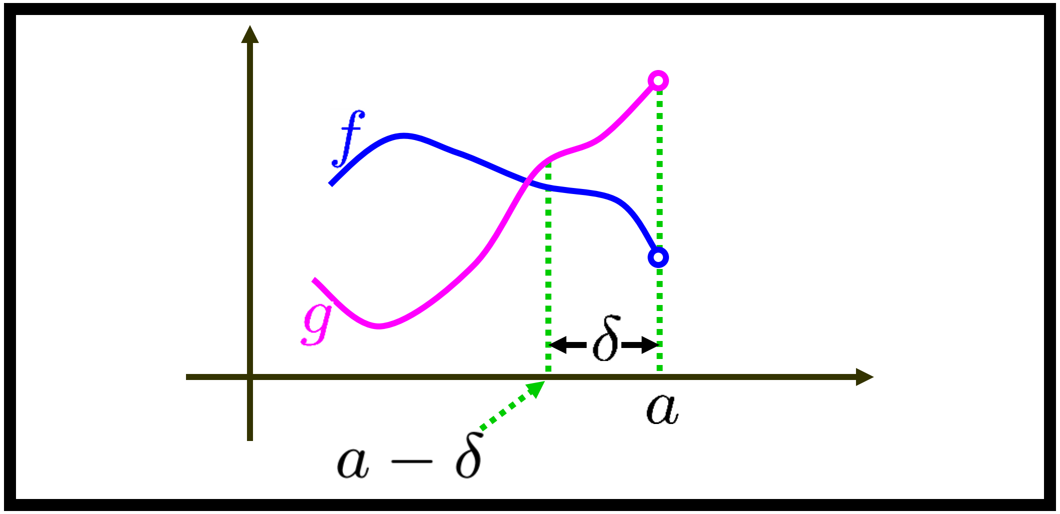


Any smaller positive number would work for δ, too.

TH'M (left montonicity of limit):

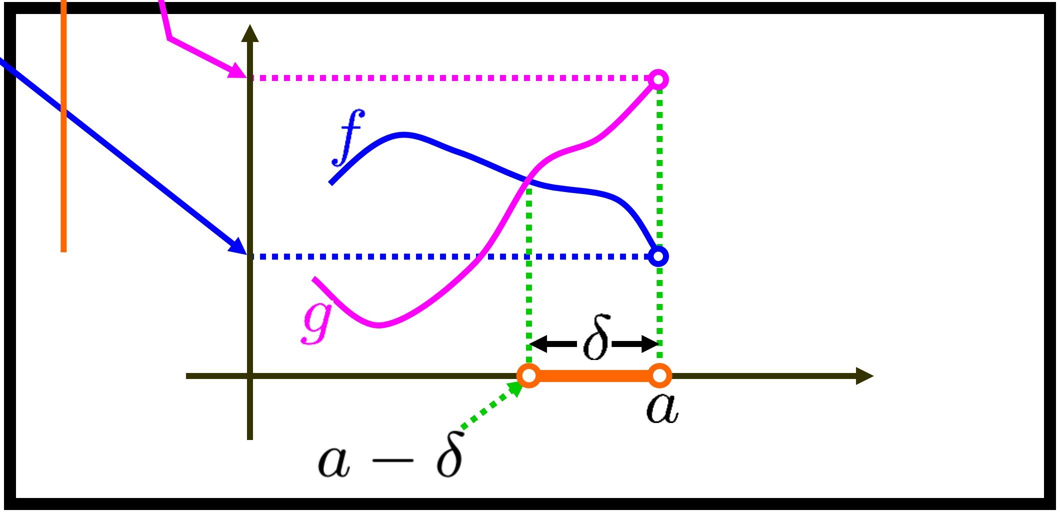
If, for some $\delta > 0$,

$$f(x) \le g(x) \quad \text{on} \quad a - \delta < x < a,$$

then $\lim_{x \to a^-} f(x) \le \lim_{x \to a^-} g(x)$,

provided both of these limits exist.

left δ-nbd of a
$x \in (a - \delta, a)$



No larger positive number would work for δ.

Exercise: right monotonicity

Next: two-sided monotonicity

TH'M (monotonicity of limit):

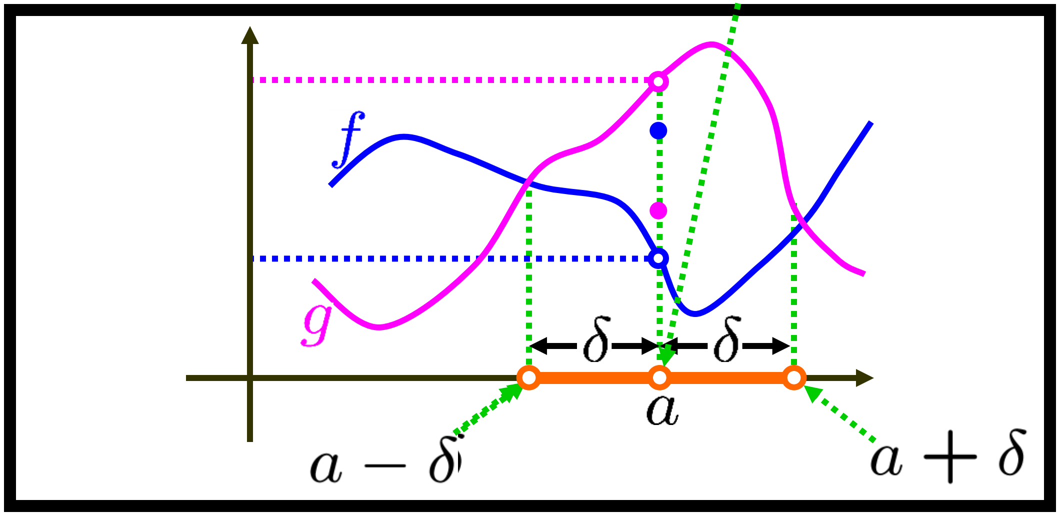
If, for some $\delta > 0$,

punctured δ-nbd of a

$$f(x) \leq g(x) \quad \text{on} \quad x \in (a-\delta, a+\delta) \setminus \{a\},$$

then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$,

provided both of these limits exist.



TH'M (monotonicity of limit):

If, for some $\delta > 0$,

$$f(x) \le g(x) \quad \text{on} \quad x \in (a-\delta, a+\delta)\backslash\{a\},$$

then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$,

provided both of these limits exist.

$$L = \lim_{x\to a} f(x) \le \underbrace{\lim_{x\to a} g(x)}_{\substack{\parallel \\ L}} \le \lim_{x\to a} h(x) = L$$

Pf is a little harder.
(We omit it.)

§4.3, p. 67: THEOREM 4.1 (squeeze th'm)

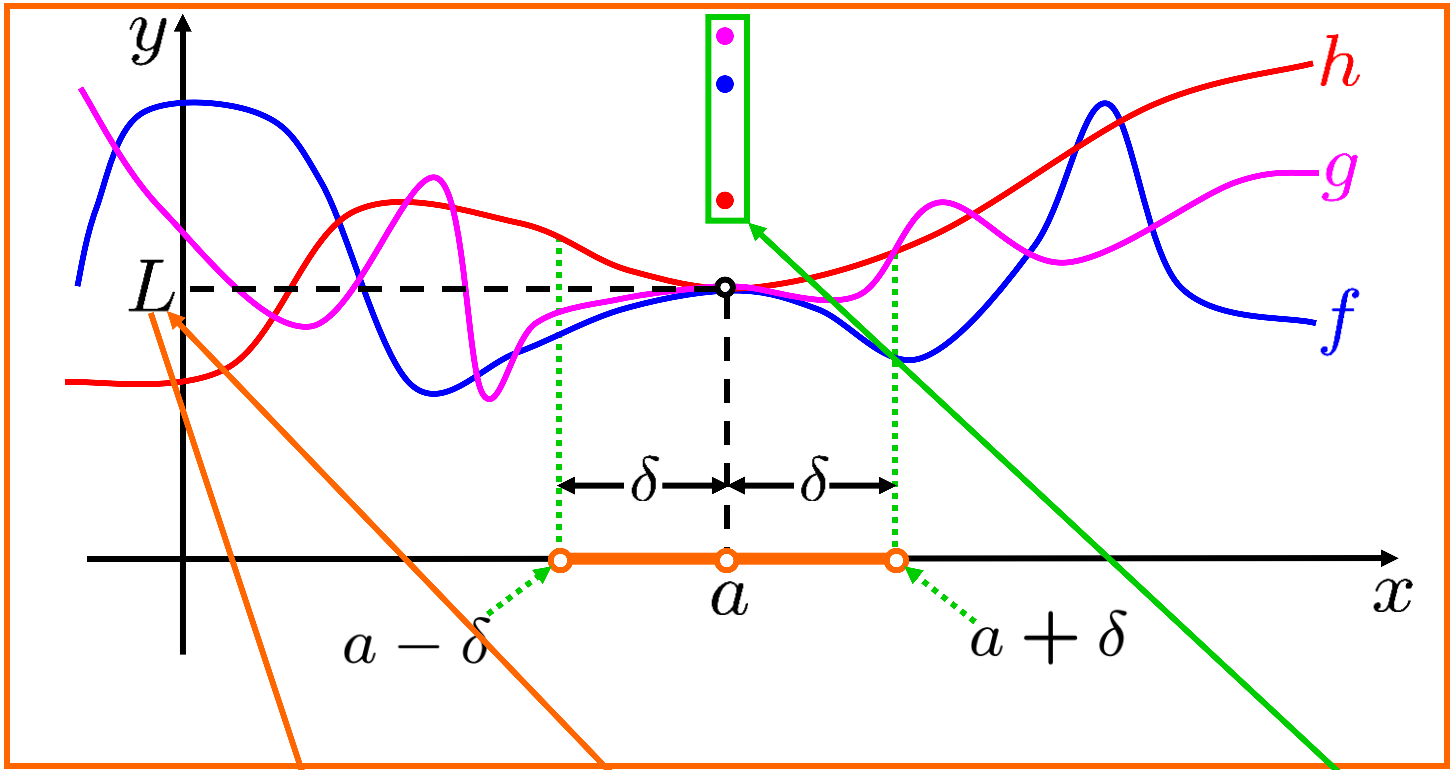
If, for some $\delta > 0$,

$$f(x) \le g(x) \le h(x) \quad \text{on} \quad x \in (a-\delta, a+\delta)\backslash\{a\},$$

and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$,

then $\lim_{x\to a} g(x) = L$.

not needed: assuming $\lim_{x\to a} g(x)$ exists



§4.3, p. 67: THEOREM 4.1 (squeeze th'm)

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then $\lim_{x \to a} g(x) = L$.

§2.3 Exercise: left and right squeeze theorem

Fact: If $\phi = \psi$ on a punctured nbd of a,

then $\lim_{x \to a} \phi(x) = \lim_{x \to a} \psi(x)$,

provided the RHS exists.

Proof: $\psi(x) \le \phi(x) \le \psi(x)$ on $x \in (a - \delta, a + \delta) \backslash \{a\}$,

so $\lim_{x \to a} \phi(x) = \lim_{x \to a} \psi(x)$.

QED

§4.3, p. 67: THEOREM 4.1 (squeeze th'm)

If, for some $\delta > 0$,

$$f(x) \le g(x) \le h(x) \quad \text{on} \quad x \in (a - \delta, a + \delta) \backslash \{a\},$$

and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$,

then $\lim_{x \to a} g(x) = L$.

$\| \lim_{x \to a} h(x)$

Fact: If $\phi = \psi$ on a punctured nbd of a,

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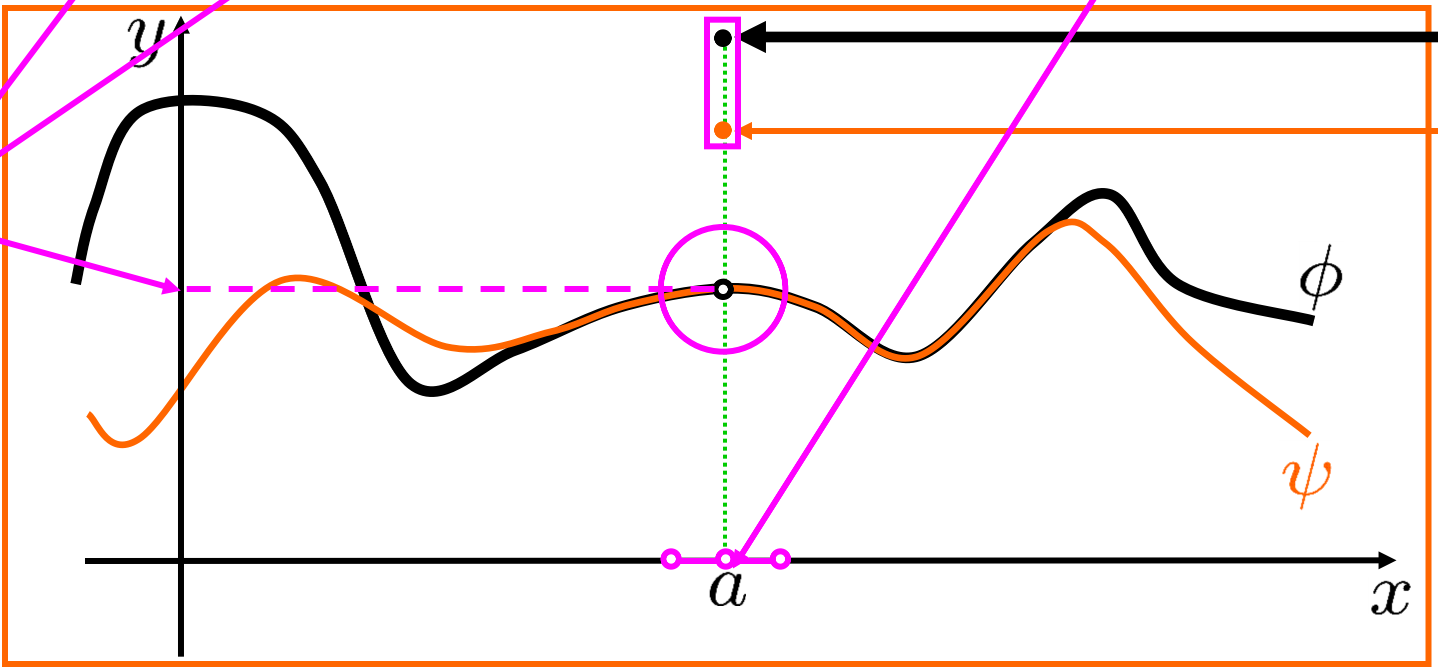
QED



Spp

EXAMPLE: Find $\lim_{h\to 0}\frac{(3+h)^2-9}{h}$.

Fact: If $\phi = \psi$ on a punctured nbd of a,

then $\lim_{x\to a}\phi(x) = \lim_{x\to a}\psi(x)$,

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then $\lim_{x\to a}\phi(x) = \lim_{x\to a}\psi(x)$,

provided the RHS exists.

SKILL
lim diff quot

EXAMPLE: Find $\displaystyle\lim_{h\to 0} \frac{(3+h)^2-9}{h}$.

$h \neq 0$

$$\frac{(3+h)^2-9}{h} = \frac{(9+6h+h^2)-9}{h} = 6+h$$

$$\lim_{h\to 0} 6+h = [6+h]_{h:\to 0} = 6$$

polynomial in h,
so continuous

$x :\to h,\ a :\to 0$

Fact: If $\phi = \psi$ on a punctured nbd of a,

then $\displaystyle\lim_{x\to a} \phi(x) = \lim_{x\to a} \psi(x)$,

provided the RHS exists.

SKILL
lim diff quot

EXAMPLE: Find $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$.

$h \neq 0$

$$\phi(h) := \frac{(3+h)^2-9}{h} = \frac{(\not{9}+6h+h^2)-\not{9}}{h} = 6+h =: \psi(h)$$

$\phi = \psi$ on every punctured nbd of 0

$$\lim_{h\to 0}\frac{(3+h)^2-9}{h} = \lim_{h\to 0} 6+h = [6+h]_{h:\to 0} = \boxed{6} \quad \blacksquare$$

polynomial in h,
so continuous

$x :\to h$, $a :\to 0$

Fact: If $\phi = \psi$ on a punctured nbd of 0, ☺

then $\lim_{h\to 0} \phi(h) = \lim_{h\to 0} \psi(h)$,

provided the RHS exists.

SKILL
Compute limits
Whitman problems
§2.3, p. 34, #1-15

SKILL
Oscillatory limit
Whitman problems
§2.3, p. 35, #16

SKILL
Find δ
Whitman problems
§2.3, p. 35, #17

SKILL
Limits from gph
Whitman problems
§2.3, p. 35, #18

SKILL
Calculator estimation of limits
Whitman problems
§2.3, p. 35, #19