# CALCULUS
Limit problems

# SOME EXAMPLE LIMIT PROBLEMS:

(1) $\lim_{x \to 5} (7x^2 - 4x + 3) = [7x^2 - 4x + 3]_{x :\to 5}$

polynomial, so continuous

$= 7(5^2) - 4(5) + 3$

$= 158$ ■

SKILL lim poly

---

Fact: Rational functions are continuous (at every point of their domain).

Fact: Polynomials are continuous (at every real number).

Def'n 2.19, p. 42:
$f$ is **continuous** if,
$\forall a \in \mathrm{dom}[f]$, $f$ is contin. at $a$.

---

NOTE: Continuity is a powerful tool.
Let's try it some more!

# SOME EXAMPLE LIMIT PROBLEMS:

(1) $\lim_{x\to 5}(7x^2 - 4x + 3) = [7x^2 - 4x + 3]_{x:\to 5}$

polynomial, so continuous

$= 7(5^2) - 4(5) + 3$

$= 158$ ■

SKILL lim poly

Def'n: We say $f(x)$ is **continuous in** $x$ if $f$ is continuous.

Def'n: We say $f(s)$ is **continuous in** $s$ if $f$ is continuous.

*etc., etc., etc.*

> Def'n 2.19, p. 42:
> $f$ is **continuous** if,
> $\forall a \in \mathrm{dom}[f]$, $f$ is contin. at $a$.

NOTE: Continuity is a powerful tool.
Let's try it some more!

# SOME EXAMPLE LIMIT PROBLEMS:

(1) $\lim_{x\to 5}(7x^2 - 4x + 3) = [7x^2 - 4x + 3]_{x:\to 5}$

polynomial, so continuous

$= 7(5^2) - 4(5) + 3$

$= 158$ ■

SKILL lim poly

---

Def'n: We say $f(x)$ is **continuous in** $x$ if $f$ is continuous.

Def'n: We say $f(s)$ is **continuous in** $s$ if $f$ is continuous.

*etc., etc., etc.*

---

Fact: If $f(x) \geq 0$, $\forall x \in \mathbb{R}$
and if $f(x)$ is continuous in $x$
then $\sqrt{f(x)}$ is continuous in $x$.

---

NOTE: Continuity is a powerful tool.
Let's try it some more!

# SOME EXAMPLE LIMIT PROBLEMS:

(1) $\lim\limits_{x \to 5} (7x^2 - 4x + 3) = [7x^2 - 4x + 3]_{x:\to 5}$

polynomial, so continuous

$= 7(5^2) - 4(5) + 3$

$= 158$ ■

SKILL
lim poly

---

$x^2 \geq 0, \forall x \in \mathbb{R}$.

$x^2$ is continuous in $x$.

So $\sqrt{x^2}$ is continuous in $x$,

*i.e.*, $|x|$ is continuous in $x$,

*i.e.*, $|\bullet|$ is continuous.

---

Fact: If $f(x) \geq 0, \forall x \in \mathbb{R}$

and if $f(x)$ is continuous in $x$

then $\sqrt{f(x)}$ is continuous in $x$.

---

NOTE: Continuity is a powerful tool.
Let's try it some more!

# SOME EXAMPLE LIMIT PROBLEMS:

(2) $\lim_{x \to 2} (4x + |x - 2|)$

$|\bullet|$ is continuous.

$|\bullet|$ is continuous.

NOTE: Continuity is a powerful tool.
Let's try it some more!

# SOME EXAMPLE LIMIT PROBLEMS:

(2) $\lim_{x \to 2} (4x + |x - 2|)$

---

Fact: If $f$ is continuous
and if $g(x)$ is continuous in $x$,
then $f(g(x))$ is continuous in $x$.

---

$|\bullet|$ is continuous.

$x - 2$ is continuous in $x$.

So $|x - 2|$ is continuous in $x$.

---

NOTE: Continuity is a powerful tool.
Let's try it some more!

# SOME EXAMPLE LIMIT PROBLEMS:

(2) $\lim_{x \to 2} (4x + |x - 2|)$

SKILL
lim w/ abs val

$$= [4x + |x-2|]_{x:\to 2} = 4 \cdot 2 + |2 - 2| = 8 \blacksquare$$

Fact: A lin. comb. of continuous expressions is continuous.

$4x$ is continuous in $x$.
$|x - 2|$ is continuous in $x$.

So $4x + |x - 2|$ is continuous in $x$.

NOTE: Continuity is a powerful tool.
Let's try it some more!

# SOME EXAMPLE LIMIT PROBLEMS:

(3) $$\lim_{x \to -3} \frac{x^3 + 4x - 7}{5 - 3x} = \left[\frac{x^3 + 4x - 7}{5 - 3x}\right]_{x :\to -3}$$

rational,
so continuous (at
numbers in domain)

$$= \frac{(-3)^3 + 4(-3) - 7}{14}$$

$[5 - 3x]_{x :\to -3}$
$= 5 - 3(-3) = 14$
nonzero

$$= -\frac{46}{14} = -\frac{23}{7} \blacksquare$$

SKILL
lim rat'l

NOTE: Continuity is a powerful tool.
Let's try it some more!

# SOME EXAMPLE LIMIT PROBLEMS:

(4) $$\lim_{x\to 1} \frac{x^4 - 5x^3 + 9x^2 - 7x + 2}{x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3}$$

$$\left[x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3\right]_{x:\to 1} = 0$$

$$\begin{array}{r|rrrrrr} 1 & 1 & -2 & 5 & -11 & 10 & -3 \\ & & 1 & -1 & 4 & -7 & 3 \\ \hline & 1 & -1 & 4 & -7 & 3 & \boxed{0} \end{array}$$

Strategy: Factor $x - 1$ from numerator and denominator as many times as possible, then cancel $x - 1$ in the numerator with $x - 1$ in the denominator as many times as possible.

Let's finish up the denominator first. . .

# SOME EXAMPLE LIMIT PROBLEMS:

(4) $$\lim_{x\to 1} \frac{x^4 - 5x^3 + 9x^2 - 7x + 2}{x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3}$$

$$\left[x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3\right]_{x:\to 1} = 0$$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | −2 | 5 | −11 | 10 | −3 |
| | | 1 | −1 | 4 | −7 | 3 |
| 1 | 1 | −1 | 4 | −7 | 3 | 0 |
| | | 1 | 0 | 4 | −3 | |
| 1 | 1 | 0 | 4 | −3 | 0 | |
| | | 1 | 1 | 5 | | |
| | 1 | 1 | 5 | 2 $\neq 0$ | | |

$$x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3 = (x-1)^2(x^3 + 4x - 3)$$

$$\left[x^3 + 4x - 3\right]_{x:\to 1} = 2 \neq 0$$

§2.3

# SOME EXAMPLE LIMIT PROBLEMS:

(4) $\displaystyle \lim_{x \to 1} \frac{x^4 - 5x^3 + 9x^2 - 7x + 2}{x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3}$

Let's factor $x - 1$ from the numerator now. . .

Exercise: Use repeated synthetic division to get:

$$x^4 - 5x^3 + 9x^2 - 7x + 2$$
$$= (x-1)^3(x-2)$$
$$[x-2]_{x:\to 1} = -1 \neq 0$$

$$x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3$$
$$= (x-1)^2(x^3 + 4x - 3)$$
$$\left[x^3 + 4x - 3\right]_{x:\to 1} = 2 \neq 0$$

# SOME EXAMPLE LIMIT PROBLEMS:

(4) $$\lim_{x\to 1} \frac{x^4 - 5x^3 + 9x^2 - 7x + 2}{x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3}$$

$$= \lim_{x\to 1} \left[\frac{(x-1)^3}{(x-1)^2}\right]\left[\frac{x-2}{x^3+4x-3}\right]$$

---

$$x^4 - 5x^3 + 9x^2 - 7x + 2$$
$$= (x-1)^3(x-2)$$
$$[x-2]_{x:\to 1} = -1 \neq 0$$

---

$$x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3$$
$$= (x-1)^2(x^3+4x-3)$$
$$\left[x^3+4x-3\right]_{x:\to 1} = 2 \neq 0$$

# SOME EXAMPLE LIMIT PROBLEMS:

(4) $$\lim_{x\to 1} \frac{x^4 - 5x^3 + 9x^2 - 7x + 2}{x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3}$$

$$= \lim_{x\to 1} \left[\frac{(x-1)^3}{(x-1)^2}\right]\left[\frac{x-2}{x^3+4x-3}\right]$$

$$= \lim_{x\to 1} \, [x-1]\left[\frac{x-2}{x^3+4x-3}\right]$$

$$[x-2]_{x:\to 1} = -1 \neq 0$$

---

$$[x-2]_{x:\to 1} = -1 \neq 0$$

$$\left[x^3+4x-3\right]_{x:\to 1} = 2 \neq 0$$

# SOME EXAMPLE LIMIT PROBLEMS:

(4) $$\lim_{x\to 1} \frac{x^4 - 5x^3 + 9x^2 - 7x + 2}{x^5 - 2x^4 + 5x^3 - 11x^2 + 10x - 3}$$

$$= \lim_{x\to 1} \left[\frac{(x-1)^3}{(x-1)^2}\right]\left[\frac{x-2}{x^3+4x-3}\right]$$

$$= \lim_{x\to 1} [x-1]\left[\frac{x-2}{x^3+4x-3}\right]$$

$$= [0]\left[\frac{-1}{2}\right] = 0 \blacksquare$$

SKILL
lim rat'l

$$[x-2]_{x:\to 1} = -1 \neq 0$$

$$\left[x^3+4x-3\right]_{x:\to 1} = 2 \neq 0$$

# SOME EXAMPLE LIMIT PROBLEMS:

(5) $$\lim_{x\to 2} \frac{2x^5 - 8x^4 + 4x^3 + 15x^2 - 12x - 4}{5x^4 - 24x^3 + 29x^2 + 12x - 28}$$

$$= \lim_{x\to 2} \left[\frac{(x-2)^2}{(x-2)^2}\right]\left[\frac{2x^3 - 4x - 1}{5x^2 - 4x - 7}\right] = \frac{7}{5}$$ ■

SKILL lim rat'l

$$2x^5 - 8x^4 + 4x^3 + 15x^2 - 12x - 4$$
$$= (x-2)^2(2x^3 - 4x - 1)$$

(Exercise)

$$\left[2x^3 - 4x - 1\right]_{x:\to 2} = 7 \neq 0$$

$$5x^4 - 24x^3 + 29x^2 + 12x - 28$$
$$= (x-2)^2(5x^2 - 4x - 7)$$

(Exercise)

$$\left[5x^2 - 4x - 7\right]_{x:\to 2} = 5 \neq 0$$

# SOME EXAMPLE LIMIT PROBLEMS:

(6) $$\lim_{x \to -3} \frac{2x^5 + 18x^4 + 58x^3 + 90x^2 + 108x + 108}{x^6 + 8x^5 - 15x^4 - 360x^3 - 1485x^2 - 2592x - 1701}$$

$$= \lim_{x \to -3} \left[ \frac{\cancel{(x+3)^3}\ 1}{(x+3)^{\cancel{5}\,2}} \right] \left[ \frac{2x^2+4}{x-7} \right] = -\infty$$ ■

SKILL
lim rat'l

$\frac{\text{pos}}{\text{small pos}}$

"$[\infty] \left[ \frac{22}{-10} \right]$"

---

$$2x^5 + 18x^4 + 58x^3 + 90x^2 + 108x + 108$$
$$= (x+3)^3(2x^2+4)$$

(Exercise)

$$[2x^2+4]_{x:\to -3} = 22 \neq 0$$

---

$$x^6 + 8x^5 - 15x^4 - 360x^3 - 1485x^2 - 2592x - 1701$$
$$= (x+3)^5(x-7)$$

(Exercise)

$$[x-7]_{x:\to -3} = -10 \neq 0$$

# SOME EXAMPLE LIMIT PROBLEMS:

SKILL
lim rat'l

(7) $$\lim_{x\to -1} \frac{x^5+3x^4-5x^3-23x^2-24x-8}{-x^7-5x^6-9x^5-5x^4+5x^3+9x^2+5x+1}$$

$$= \lim_{x\to -1} \left[\frac{\overset{1}{\cancel{(x+1)^3}}}{(x+1)^{\cancel{6}\,3}}\right]\left[\frac{x^2-8}{-x+1}\right]$$

left: pos/sm neg
right: pos/sm pos

left: $-\infty$
right: $\infty$

$\frac{-7}{2}$

left: $\infty$
right: $-\infty$

two-sided limit DNE ■

$$x^5+3x^4-5x^3-23x^2-24x-8$$
$$= (x+1)^3(x^2-8)$$

(Exercise)

$$\left[x^2-8\right]_{x:\to -1} = -7 \neq 0$$

$$-x^7-5x^6-9x^5-5x^4+5x^3+9x^2+5x+1$$
$$= (x+1)^6(-x+1)$$

(Exercise)

$$[-x+1]_{x:\to -1} = 2 \neq 0$$

General procedure: Let $P$ and $Q$ be polynomials, and let $a \in \mathbb{R}$.

To compute $\lim_{x\to a} \frac{P(x)}{Q(x)}$, $\lim_{x\to a^-} \frac{P(x)}{Q(x)}$ or $\lim_{x\to a^+} \frac{P(x)}{Q(x)}$,

write $P(x) = [(x-a)^m][F(x)]$,
and $Q(x) = [(x-a)^n][G(x)]$,
with $F(a) \neq 0$ and $G(a) \neq 0$, and then:

If $m > n$, $\frac{P(x)}{Q(x)} \overset{x \neq a}{=} \left[(x-a)^{m-n}\right]\left[\frac{F(x)}{G(x)}\right] \underset{x \to a}{\rightarrow} 0.$

If $m = n$, $\frac{P(x)}{Q(x)} \overset{x \neq a}{=} \frac{F(x)}{G(x)} \overset{x \to a}{\rightarrow} \frac{F(a)}{G(a)}.$

General procedure: Let $P$ and $Q$ be polynomials, and let $a \in \mathbb{R}$.

To compute $\lim\limits_{x\to a} \dfrac{P(x)}{Q(x)}$, $\lim\limits_{x\to a^-} \dfrac{P(x)}{Q(x)}$ or $\lim\limits_{x\to a^+} \dfrac{P(x)}{Q(x)}$,

write $P(x) = [(x-a)^m][F(x)]$,
and $Q(x) = [(x-a)^n][G(x)]$,
with $F(a) \neq 0$ and $G(a) \neq 0$, and then:

If $m < n$, $$\frac{P(x)}{Q(x)} \overset{x \neq a}{=} \underbrace{\left[\frac{1}{(x-a)^{n-m}}\right]}_{??} \underbrace{\left[\frac{F(x)}{G(x)}\right]}_{\xrightarrow{x \to a} \frac{F(a)}{G(a)} \neq 0}.$$

If $m < n$ and $n - m$ is even, then $\dfrac{1}{(x-a)^{n-m}} \xrightarrow{x \to a} \infty$.

If $m < n$ and $n - m$ is odd,

then $\dfrac{1}{(x-a)^{n-m}}$ $\xrightarrow{x \to a^+} \infty$ and $\xrightarrow{x \to a^-} -\infty$.

(Two-sided limit DNE.)

If $m < n$, $\dfrac{P(x)}{Q(x)} \overset{x \neq a}{=} \left[\underbrace{\dfrac{1}{(x-a)^{n-m}}}_{??}\right] \left[\underbrace{\dfrac{F(x)}{G(x)}}\right]$.

$\dfrac{F(x)}{G(x)} \xrightarrow{x \to a} \dfrac{F(a)}{G(a)} \neq 0$

Note: $\lim_{x \to \boxed{0}}$ of a rational expression of $x$ is easier than the rest, because it's so easy to factor $\underbrace{x - 0}_{x}$ from a polynomial in $x$...

---

*e.g.*: Factor $x$ as many times as possible from

$$x^8 + 4x^7 + 3x^6 - x^5$$
$$= x^5(x^3 + 4x^2 + 3x - 1)$$

$$\left[x^3 + 4x^2 + 3x - 1\right]_{x:\to 0} = -1 \neq 0 \blacksquare$$

Note: $\lim_{x \to 0}$ of a rational expression of $x$ is easier than the rest, because it's so easy to factor $\underbrace{x - 0}_{x}$ from a polynomial in $x$...

e.g.: $$\lim_{x \to 0} \frac{x^9 - 4x^8 + 3x^7 - 12x^6 + 4x^5}{7x^8 - 3x^7 + 5x^6 - 2x^5}$$

$$= \lim_{x \to 0} \frac{\cancel{x^5}\left[x^4 - 4x^3 + 3x^2 - 12x + 4\right]}{\cancel{x^5}\left[7x^3 - 3x^2 + 5x - 2\right]}$$

SKILL
lim rat'l

$$= \frac{4}{-2} = -2 \blacksquare$$

Exercise: Evaluate the limit, if it exists.

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+2t}\right)$$

rational

write as: poly/poly

$$\frac{1}{t}-\frac{1}{t^2+2t}=\frac{1}{t}-\frac{1}{t(t+2)}=\frac{t+2}{t(t+2)}-\frac{1}{t(t+2)}$$

$t^2+2t = t(t+2)$

$(t+2)$

COMMON DENOMINATOR

$t+1$

$$=\frac{t+1}{t(t+2)}=\left[\frac{1}{t}\right]\left[\frac{t+1}{t+2}\right]$$

$t\to 0^+$: $\infty$

$t\to 0^-$: $-\infty$

$\frac{1}{t}$: $t\to 0^\pm$, $\pm\infty$

$\frac{t+1}{t+2}$: $t\to 0$, $\frac{1}{2}$

Exercise: Evaluate the limit, if it exists.

$$\lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + 2t} \right) \text{ does NOT exist.} \blacksquare$$

SKILL
lim rat'l

$$\frac{1}{t} - \frac{1}{t^2 + 2t} = \frac{1}{t} - \frac{1}{t(t+2)} = \frac{t+2}{t(t+2)} - \frac{1}{t(t+2)}$$

$$= \frac{t+1}{t(t+2)} = \boxed{\left[\frac{1}{t}\right] \left[\frac{t+1}{t+2}\right]}$$

$t \to 0^+$: $\to \infty$

$t \to 0^-$: $\to -\infty$

SKILL
lim rat'l sqrt

LIMIT ZERO

EXAMPLE: Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

LIMIT ZERO

Rationalize the numerator! ☺

$$\frac{\sqrt{t^2+9}-3}{t^2} = \frac{\sqrt{t^2+9}-3}{t^2} \, \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$

RATIONAL

LIMIT NONZERO

$$= \frac{(t^2+9)-9}{t^2\left[\sqrt{t^2+9}+3\right]}$$

1

$t \neq 0$

$$= \frac{1}{\sqrt{t^2+9}+3} \to \frac{1}{\sqrt{0^2+9}+3} = \frac{1}{6}$$

continuous at $t = 0$

$t \to 0$

$\| \; \frac{1}{6}$ ■

Exercise: Evaluate the limit, if it exists.

SKILL
lim rat'l sqrt

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{4+t}}-\frac{1}{2t}\right)$$

$$\frac{1}{t\sqrt{4+t}}-\frac{1}{2t}=\frac{2}{2t\sqrt{4+t}}-\frac{\sqrt{4+t}}{2t\sqrt{4+t}}$$

COMMON DENOMINATOR

RATIONALIZE NUMERATOR

$$=\frac{2-\sqrt{4+t}}{2t\sqrt{4+t}}\;\frac{2+\sqrt{4+t}}{2+\sqrt{4+t}}$$

$$=\frac{4-1(4+t)}{2t\sqrt{4+t}(2+\sqrt{4+t})}$$

$$\overset{t\neq 0}{=}\frac{-1}{2\sqrt{4+t}(2+\sqrt{4+t})}$$

Exercise: Evaluate the limit, if it exists.

SKILL
lim rat'l sqrt

$$\lim_{t \to 0} \left( \frac{1}{t\sqrt{4+t}} - \frac{1}{2t} \right)$$

$$\frac{1}{t\sqrt{4+t}} - \frac{1}{2t} \overset{t \neq 0}{=} \frac{-1}{2\sqrt{4+t}(2+\sqrt{4+t})}$$

$$\to \frac{-1}{2\sqrt{4+0}(2+\sqrt{4+0})}$$

$t \to 0$

$$\overset{t \neq 0}{=} \frac{-1}{2\sqrt{4+t}(2+\sqrt{4+t})}$$

Exercise: Evaluate the limit, if it exists.

SKILL
lim rat'l sqrt

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{4+t}}-\frac{1}{2t}\right)=-\frac{1}{16}\ \blacksquare$$

$$\frac{1}{t\sqrt{4+t}}-\frac{1}{2t}\overset{t\neq 0}{=}\frac{-1}{2\sqrt{4+t}(2+\sqrt{4+t})}$$

$$\to\frac{-1}{2\sqrt{4+0}(2+\sqrt{4+0})}$$

$t\to 0$

$$=-\frac{1}{16}$$

Exercise:

Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]=0.$$

Illustrate by graphing the functions $f$, $g$ and $h$ (in the notation of the Squeeze Theorem) on the same screen.

---

$\sqrt{x^5+8x^2}\geq 0$ for $x\in[-2,\infty)$

$$-1\leq \sin \leq 1$$

$$-1\leq \sin\frac{2\pi}{x}\leq 1 \text{ for } x\neq 0$$

$$-\sqrt{x^5+8x^2}\leq\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]\leq\sqrt{x^5+8x^2}$$

for $x\in[-2,\infty)\setminus\{0\}$

and so for $x\in(-2,2)\setminus\{0\}$

Exercise:

Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]=0.$$

Illustrate by graphing the functions $f$, $g$ and $h$ (in the notation of the Squeeze Theorem) on the same screen.

$$-\sqrt{x^5+8x^2}\le\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]\le\sqrt{x^5+8x^2}$$

for $x\in(-2,2)\backslash\{0\}$

$x\to 0$

$$-\sqrt{x^5+8x^2}\le\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]\le\sqrt{x^5+8x^2}$$

for $x\in(-2,2)\backslash\{0\}$

Exercise:

Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]=0.$$

Illustrate by graphing the functions $f$, $g$ and $h$ (in the notation of the Squeeze Theorem) on the same screen.

$$-\sqrt{x^5+8x^2}\le\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]\le\sqrt{x^5+8x^2}$$

for $x\in(-2,2)\backslash\{0\}$

$$\sqrt{x^5+8x^2}=\left[\sqrt{x^2}\right]\left[\sqrt{x^3+8}\right]$$

$$=\left[\,|x|\,\right]\left[\sqrt{x^3+8}\right]$$

contin. in $x$ — contin. at $x=0$

$x\to 0$: $\sqrt{x^5+8x^2}\to 0$

Exercise:

Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \left[\sqrt{x^5 + 8x^2}\right]\left[\sin\frac{2\pi}{x}\right] = 0.$$

Illustrate by graphing the functions $f$, $g$ and $h$ (in the notation of the Squeeze Theorem) on the same screen.

$$-\sqrt{x^5 + 8x^2} \le \left[\sqrt{x^5 + 8x^2}\right]\left[\sin\frac{2\pi}{x}\right] \le \sqrt{x^5 + 8x^2}$$

for $x \in (-2, 2)\backslash\{0\}$

$x \to 0$ $\quad$ $x \to 0$

0 $\quad$ 0

Exercise:

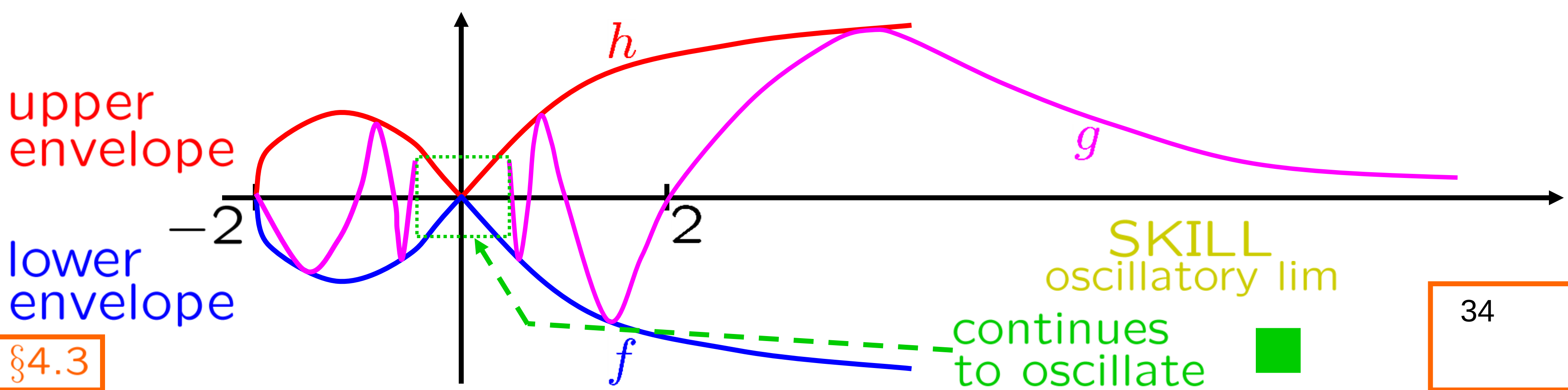
Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]=0.$$

Illustrate by graphing the functions $f$, $g$ and $h$ (in the notation of the Squeeze Theorem) on the same screen.

$$-\sqrt{x^5+8x^2}\le\left[\sqrt{x^5+8x^2}\right]\left[\sin\frac{2\pi}{x}\right]\le\sqrt{x^5+8x^2}$$

$f(x)$ $\quad g(x)$ $\quad$ inside the envelope $\quad h(x)$




SKILL
oscillatory lim

Exercise: If

$$f(x) = \begin{cases} x^4, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

$\mathbb{Q} :=$ {rational numbers}

$\forall x \in \mathbb{R} \backslash \mathbb{Q}$, $f(x) = x^4$,

$\mathbb{R} \backslash \mathbb{Q}$ = {irrational numbers}

so $0 \le f(x) \le x^4$.

$\forall x \in \mathbb{Q}$, $f(x) = 0$,

$\mathbb{Q}$ = {rational numbers}

so $0 \le f(x) \le x^4$.

Then, $\forall x \in \mathbb{R}$, $0 \le f(x) \le x^4$.

Exercise: If

$$f(x) = \begin{cases} x^4, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

Graph of $y = f(x)$?

---

Then, $\forall x \in \mathbb{R}$, $0 \leq f(x) \leq x^4$.

$x \to 0$: $0 \to 0$

SQUEEZE THEOREM: $x \to 0$: $f(x) \to 0$

$x \to 0$: $x^4 \to 0$

QED

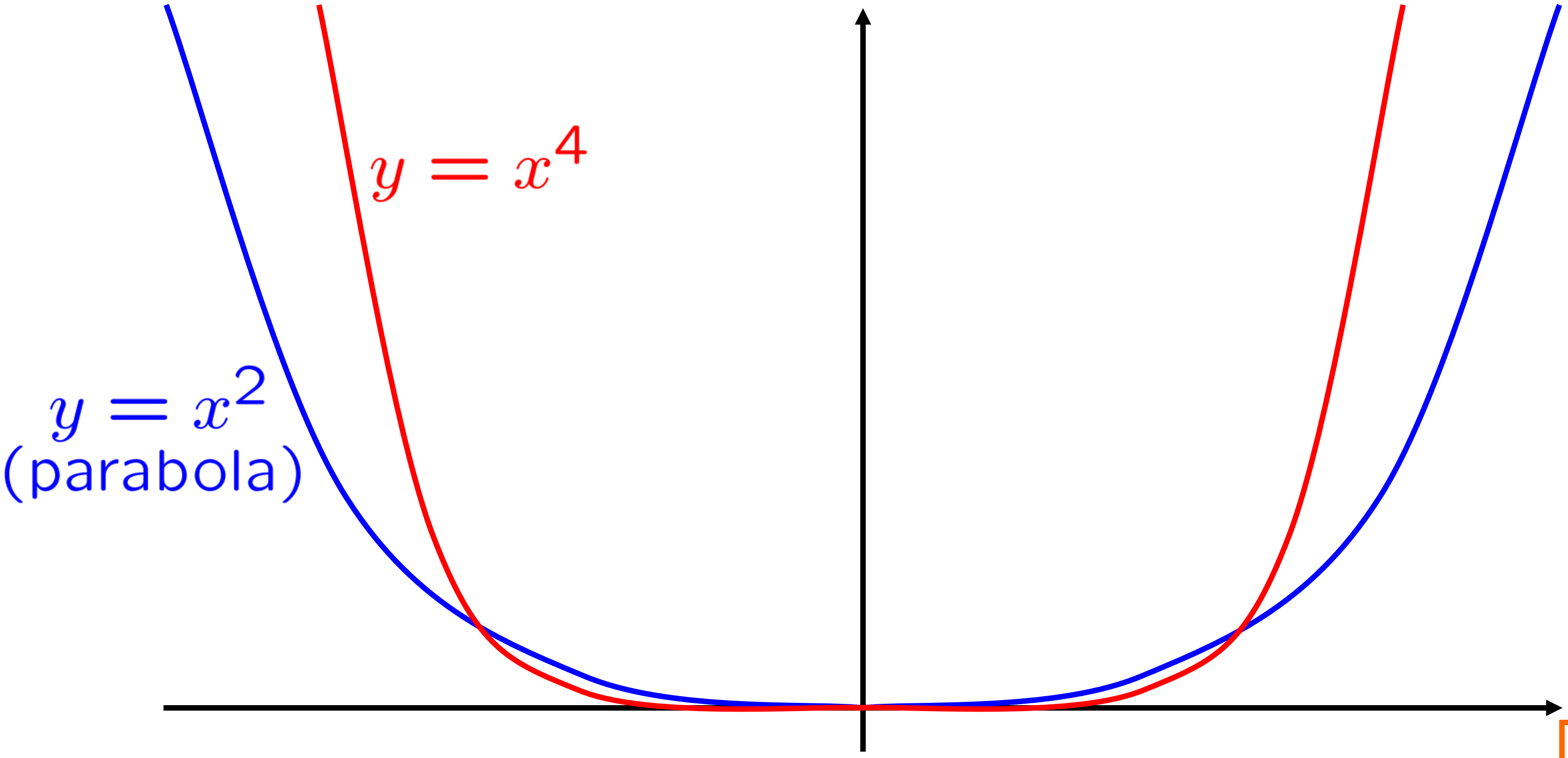
Then, $\forall x \in \mathbb{R}$, $0 \leq f(x) \leq x^4$.

Exercise: If

$$f(x) = \begin{cases} x^4, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

Graph of $y = f(x)$?

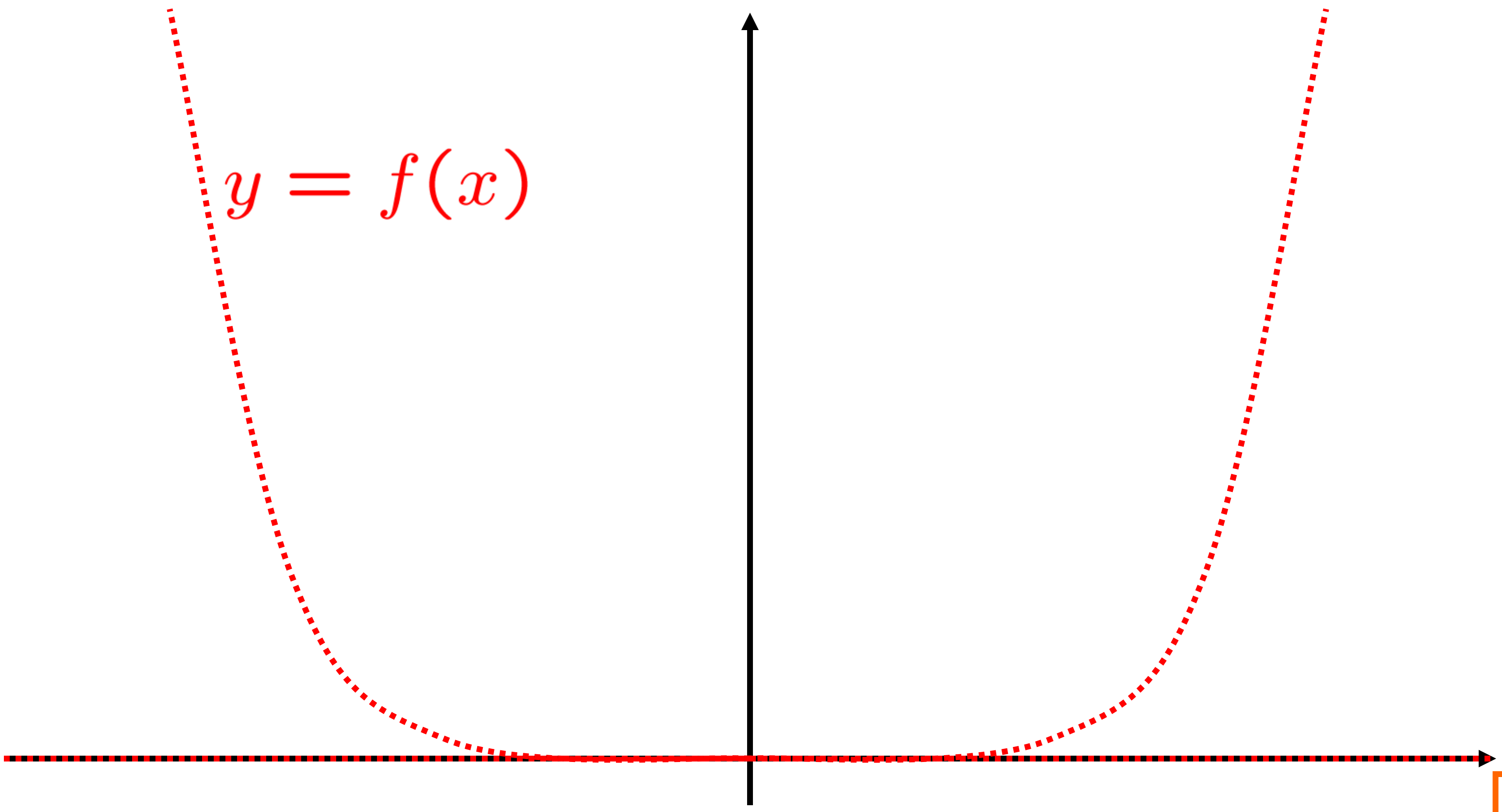


Exercise: If

$$f(x) = \begin{cases} x^4, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

Graph of $y = f(x)$?



SKILL
Compute limits
Whitman problems
§2.3, p. 34, #1-15

SKILL
Oscillatory limit
Whitman problems
§2.3, p. 35, #16

SKILL
Find $\delta$
Whitman problems
§2.3, p. 35, #17

SKILL
Limits from gph
Whitman problems
§2.3, p. 35, #18

SKILL
Calculator estimation of limits
Whitman problems
§2.3, p. 35, #19