CALCULUS

Trigonometric limits

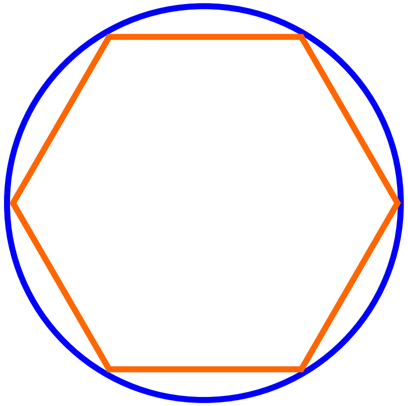
Calculation of area inside a circle...

circle of radius $r$

incribed regular $n$-gon

$n = 6$ (hexagon)

Split the hexagon into six triangles.



Spp

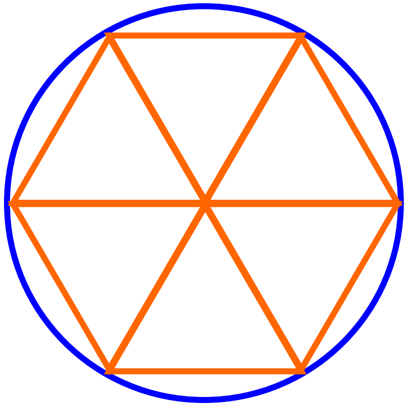
Calculation of area inside a circle. . .

circle of radius $r$

incribed regular $n$-gon

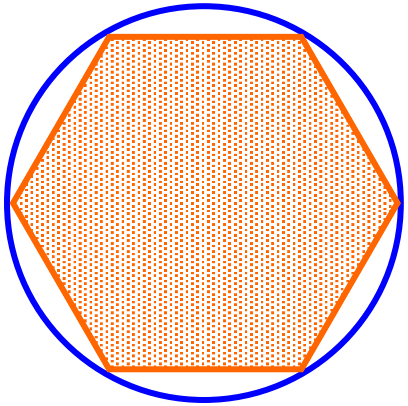
$n = 6$ (hexagon)

Split the hexagon into six triangles.
Compute area inside hexagon.



Calculation of area inside a circle. . .
circle of radius $r$
incribed regular $n$-gon
$n = 6$ (hexagon)

Split the hexagon into six triangles.
Compute area inside hexagon.
Generalize to $n$-gon



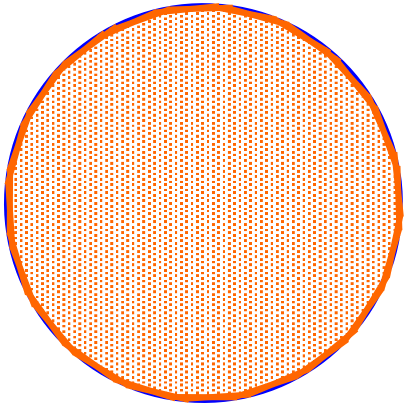
Calculation of area inside a circle...

circle of radius $r$

incribed regular $n$-gon

$n = 20$

Split the hexagon into six triangles.
Compute area inside hexagon.
Generalize to $n$-gon
Let $n \to \infty$.
Get area inside circle.



$n = 20$

$n = 6$

LET'S GO BACK TO THE HEXAGON...

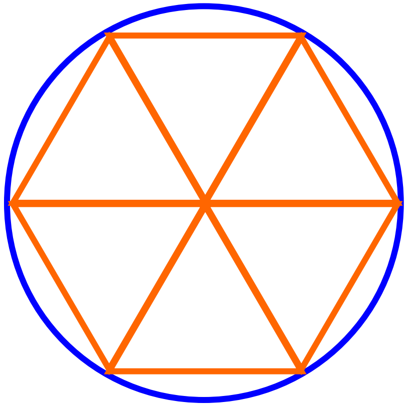
Calculation of area inside a circle...

circle of radius $r$

incribed regular $n$-gon

$n = 6$ (hexagon)

Split the hexagon into six triangles.
Compute area inside hexagon.
Generalize to $n$-gon
Let $n \to \infty$.
Get area inside circle.



LET'S GO BACK TO THE HEXAGON...

Calculation of area inside a circle. . .

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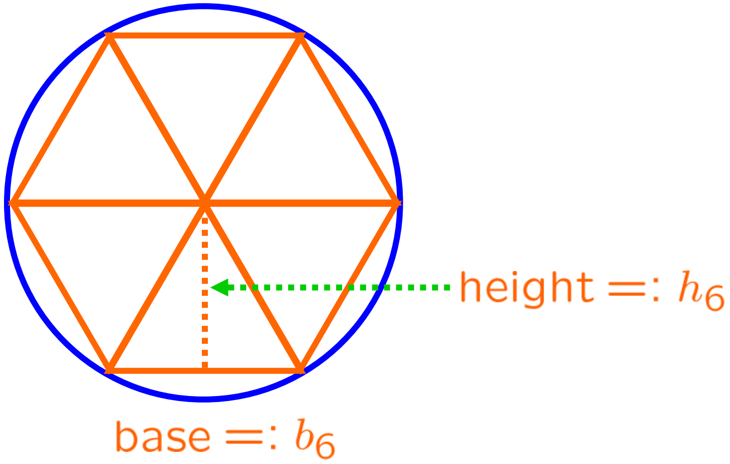

Calculation of area inside a circle. . .

circle of radius $r$

incribed regular $n$-gon

$n = 6$ (hexagon)

Split the hexagon into six triangles.
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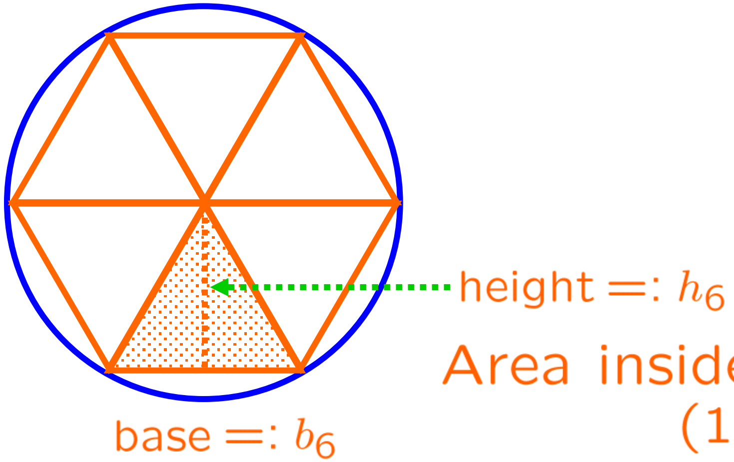



Area inside one triangle:
$(1/2)b_6h_6$

# Calculation of area inside a circle. . .

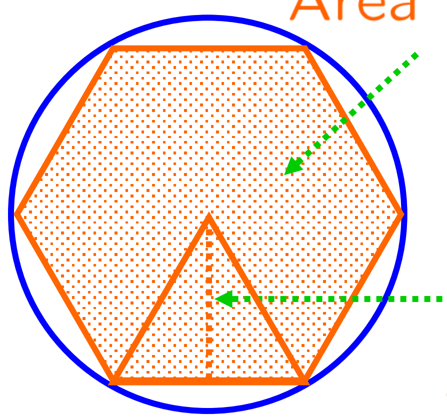
circle of radius $r$

incribed regular $n$-gon

Back to: 20-gon

$n = 6$ (hexagon)

Split the hexagon into six triangles.
Compute area inside hexagon.
Generalize to $n$-gon
Let $n \to \infty$.
Get area inside circle.



height $=: h_6$

base $=: b_6$

Area inside inscribed hexagon:

$6(1/2)b_6 h_6$

$= (1/2)C_6 h_6$

Area inside one triangle:

$(1/2)b_6 h_6$

Spp

$C_6 :=$ circumference of hexagon $= 6b_6$

Area inside incribed $n$-gon:
$(1/2)C_n h_n$

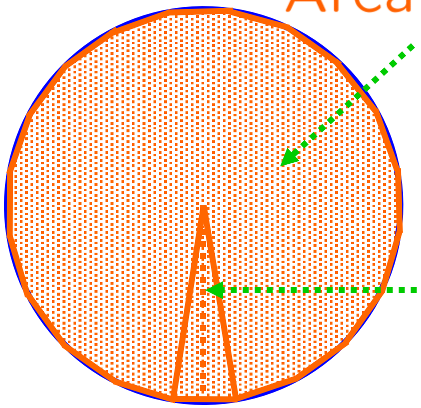
As $n \to \infty$: $C_n \to 2\pi r$ and $h_n \to r$

Next: $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$

Area inside circle of radius $r$:
$$\lim_{n\to\infty} (1/2)C_n h_n = (1/2)(2\pi r)r = \pi r^2$$

Calculation of area inside a circle. . .
circle of radius $r$
incribed regular $n$-gon
$n = 20$

Split the hexagon into six triangles.
Compute area inside hexagon.
Generalize to $n$-gon
Let $n \to \infty$.
Get area inside circle.



Area inside inscribed 20-gon:
$20(1/2)b_{20}h_{20}$
$= (1/2)C_{20}h_{20}$

height$=:$ $h_{20}$

Area inside one triangle:
$(1/2)b_{20}h_{20}$

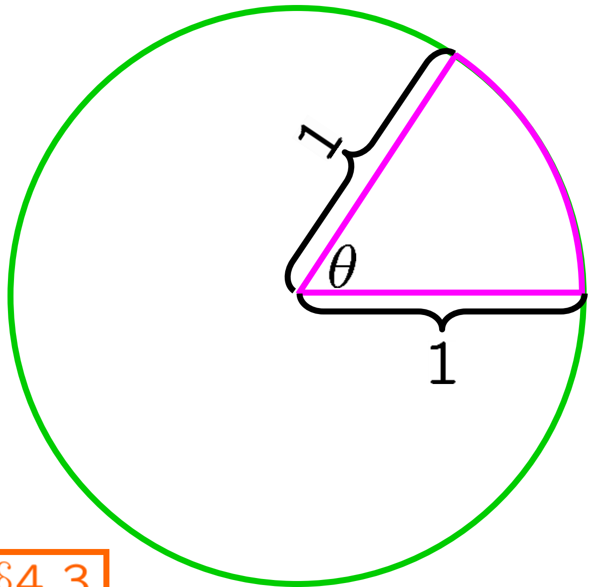
base$=:$ $b_{20}$

Spp $C_{20} :=$ circumference of hexagon $= 20b_{20}$

$$\frac{\text{area in the sector}}{\pi \cdot 1^2} = \frac{\text{area in the sector}}{\text{area in circle}} = \frac{\theta}{2\pi}$$

$$\text{area in the sector} = \left[\pi \cdot 1^2\right]\left[\frac{\theta}{2\pi}\right] = \frac{\theta}{2}$$

$0 < \theta < \pi/2$

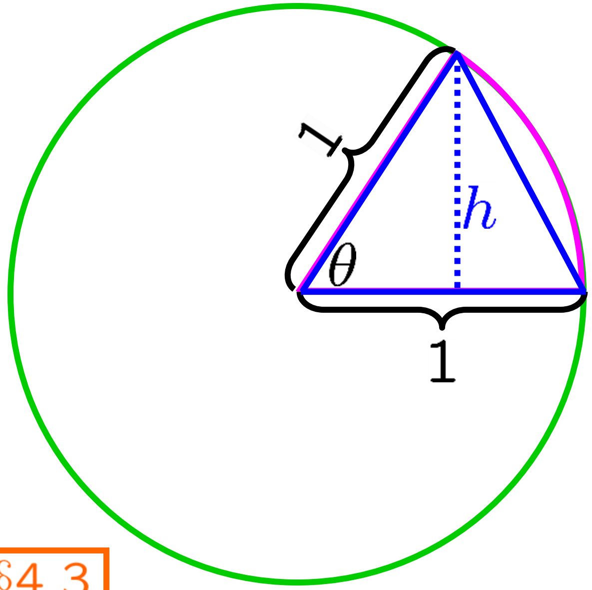


$$\text{area in the sector} = \left[\cancel{\pi} \cdot \cancel{1^2}\right] \left[\frac{\theta}{2\cancel{\pi}}\right] = \frac{\theta}{2}$$

$0 < \theta < \pi/2$

$$\text{area in the blue triangle} = \frac{\cancel{1} \cdot h}{2} = \frac{\sin\theta}{2}$$

$$\sin\theta = \frac{h}{\cancel{1}} = h$$



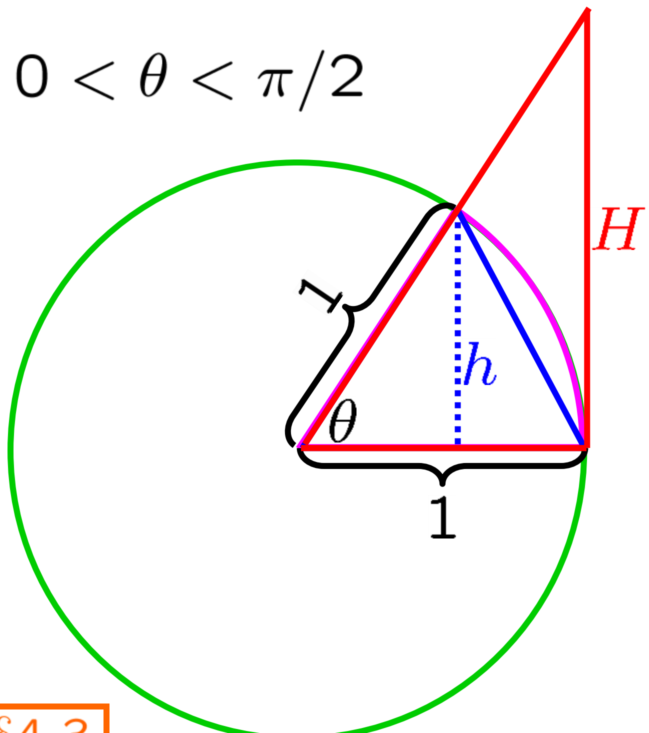
$$\tan\theta = \frac{H}{\cancel{1}} = H$$

$$\text{area in the red triangle} = \frac{\cancel{1}\cdot H}{2} = \frac{\tan\theta}{2}$$

$$\text{area in the sector} = \left[\cancel{\pi}\cdot \cancel{1^2}\right]\left[\frac{\theta}{2\cancel{\pi}}\right] = \frac{\theta}{2}$$

$$\text{area in the blue triangle} = \frac{\cancel{1}\cdot h}{2} = \frac{\sin\theta}{2}$$

$0 < \theta < \pi/2$



$$\text{area in the red triangle} = \frac{\cancel{1} \cdot H}{2} = \frac{\tan\theta}{2}$$

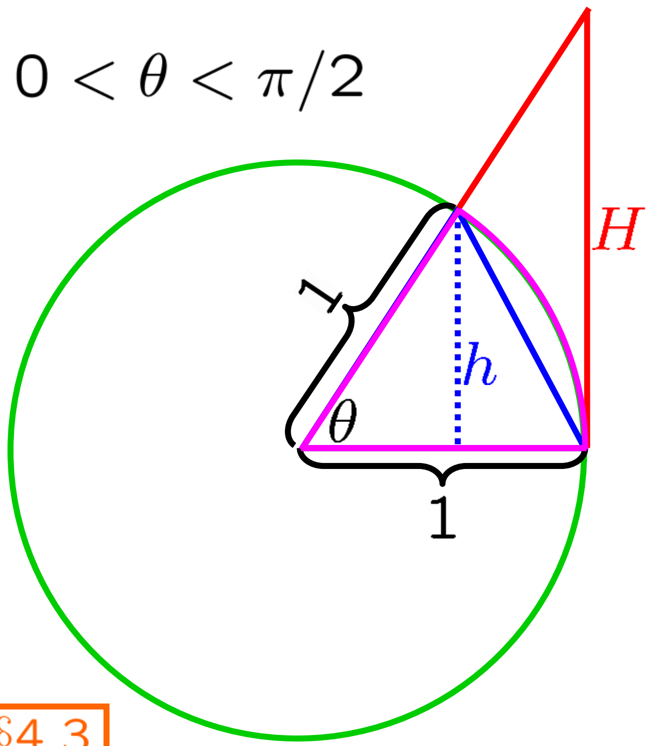
$$\vee$$

$$\text{area in the sector} = \left[\cancel{\pi} \cdot \cancel{1^2}\right]\left[\frac{\theta}{2\cancel{\pi}}\right] = \frac{\theta}{2}$$

$$\vee$$

$$\text{area in the blue triangle} = \frac{\cancel{1} \cdot h}{2} = \frac{\sin\theta}{2}$$

$0 < \theta < \pi/2$

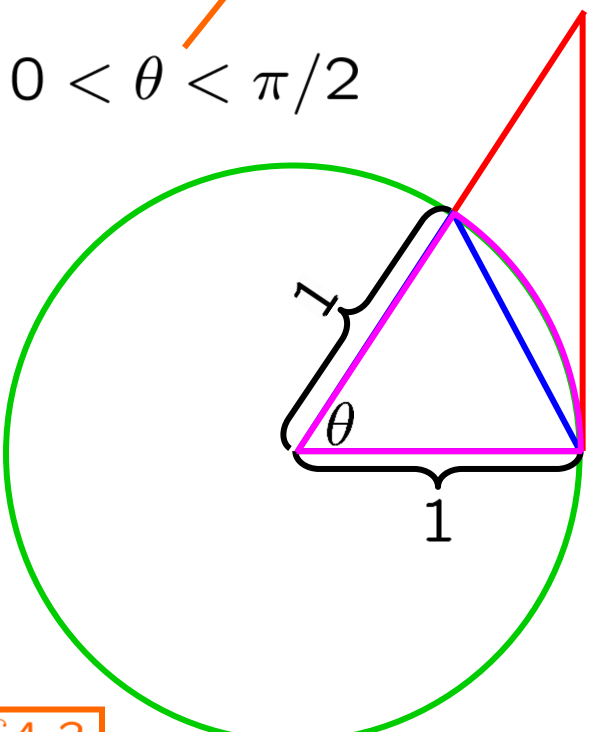



$$2 \times \left(\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}\right)$$

$$\sin\theta < \theta < \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\forall \theta \in (0, \pi/2), \quad \sin\theta < \theta < \frac{\sin\theta}{\cos\theta}$$

$0 < \theta < \pi/2$




$$\sin\theta \; < \; \theta \; < \; \tan\theta \; = \; \frac{\sin\theta}{\cos\theta}$$

$$\forall \theta \in (0, \pi/2), \quad \sin\theta < \theta < \frac{\sin\theta}{\cos\theta}$$

DIVIDE BY $\sin\theta$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

RECIPROCATE

$\theta \to 0^+$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

SQUEEZE THEOREM

$$1 \qquad 1 \qquad 1$$

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$$

$\theta \to 0^+$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$1 \qquad 1 \qquad 1$$

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1$$

$$\left[\frac{\sin\theta}{\theta}\right]_{\theta :\to -0.001}$$

$$= \left[\frac{\sin(-0.001)}{-0.001}\right]$$

$$= \left[\frac{+\sin(0.001)}{+0.001}\right]$$

$$= \left[\frac{\sin(0.001)}{0.001}\right]$$

$$\approx 1$$

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

One of two important trig limits. The other is the limit as $\theta \to 0$ in...

$$\frac{1 - \cos\theta}{\theta}$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

One of two important trig limits. The other is the limit as $\theta \to 0$ in...

$$\frac{1-\cos\theta}{\theta}=\frac{1-\cos\theta}{\theta}\,\frac{1+\cos\theta}{1+\cos\theta}=\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$$

no cos; only sin

$$=\frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

doesn't tend to 0

$$\underset{\theta\to 0}{\sim}\frac{\theta^2}{\theta(1+\cos\theta)}\underset{\theta\neq 0}{=}\frac{\theta}{1+\cos\theta}\underset{\theta\to 0}{\to}\frac{0}{1+\cos 0}=0$$

$$\sin\theta \underset{\theta\to 0}{\sim} \theta$$

transcendental polynomial

$$\sin^2\theta \underset{\theta\to 0}{\sim} \theta^2$$

**Fact:** $f(\theta)\underset{\theta\to a}{\sim} g(\theta)\underset{\theta\to a}{\to} L \quad\Rightarrow\quad f(\theta)\underset{\theta\to a}{\to} L$

**Def'n:** $f(\theta)\underset{\theta\to a}{\sim} g(\theta)$ means $\lim_{\theta\to a}\left[\frac{f(\theta)}{g(\theta)}\right]=1.$

§4.3

§4.3, p. 68, l.-7: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

§4.3, p. 69, l.-10: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ 1 $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$

$$\frac{1-\cos\theta}{\theta} = \frac{1-\cos\theta}{\theta}\,\frac{1+\cos\theta}{1+\cos\theta} = \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

$$\underset{\theta\to 0}{\sim} \frac{\theta^2}{\theta(1+\cos\theta)} \underset{\theta\neq 0}{=} \frac{\theta}{1+\cos\theta}$$

$$\underset{\theta\to 0}{\to} \frac{0}{1+\cos 0}$$

$$= 0$$

$\sin\theta \underset{\theta\to 0}{\sim} \theta$

transcendental polynomial

$\sin^2\theta \underset{\theta\to 0}{\sim} \theta^2$

Fact: $f(\theta) \underset{\theta\to a}{\sim} g(\theta) \underset{\theta\to a}{\to} L \quad\Rightarrow\quad f(\theta) \underset{\theta\to a}{\to} L$

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§4.3

§4.3, p. 68, l.-7: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

§4.3, p. 69, l.-10: $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$

We will meet these limits again!

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Asymptotics alert: $f \underset{a}{\sim} g$ and $p \underset{a}{\sim} q$

implies both $fp \underset{a}{\sim} gq$ and $f/p \underset{a}{\sim} g/q$

Fact: $f(\theta) \underset{\theta \to a}{\sim} g(\theta) \underset{\theta \to a}{\to} L \quad \Rightarrow \quad f(\theta) \underset{\theta \to a}{\to} L$

Def'n: $f(\theta) \underset{\theta \to a}{\sim} g(\theta)$ means $\lim_{\theta \to a} \left[ \frac{f(\theta)}{g(\theta)} \right] = 1$.

Spp

*e.g.*: $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{4x^2+x^3}} = \lim_{x\to 0^+} \frac{x}{2x} = \lim_{x\to 0^+} \frac{1}{2} = \frac{1}{2}$ ■

$$\sin x \underset{x\to 0^+}{\sim} x$$

and $$\sqrt{4x^2+x^3} \underset{x\to 0^+}{\sim} \sqrt{4x^2} \underset{x>0}{=} 2x$$

---

Asymptotics alert: $f \underset{a}{\sim} g$ and $p \underset{a}{\sim} q$

implies both $fp \underset{a}{\sim} gq$ and $f/p \underset{a}{\sim} g/q$

BUT implies neither $f+p \underset{a}{\sim} g+q$ nor $f-p \underset{a}{\sim} g-q$.

Fact: $f(\theta) \underset{\theta\to a}{\sim} g(\theta) \underset{\theta\to a}{\to} L \quad \Rightarrow \quad f(\theta) \underset{\theta\to a}{\to} L$

Def'n: $f(\theta) \underset{\theta\to a}{\sim} g(\theta)$ means $\lim_{\theta\to a}\left[\frac{f(\theta)}{g(\theta)}\right] = 1$.

Spp

e.g.: $\displaystyle\lim_{t\to 0^+} \frac{\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}}{\sin^2 t}$

$$\sqrt{t^2+5t^3} \underset{t\to 0}{\sim} \sqrt{t^2}$$

$$\sqrt{t^2+t^3+t^4} \underset{t\to 0}{\sim} \sqrt{t^2}$$

Would like to subtract,
but not allowed.

Asymptotics alert: $f \underset{a}{\sim} g$ and $p \underset{a}{\sim} q$

implies both $fp \underset{a}{\sim} gq$ and $f/p \underset{a}{\sim} g/q$

BUT implies neither $f+p \underset{a}{\sim} g+q$ nor $f-p \underset{a}{\sim} g-q$.

Fact: $f(\theta) \underset{\theta\to a}{\sim} g(\theta) \underset{\theta\to a}{\to} L \Rightarrow f(\theta) \underset{\theta\to a}{\to} L$

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Spp

*e.g.*: $\lim_{t\to 0^+} \dfrac{\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}}{\sin^2 t}$

---

$$\sqrt{t^2+5t^3} \underset{t\to 0}{\sim} \sqrt{t^2}$$

$$\sqrt{t^2+t^3+t^4} \underset{t\to 0}{\sim} \sqrt{t^2}$$

$$\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4} \underset{t\to 0^+}{\sim} 2\sqrt{t^2} \underset{t>0}{=} 2t$$

---

Asymptotics alert: $f \underset{a}{\sim} g$ and $p \underset{a}{\sim} q$

implies both $fp \underset{a}{\sim} gq$ and $f/p \underset{a}{\sim} g/q$

BUT implies neither $f+p \underset{a}{\sim} g+q$ nor $f-p \underset{a}{\sim} g-q$.

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$\sqrt{f} \underset{a}{\sim} \sqrt{g}$ and $\sqrt{p} \underset{a}{\sim} \sqrt{q}$

implies $\sqrt{f}+\sqrt{p} \underset{a}{\sim} \sqrt{g}+\sqrt{q}$

but not $\sqrt{f}-\sqrt{p} \underset{a}{\sim} \sqrt{g}-\sqrt{q}$

e.g.: $\displaystyle\lim_{t\to 0^+}\frac{\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}}{\sin^2 t}$

$$\frac{\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}}{\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}} \underset{t\to 0^+}{\sim} 2\sqrt{t^2} \underset{t>0}{=} 2t$$

$$=\frac{\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}}{1}\,\frac{\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}}{\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}}$$

$$\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4} \underset{t\to 0^+}{\sim} 2\sqrt{t^2} \underset{t>0}{=} 2t$$

e.g.: $\displaystyle\lim_{t\to 0^+} \frac{\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}}{\sin^2 t}$

$$= \lim_{t\to 0^+} \frac{4t^3-t^4}{\left[\sin^2 t\right]\left[\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}\right]}$$

$$\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}$$

$$= \frac{\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}}{1}\,\frac{\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}}{\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}}$$

$$= \frac{\left(\cancel{t^2}+5t^3\right)-\left(\cancel{t^2}+t^3+t^4\right)}{\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}} = \frac{4t^3-t^4}{\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}}$$

$$\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4} \underset{t\to 0^+}{\sim} 2\sqrt{t^2} \underset{t>0}{=} 2t$$

$$\text{e.g.: } \lim_{t\to 0^+} \frac{\sqrt{t^2+5t^3}-\sqrt{t^2+t^3+t^4}}{\sin^2 t}$$

More on asymptotics later...

STOP

$$= \lim_{t\to 0^+} \frac{4t^3 - t^4}{\left[\sin^2 t\right]\left[\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4}\right]}$$

$$= \lim_{t\to 0^+} \frac{4t^3}{\left[t^2\right][2t]} = \lim_{t\to 0^+} \frac{4}{2} = \frac{4}{2} = 2 \blacksquare$$

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$$4t^3 - t^4 \underset{t\to 0^+}{\sim} 4t^3$$

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$$\sin^2 t \underset{t\to 0^+}{\sim} t^2$$

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$$\sqrt{t^2+5t^3}+\sqrt{t^2+t^3+t^4} \underset{t\to 0^+}{\sim} 2\sqrt{t^2} \underset{t>0}{=} 2t$$



Spp