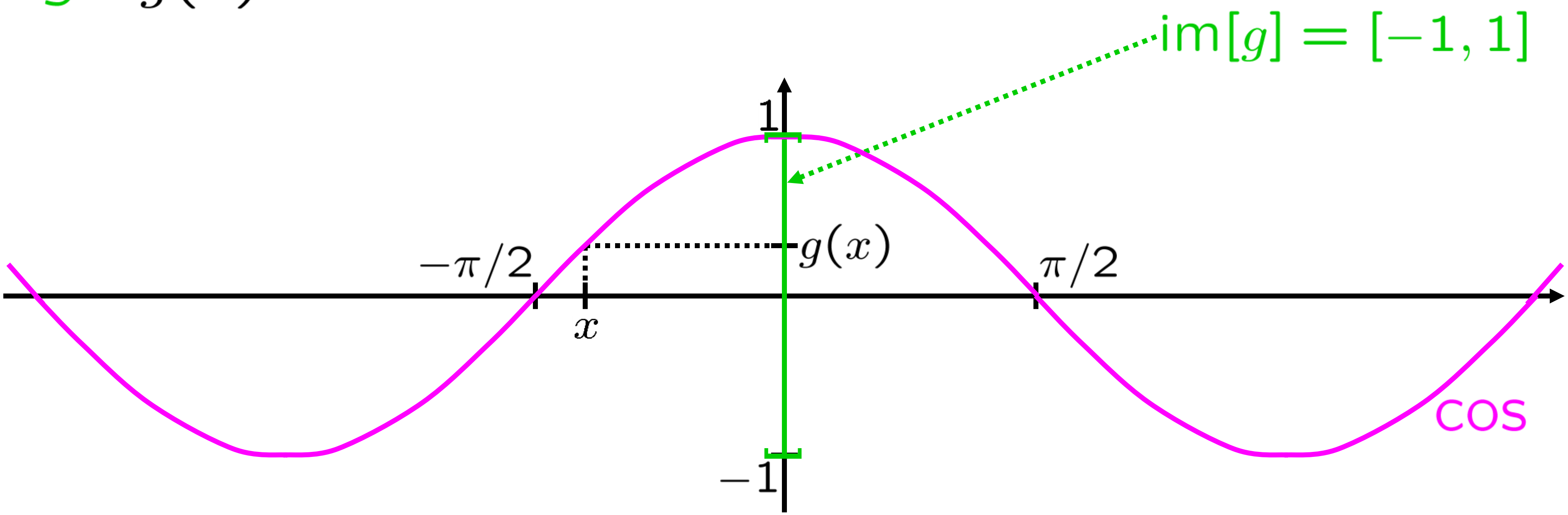
CALCULUS
Bounded functions and horizontal asymptotes

Def'n: The **image of** f is $\boxed{\mathrm{im}[f]} := \{f(x) \mid x \in \mathrm{dom}[f]\}$.

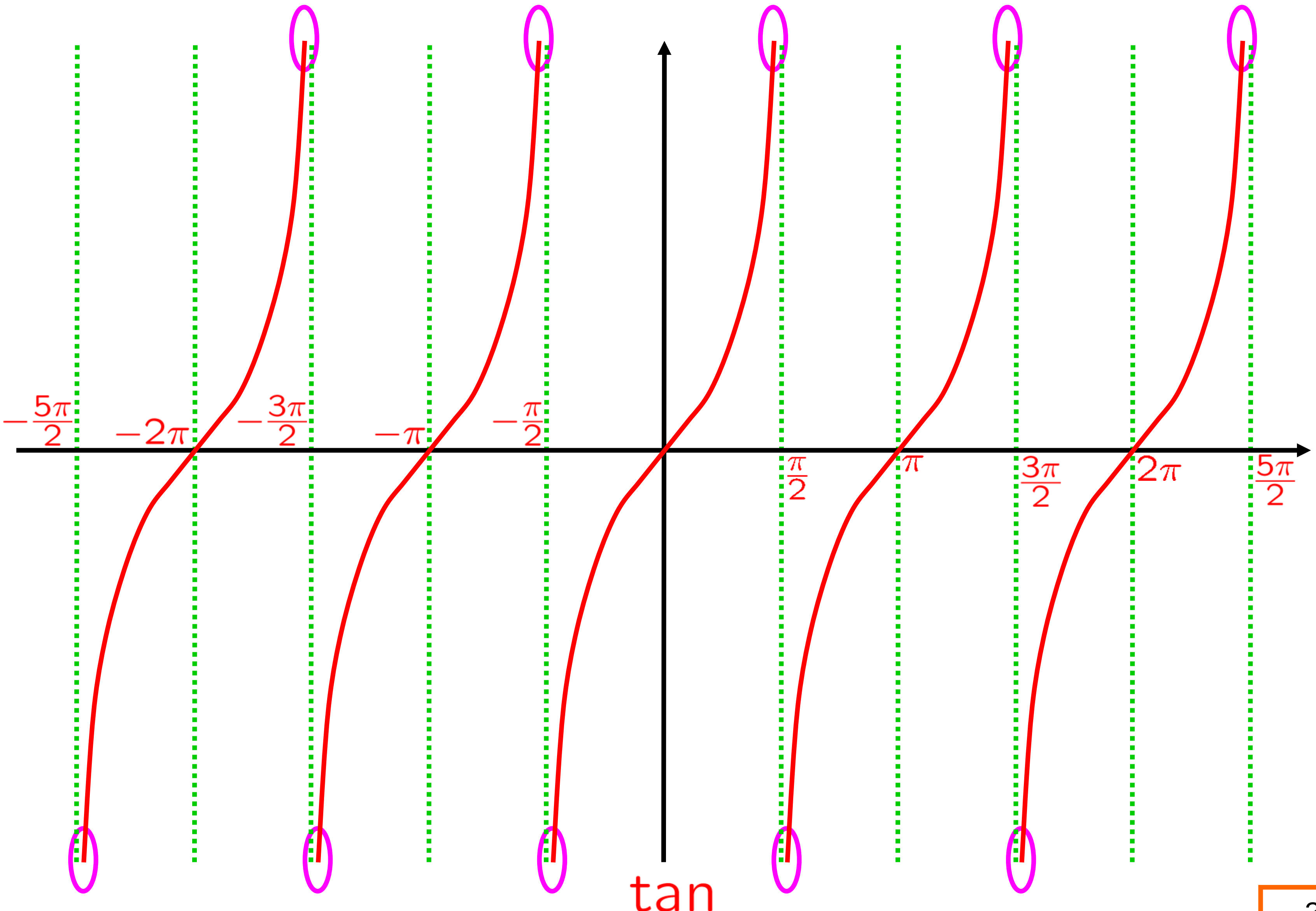
cf. §2.5, p. 40, Def'n 2.17: A function f is **bounded** if $\mathrm{im}[f]$ is contained in a bounded interval.

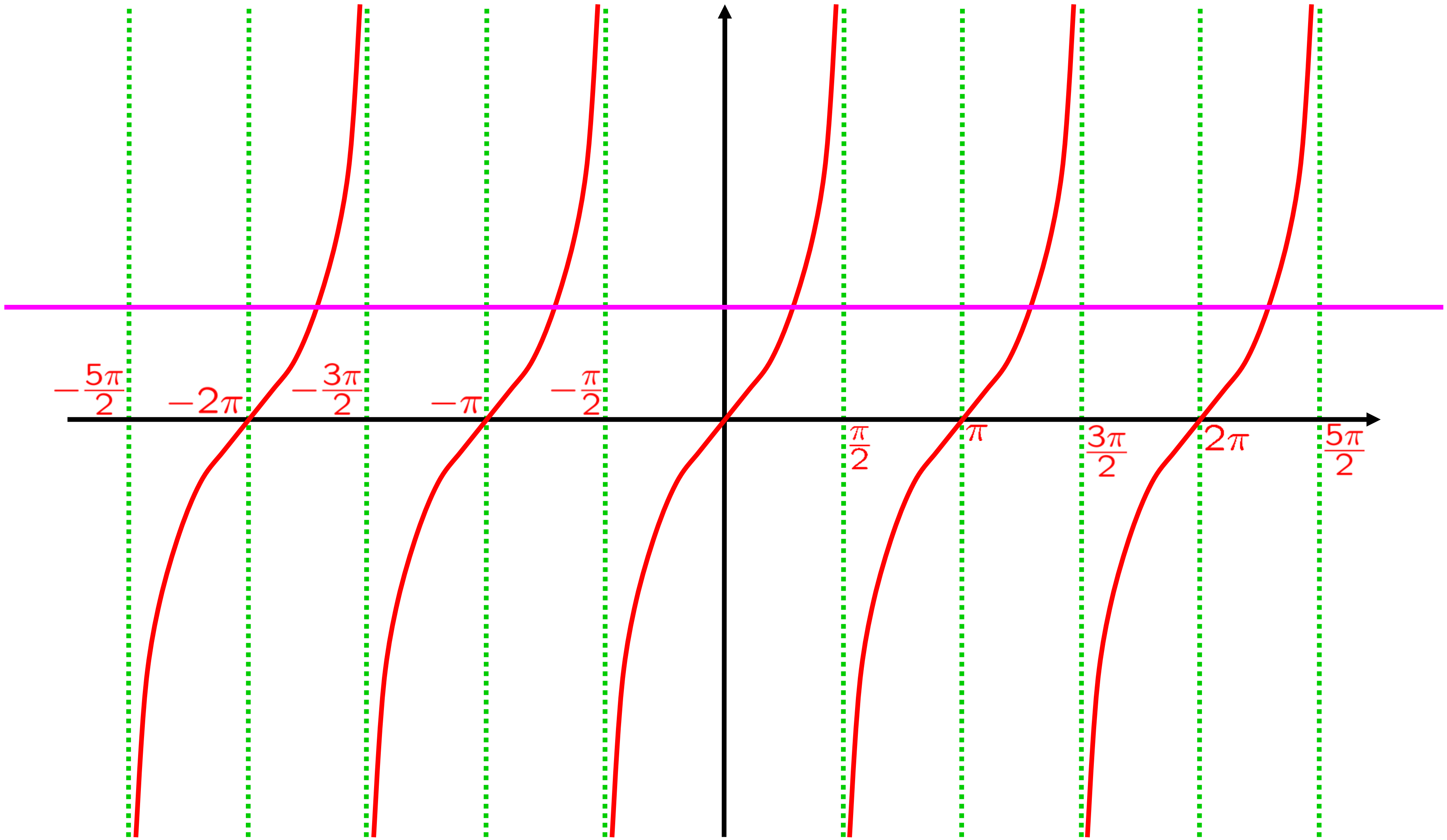
e.g.: $g(x) := \cos x$ is bounded.



cos is oscillatory; it doesn't grow,
but it also doesn't settle into a limit.

non-e.g.: $h(x) := \tan x$ is unbounded.

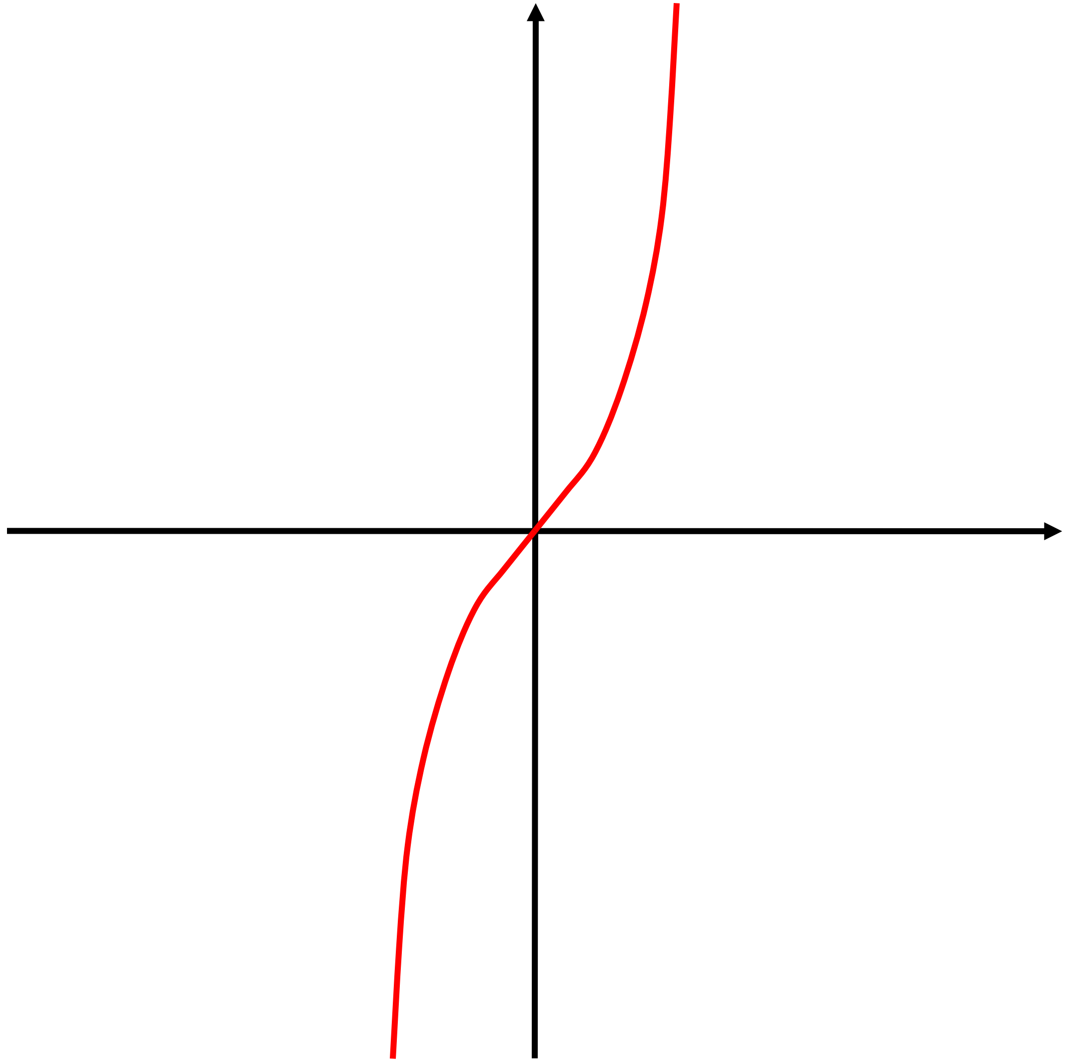




NOT 1-1

tan

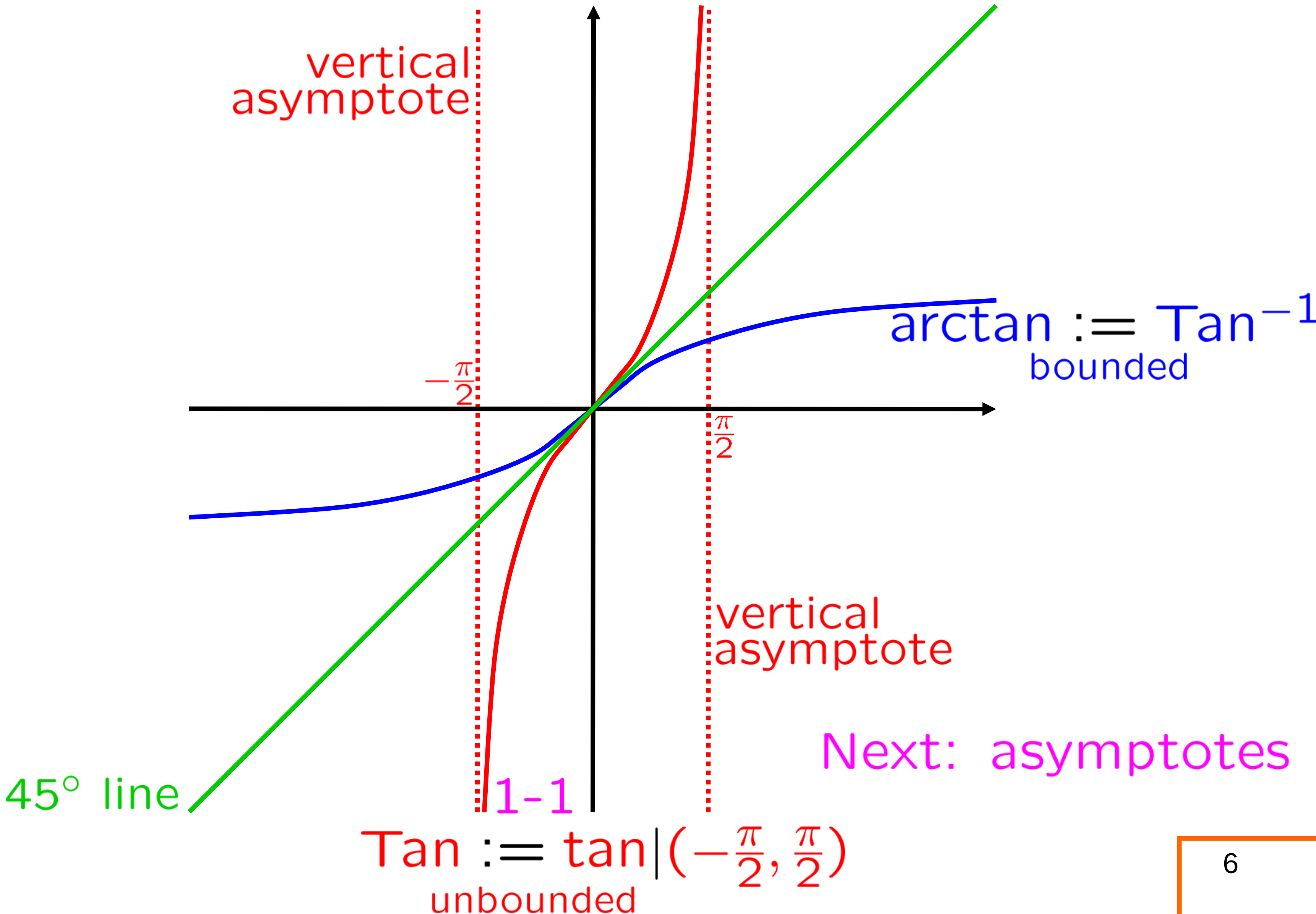
$\mathrm{Tan} := \mathrm{tan}|(-\frac{\pi}{2}, \frac{\pi}{2})$



$\mathrm{Tan} := \tan|_{(-\frac{\pi}{2}, \frac{\pi}{2})}$

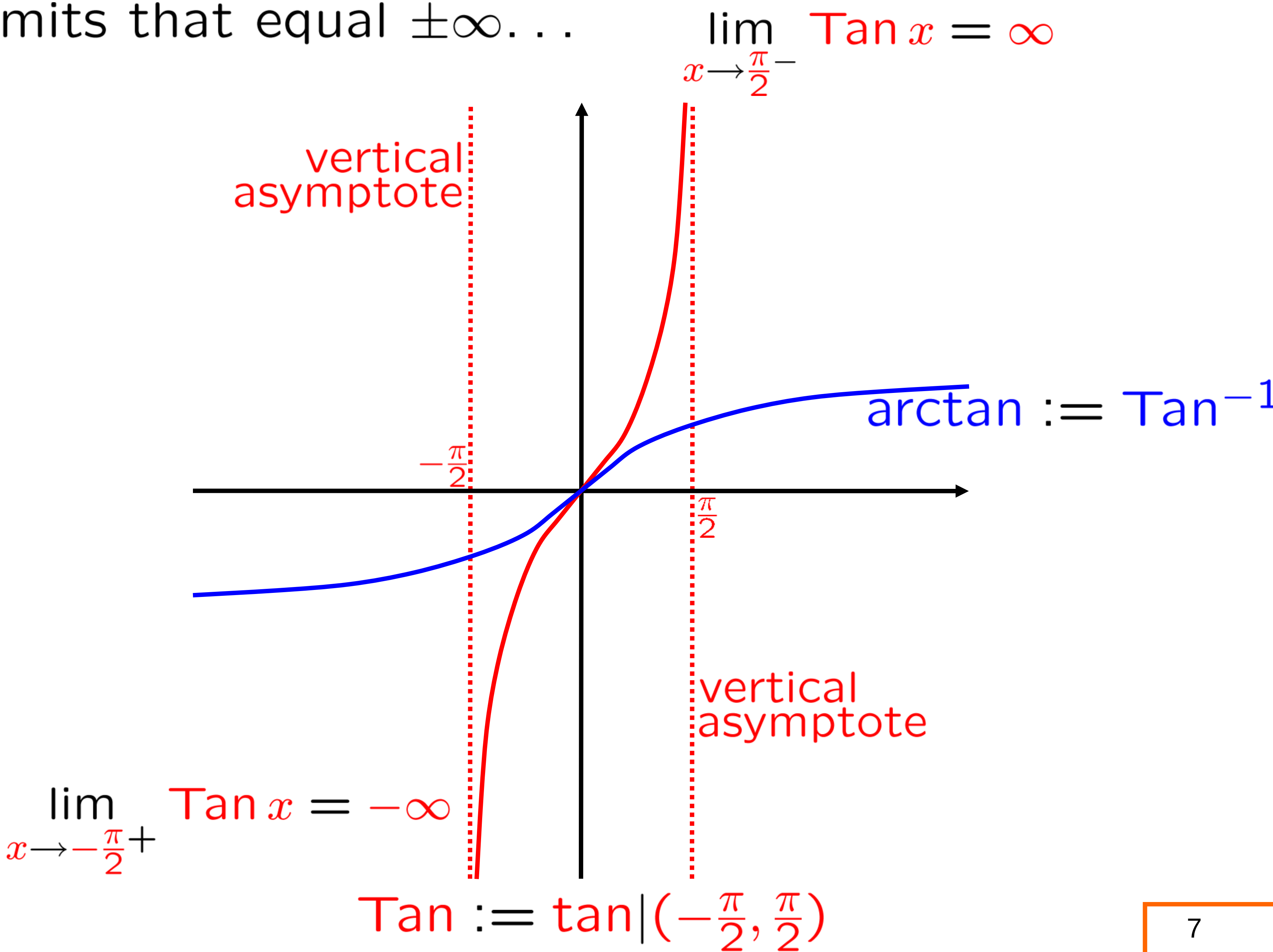
unbounded

Vertical asymptotes are related to limits that equal $\pm\infty$...



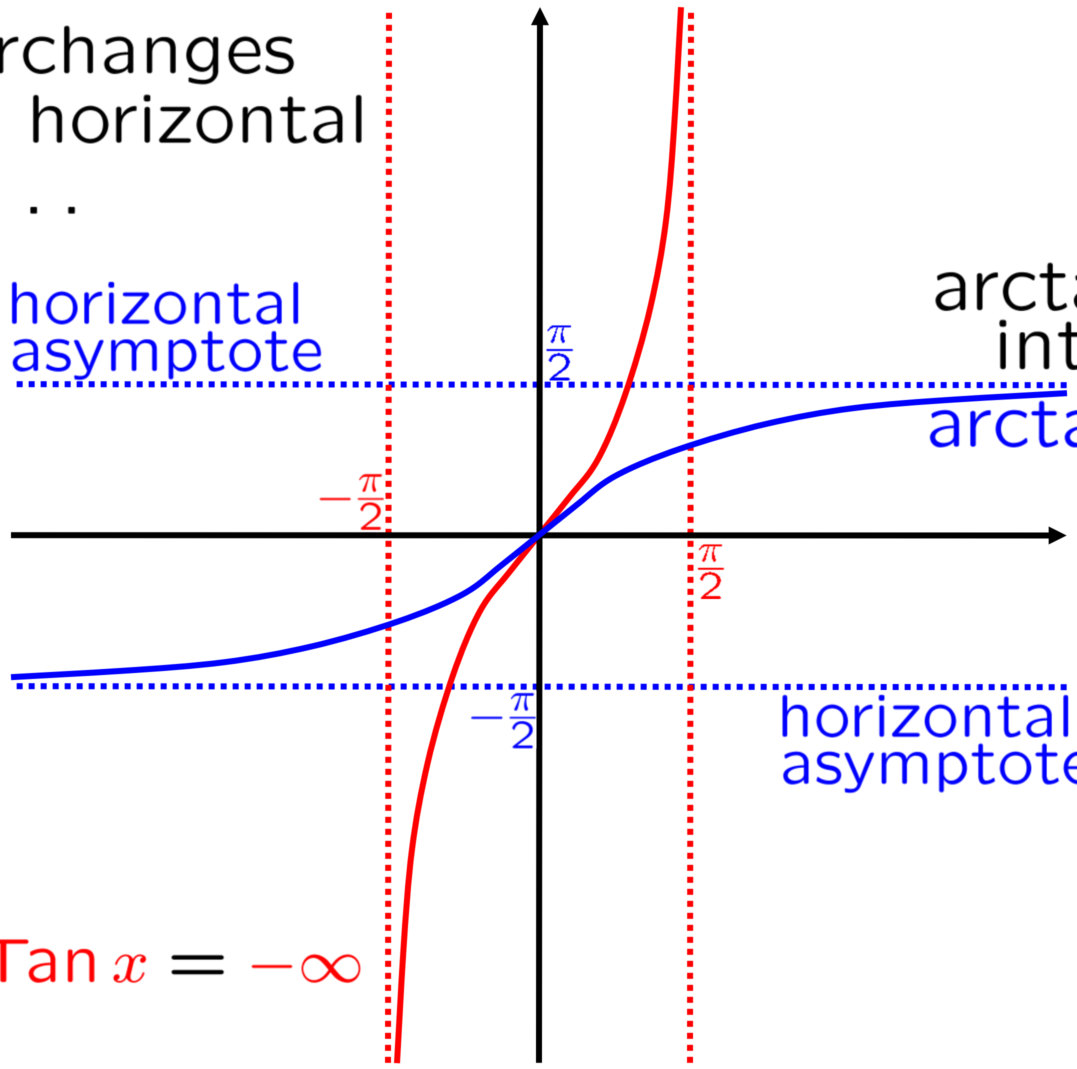
Next: asymptotes

Vertical asymptotes are related to limits that equal $\pm\infty$...



$$\lim_{x\to\frac{\pi}{2}^-} \text{Tan}\, x = \infty$$

Inverse interchanges vertical and horizontal asymptotes. . .



arctan settles into a limit.

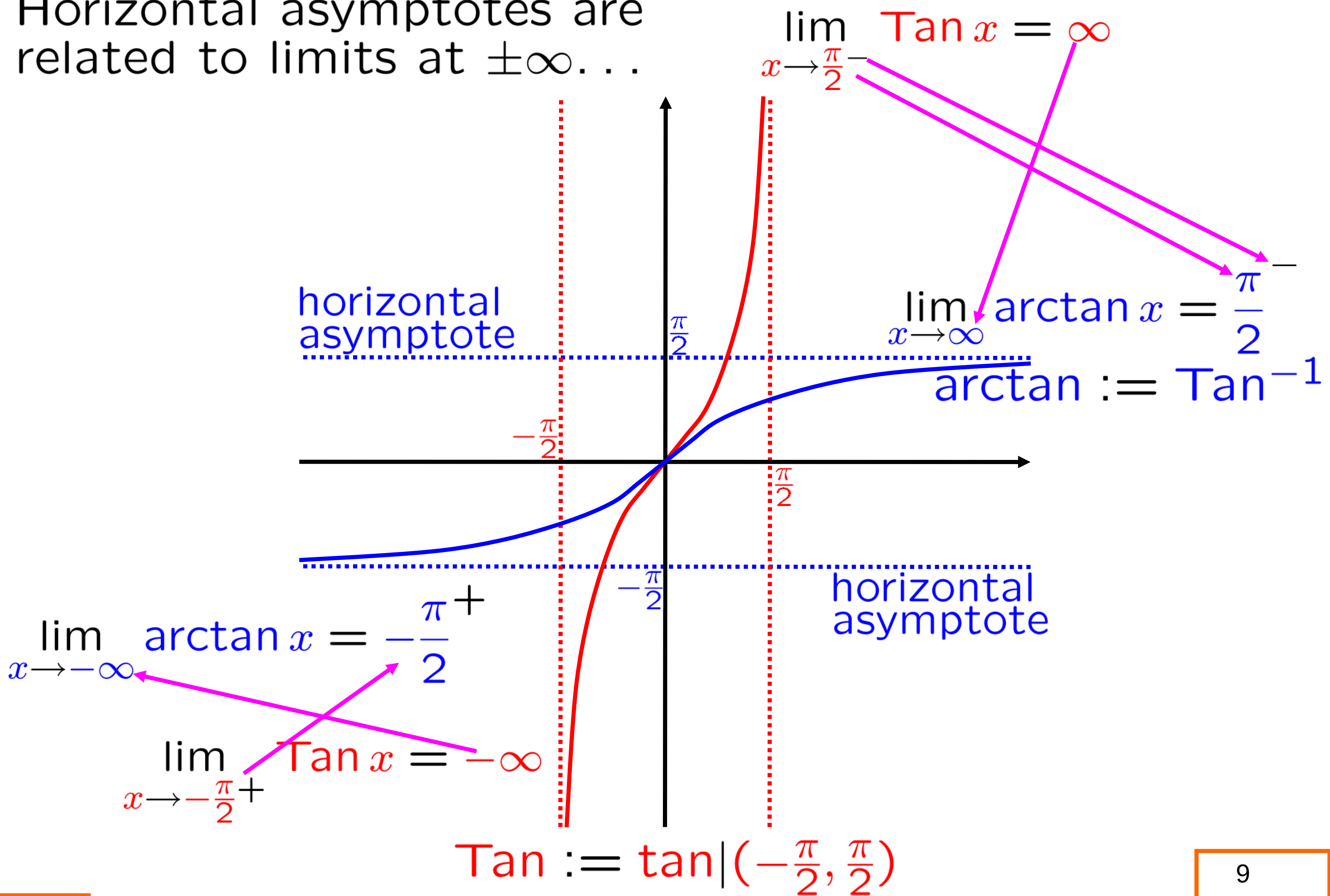
$$\arctan := \text{Tan}^{-1}$$

$$\lim_{x\to-\frac{\pi}{2}^+} \text{Tan}\, x = -\infty$$

$$\text{Tan} := \tan\Big|\left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

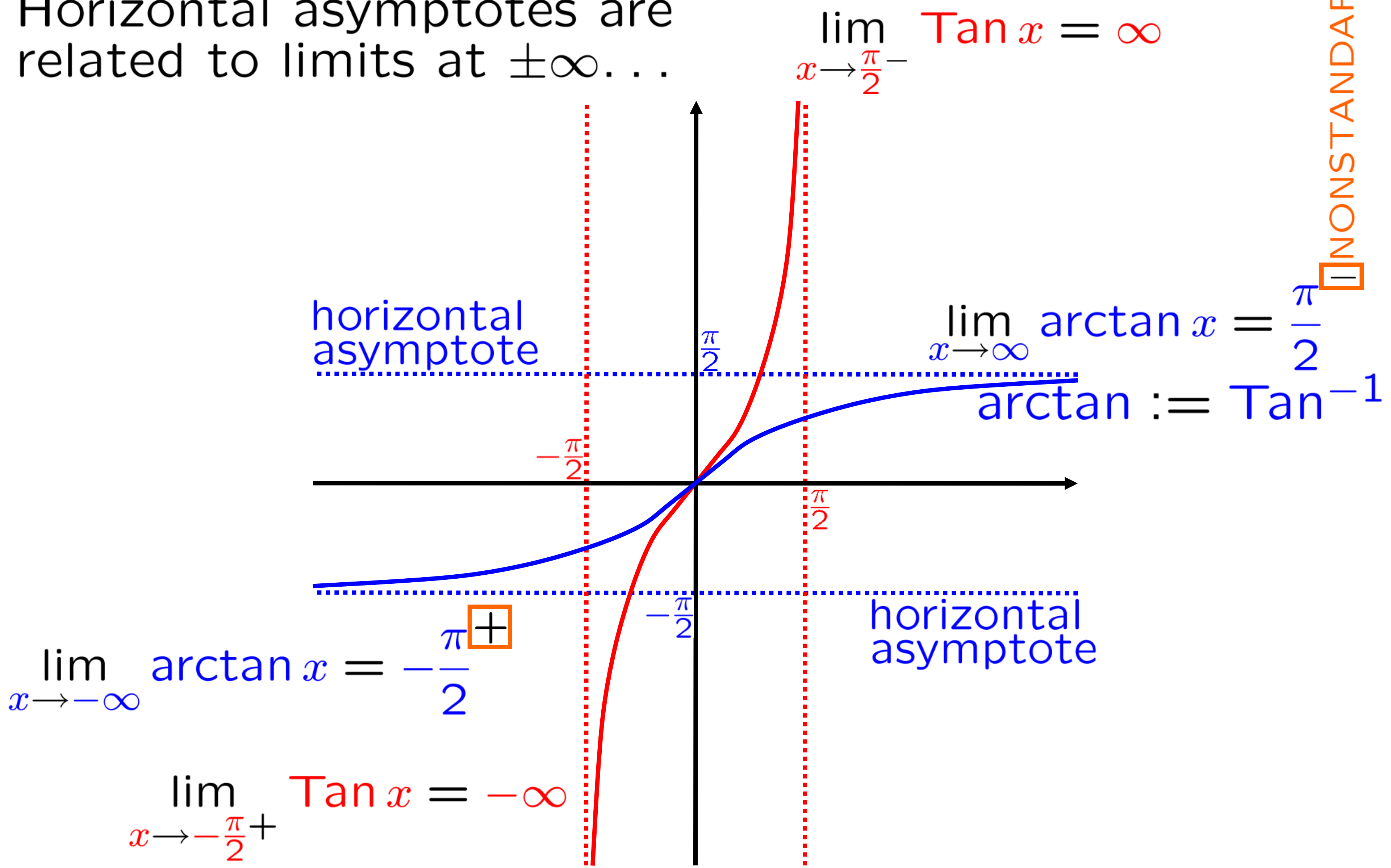
Spp

Horizontal asymptotes are related to limits at $\pm\infty$...



Definition of horizontal asymptote?

Horizontal asymptotes are related to limits at $\pm\infty$...



$\mathrm{Tan} := \tan|(-\frac{\pi}{2}, \frac{\pi}{2})$

NONSTANDARD

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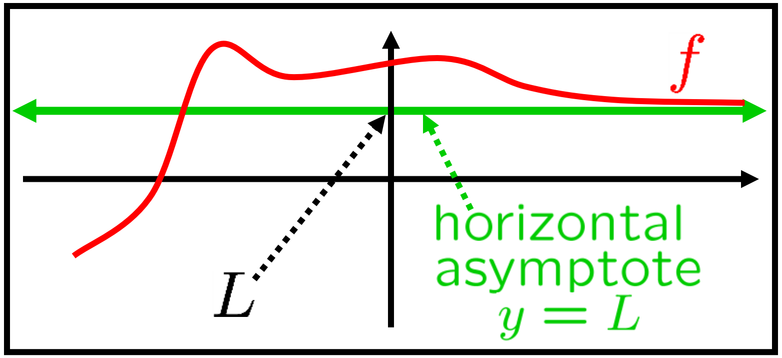
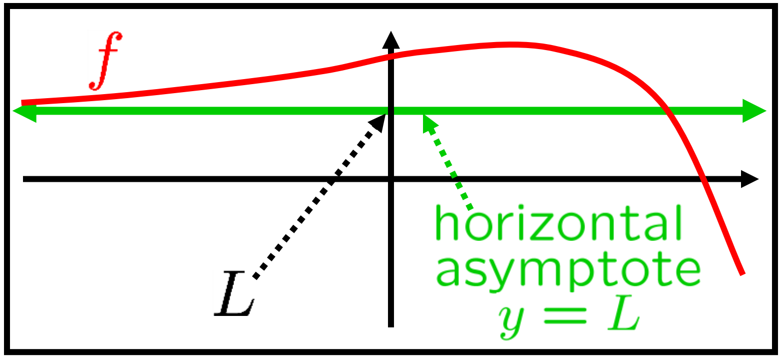
DEFINITION: The line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$ if

either $\lim_{x \to \infty} f(x) = L$



or $\lim_{x \to -\infty} f(x) = L$



Key point: Finding horizontal asymptotes is the same as computing limits at ∞ and at $-\infty$...

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Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = 1.$

$-\infty \leq a \leq \infty$

The Joy of Asymptotics. . .

PROBLEM: $\lim_{x\to a} f(x) = L$

$$f(x) = \frac{[f_1(x)]\left[\sqrt{f_2(x)}\right]\left[(f_3(x))^{2/7}\right]}{[f_4(x)]\left[\sqrt[4]{f_5(x)}\right]}$$

$$\underset{x \to a}{\sim} \frac{[g_1(x)]\left[\sqrt{g_2(x)}\right]\left[(g_3(x))^{2/7}\right]}{[g_4(x)]\left[\sqrt[4]{g_5(x)}\right]}$$

$$f_1(x) \underset{x \to a}{\sim} g_1(x)$$

$$\sqrt{f_2(x)} \underset{x \to a}{\sim} \sqrt{g_2(x)}$$

$$(f_3(x))^{2/7} \underset{x \to a}{\sim} (g_3(x))^{2/7}$$

$$f_4(x) \underset{x \to a}{\sim} g_4(x)$$

$$\sqrt[4]{f_5(x)} \underset{x \to a}{\sim} \sqrt[4]{g_5(x)}$$

ASYMPTOTICS ALERT!

$$\underset{x \to a}{\to} L$$

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Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\simeq} g(x)$ means $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = 1.$

$-\infty \leq a \leq \infty$

The Joy of Asymptotics. . .

PROBLEM: $\lim_{x\to a} f(x)$

$$f(x) = (f_1(x)) + (f_2(x))$$

ASYMPTOTICS ALERT!

(nowhere to go)

$$f_1(x) \underset{x \to a}{\simeq} g_1(x)$$

$$f_2(x) \underset{x \to a}{\simeq} g_2(x)$$

Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\simeq} g(x)$ means $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = 1.$

$-\infty \le a \le \infty$

The Joy of Asymptotics. . .

PROBLEM: $\lim_{x\to a} f(x) = L$

$$f(x) = \sqrt{f_1(x)} + \sqrt{f_2(x)}$$

$$\underset{x \to a}{\simeq} \sqrt{g_1(x)} + \sqrt{g_2(x)}$$

$$\underset{x \to a}{\to} L$$

$$\sqrt{f_1(x)} \underset{x \to a}{\simeq} \sqrt{g_1(x)}$$

$$\sqrt{f_2(x)} \underset{x \to a}{\simeq} \sqrt{g_2(x)}$$

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Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = 1.$

$-\infty \le a \le \infty$

A polynomial is asymptotic at 0
to its lowest order term.

$$\text{rational} = \frac{\text{polynomial}}{\text{polynomial}}$$

asymptotics of polynomials at $\pm\infty$?

First, let's review
asymptotics of polynomials at 0. . .

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Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = 1.$

How about asymptotics at $\pm\infty$?

A polynomial is asymptotic at 0 to its lowest order term.

e.g.: $3x^5 - 7x^4 + 2x^3 \boxed{- x^2} \underset{x \to 0}{\sim} \boxed{- x^2}$

Pf: $\frac{3x^5 - 7x^4 + 2x^3 - x^2}{-x^2} \underset{x \neq 0}{=} -3x^3 + 7x^2 - 2x + 1$

$\underset{x \to 0}{\to} -3 \cdot 0^3 + 7 \cdot 0^2 - 2 \cdot 0 + 1 = 1$

QED

Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\simeq} g(x)$ means $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = 1.$ $-\infty \leq a \leq \infty$

How about asymptotics at $\pm\infty$?

A polynomial is asymptotic at ∞ $-\infty$ to its highest order term.

leading term

e.g.: $3x^5 - 7x^4 + 2x^3 - x^2 \underset{x \to \infty}{\simeq} 3x^5$

Pf: $$\frac{3x^5 - 7x^4 + 2x^3 - x^2}{3x^5} = 1 - \frac{7}{3}\cdot\frac{1}{x} + \frac{2}{3}\cdot\frac{1}{x^2} - \frac{1}{3}\cdot\frac{1}{x^3}$$

$$\underset{x\to\infty}{\to} 1 - \frac{7}{3}\cdot 0 + \frac{2}{3}\cdot 0 - \frac{1}{3}\cdot 0 = 1$$

QED

same result and same proof for $-\infty$...

Spp

Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = 1.$

$-\infty \le a \le \infty$

A polynomial is asymptotic at $\pm\infty$ $-\infty$ to its highest order term.

leading term

e.g.: $3x^5 - 7x^4 + 2x^3 - x^2 \underset{x \to -\infty}{\sim} 3x^5$

Pf: $\frac{3x^5 - 7x^4 + 2x^3 - x^2}{3x^5} = 1 - \frac{7}{3} \cdot \frac{1}{x} + \frac{2}{3} \cdot \frac{1}{x^2} - \frac{1}{3} \cdot \frac{1}{x^3}$

$\underset{x \to -\infty}{\to} 1 - \frac{7}{3} \cdot 0 + \frac{2}{3} \cdot 0 - \frac{1}{3} \cdot 0 = 1$

QED

Spp

Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = 1.$

$-\infty \leq a \leq \infty$

A polynomial is asymptotic at $\pm\infty$ to its highest order term.

leading term

e.g.: $$\lim_{x \to -\infty} \frac{8x^7 + 2x^2 - x + 3}{2x^7 + x^5 + 5x^4 - 8x^3 + 2x + 9}$$

$$= \lim_{x \to -\infty} \frac{8x^7}{2x^7} = \lim_{x \to \infty} \frac{8}{2} = \frac{8}{2} = 4 \blacksquare$$

same degree $\Rightarrow$ limit at $\pm\infty$ is quotient of leading coefficients

e.g.: $$\lim_{x \to \infty} \frac{3x^9 - 8x^7 - 5x^2 + 8x + 3}{6x^9 - 7x^5 - x^4 + 7x^3 + 8x - 9} = \frac{3}{6} = \frac{1}{2} \blacksquare$$

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Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = 1.$

$-\infty \le a \le \infty$

A polynomial is asymptotic at $\pm\infty$
to its highest order term.

leading term

e.g.: $\lim_{x \to \infty, -\infty} \frac{1000000x^7 + 5000x^2 + 3000000000}{x^8 - 9999999x^3 - 50000000000}$

$= \lim_{x \to \infty, -\infty} \frac{1000000x^7}{x^8} = \lim_{x \to \infty, -\infty} 1000000 \cdot \frac{1}{x} = 1000000 \cdot 0 = 0$

Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = 1.$

$-\infty \leq a \leq \infty$

A polynomial is asymptotic at $\pm\infty$ to its highest order term.

leading term

How about larger degree in numerator?

e.g.: $$\lim_{x \to -\infty} \frac{1000000x^7 + 5000x^2 + 3000000000}{x^8 - 9999999x^3 - 50000000000}$$

$$= \lim_{x \to -\infty} \frac{1000000x^7}{x^8} = \lim_{x \to -\infty} 1000000 \cdot \frac{1}{x} = 1000000 \cdot 0 = 0$$

larger degree in denominator $\Rightarrow$ limit at $\pm\infty$ is 0

e.g.: $$\lim_{x \to -\infty} \frac{3x^5 - 2x^3 + 4x^2 - x + 9}{5x^9 - 7x^8 - x^6 - 2x^3 - 7x - 9} = 0$$

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Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = 1.$

$-\infty \le a \le \infty$

A polynomial is asymptotic at $\pm\infty$ to its highest order term.

leading term

How about larger degree in numerator?

e.g.: $\lim_{x\to\infty}\frac{-2x^8+3x^3-x^2+8}{-999999x^5+x} = \lim_{x\to\infty}\frac{-2x^8}{-999999x^5}$

$= \lim_{x\to\infty}\frac{-2}{-999999}\cdot x^3$

$= +\infty$ ■

How about the limit at $-\infty$?

Spp

larger degree in numerator $\Rightarrow$ limit at ∞ is $\pm\infty$

Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = 1.$

$-\infty \leq a \leq \infty$

A polynomial is asymptotic at $\pm\infty$ to its highest order term.

leading term

e.g.: $\lim_{x\to-\infty} \frac{-2x^8 + 3x^3 - x^2 + 8}{999999x^5 + x} = \lim_{x\to-\infty} \frac{-2x^8}{999999x^5}$

$= \lim_{x\to-\infty} \frac{-2}{999999} \cdot x^3$

$= +\infty$ ■

How about if (degree of numerator)−(degree of denominator) is even?

Spp

larger degree in numerator $\Rightarrow$ limit at ∞ is $\pm\infty$

Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = 1.$

$-\infty \leq a \leq \infty$

A polynomial is asymptotic at $\pm\infty$
to its highest order term.

leading term

e.g.: $\lim_{x \to -\infty} \frac{-2x^9 + 3x^3 - x^2 + 8}{999999x^5 + x} = \lim_{x \to -\infty} \frac{-2x^9}{999999x^5}$

$= \lim_{x \to -\infty} \frac{-2}{999999} \cdot x^4$

$= -\infty$ ■

How about if
(degree of numerator)$-$(degree of denominator)
is even?

Spp

larger degree in numerator $\Rightarrow$ limit at ∞ is $\pm\infty$

Limits at $\pm\infty$ of rational functions

Def'n: $f(x) \underset{x \to a}{\sim} g(x)$ means $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = 1.$

$-\infty \leq a \leq \infty$

A polynomial is asymptotic at $\pm\infty$
to its highest order term.

leading term

e.g.: $$\lim_{x \to -\infty} \frac{-2x^9 + 3x^3 - x^2 + 8}{999999x^5 + x} = \lim_{x \to -\infty} \frac{-2x^9}{999999x^5}$$

$$= \lim_{x \to -\infty} \frac{-2}{999999} \cdot x^4$$

$$= -\infty \quad \blacksquare$$

SKILL
lim rat'l

larger degree in numerator $\Rightarrow$ limit at $-\infty$ is $\pm\infty$
larger degree in numerator $\Rightarrow$ limit at ∞ is $\pm\infty$

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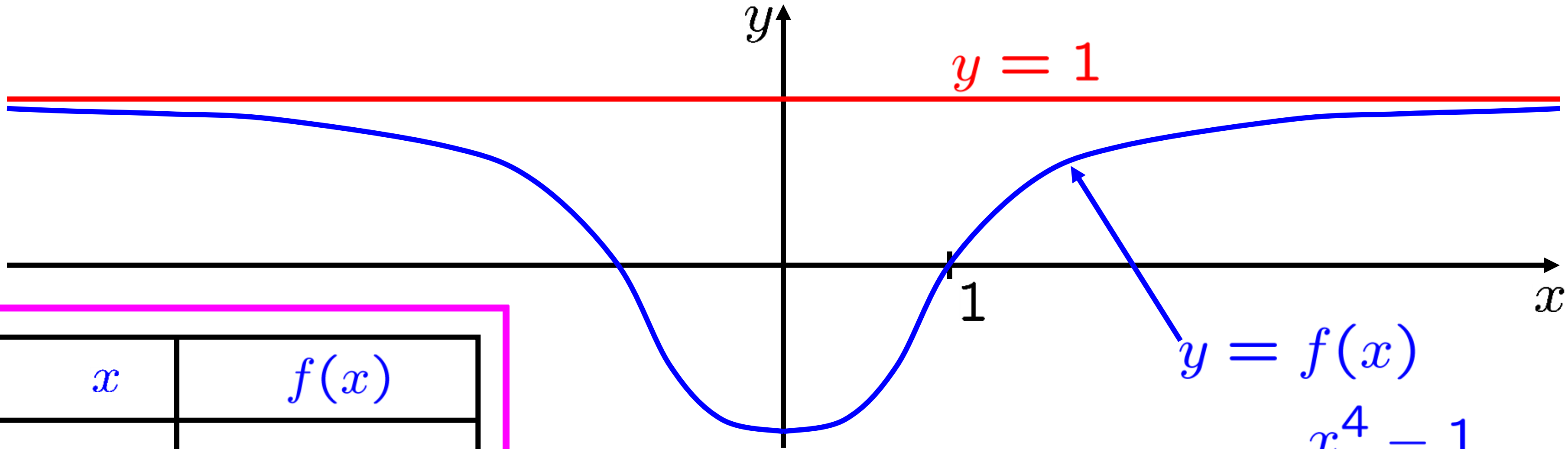
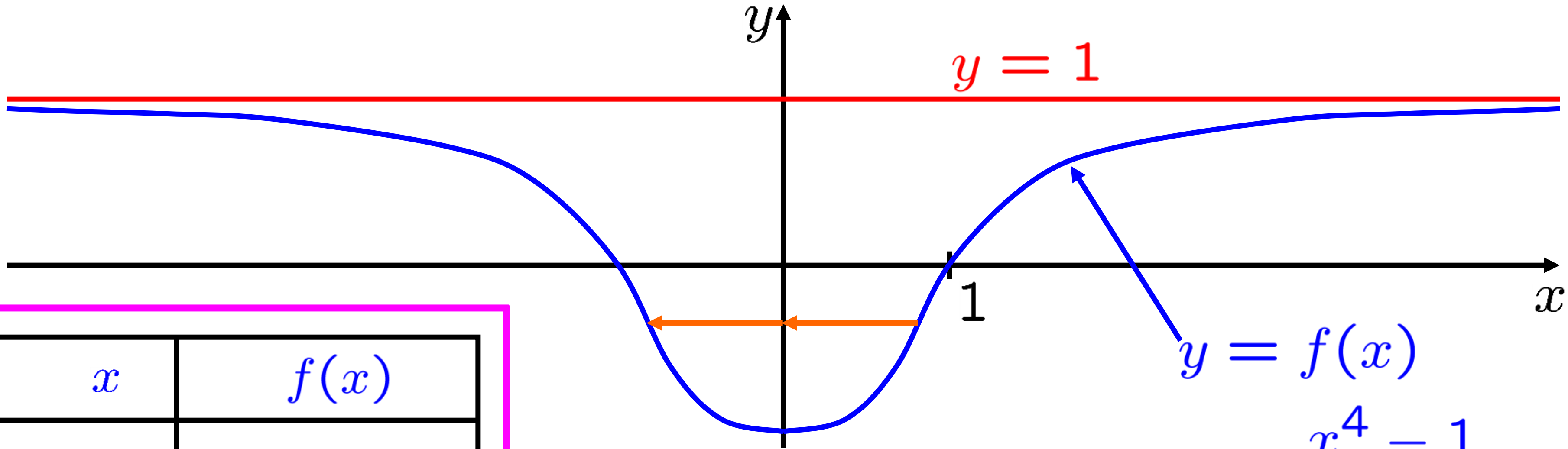


x	$f(x)$
0	-1
1	0
2	0.882353
3	0.975610
4	0.992218
5	0.996805
10	0.999800
20	0.999988
30	0.999998
40	0.999999

$$f(x) = \frac{x^4 - 1}{x^4 + 1}$$

$$\lim_{x \to \infty} \frac{x^4 - 1}{x^4 + 1} = 1$$

Spp



$$= \frac{x^4 - 1}{x^4 + 1}$$

x	$f(x)$
0	-1
1	0
2	0.882353
3	0.975610
4	0.992218
5	0.996805
10	0.999800
20	0.999988
30	0.999998
40	0.999999

$$f(-x) = f(x)$$

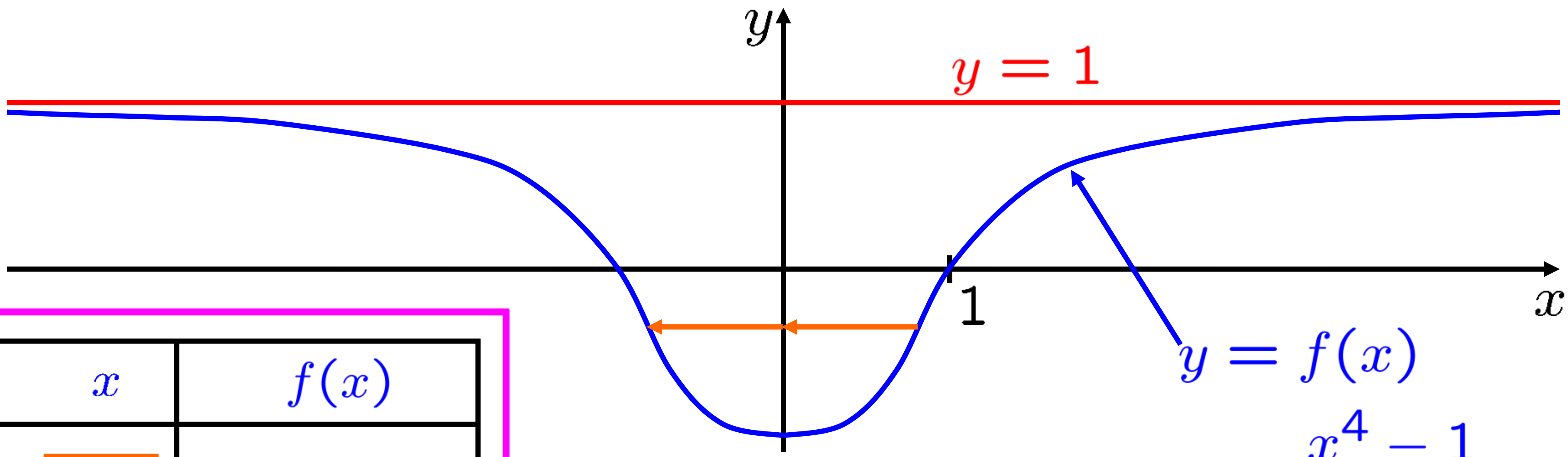
$$(-x)^2 = x^2,\ (-x)^4 = x^4,\ (-x)^6 = x^6,\ \ldots$$

"even"

$$f(x) = \frac{x^4 - 1}{x^4 + 1}$$

$$\lim_{x \to -\infty} \frac{x^4 - 1}{x^4 + 1} = 1$$

Spp



$$= \frac{x^4 - 1}{x^4 + 1}$$

x	$f(x)$
0	−1
±1	0
±2	0.882353
±3	0.975610
±4	0.992218
±5	0.996805
±10	0.999800
±20	0.999988
±30	0.999998
±40	0.999999

$$f(x) = \frac{x^4 - 1}{x^4 + 1}$$

$$\lim_{x \to -\infty} \frac{x^4 - 1}{x^4 + 1} = 1$$

Spp

Exercise: Find the vert. and horiz. asymptotes of

$$f(x) = \frac{\sqrt{4x^2+1}}{\boxed{6x-3}}.$$

zero at $x = 1/2$

SKILL
limit rat'l sqrt

Vertical: Check $\lim_{x \to 1/2^{\pm}}$.

Horizontal: Check $\lim_{x \to \pm\infty}$.

$$\lim_{x \uparrow 1/2} f(x)$$

Exercise: Find the vert. and horiz. asymptotes of

$$f(x) = \frac{\sqrt{4x^2+1}}{6x-3}.$$

positive

small negative

Vertical: Check $\lim_{x \to 1/2^{\pm}}$.

SKILL
limit rat'l sqrt

Horizontal: Check $\lim_{x \to \pm\infty}$.

$\lim_{x \uparrow 1/2} f(x) = -\infty$ $\lim_{x \downarrow 1/2} f(x)$

$x = 1/2$ is the only vertical asymptote.

For completeness, we check $\lim_{x \downarrow 1/2} \ldots$

Spp

Exercise: Find the vert. and horiz. asymptotes of

$$f(x) = \frac{\sqrt{4x^2+1}}{6x-3}.$$

positive

small positive

Vertical: Check $\lim_{x \to 1/2^{\pm}}$.

SKILL
limit rat'l sqrt

Horizontal: Check $\lim_{x \to \pm\infty}$.

$\lim_{x \uparrow 1/2} f(x) = -\infty$ $\lim_{x \downarrow 1/2} f(x) = \infty$

$x = 1/2$ is the only vertical asymptote.

For completeness, we check $\lim_{x \downarrow 1/2}$ . . .

Spp

Exercise: Find the vert. and horiz. asymptotes of

$$f(x) = \frac{\sqrt{4x^2+1}}{6x-3}.$$

$$\frac{\sqrt{4x^2+1}}{6x-3} \underset{x\to\pm\infty}{\sim} \frac{\sqrt{4x^2}}{6x} = \frac{2|x|}{6x} = \frac{|x|}{3x}$$

$$\frac{|x|}{3x} \underset{x>0}{=} \frac{x}{3x} \underset{x\neq 0}{=} \frac{1}{3} \underset{x\to\infty}{\longrightarrow} \frac{1}{3}$$

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{6x-3} = \frac{1}{3}$$

$y = 1/3$ is the horiztonal asymptote at ∞. $-\infty$?

Spp

Exercise: Find the vert. and horiz. asymptotes of

$$f(x) = \frac{\sqrt{4x^2+1}}{6x-3}.$$

$$\frac{\sqrt{4x^2+1}}{6x-3} \underset{x\to\pm\infty}{\sim} \frac{\sqrt{4x^2}}{6x} = \frac{2|x|}{6x} = \frac{|x|}{3x}$$

$$\frac{|x|}{3x} \underset{x<0}{=} \frac{-x}{3x} \underset{x\neq 0}{=} -\frac{1}{3} \underset{x\to-\infty}{\longrightarrow} -\frac{1}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{6x-3} = -\frac{1}{3}$$

$y = -1/3$ is the horiztonal $-\infty$?
asymptote at $-\infty$. ■

Spp

Exercise: Compute $\lim_{x\to\infty}\left(\sqrt{9x^2+6}-3x\right)$.

SKILL
limit rat'l sqrt

$\sqrt{9x^2+6}$: $x \to \infty$ gives ∞; $3x$: $x \to \infty$ gives ∞

$\infty - \infty$ is indeterminate.

$$\sqrt{9x^2+6}-3x = \frac{\sqrt{9x^2+6}-3x}{1}\,\frac{\sqrt{9x^2+6}+3x}{\sqrt{9x^2+6}+3x}$$

STOP

$$= \frac{(9x^2+6)-9x^2}{\sqrt{9x^2+6}+3x}$$

$x \to \infty$: $\to 0$ ■

Denominator: $x \to \infty$ gives ∞

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