CALCULUS
Derivative of the natural logarithm

Goal: $\frac{d}{dx}[\ln x]$

$$e^{\ln x} \overset{x>0}{=} x$$

$$\left[e^{\ln x}\right]\left[\frac{d}{dx}[\ln x]\right] \overset{\text{CHAIN RULE}}{=} \frac{d}{dx}\left[e^{\ln x}\right] \overset{x>0}{=} \frac{d}{dx}[x] = 1$$

DIVDE by $e^{\ln x}$

$$\frac{d}{dx}[\ln x] \overset{x>0}{=} \frac{1}{e^{\ln x}} \overset{x>0}{=} \frac{1}{x}$$

$$e^{\ln x} \overset{x>0}{=} x$$

Goal: $\frac{d}{dx}[\ln x]$

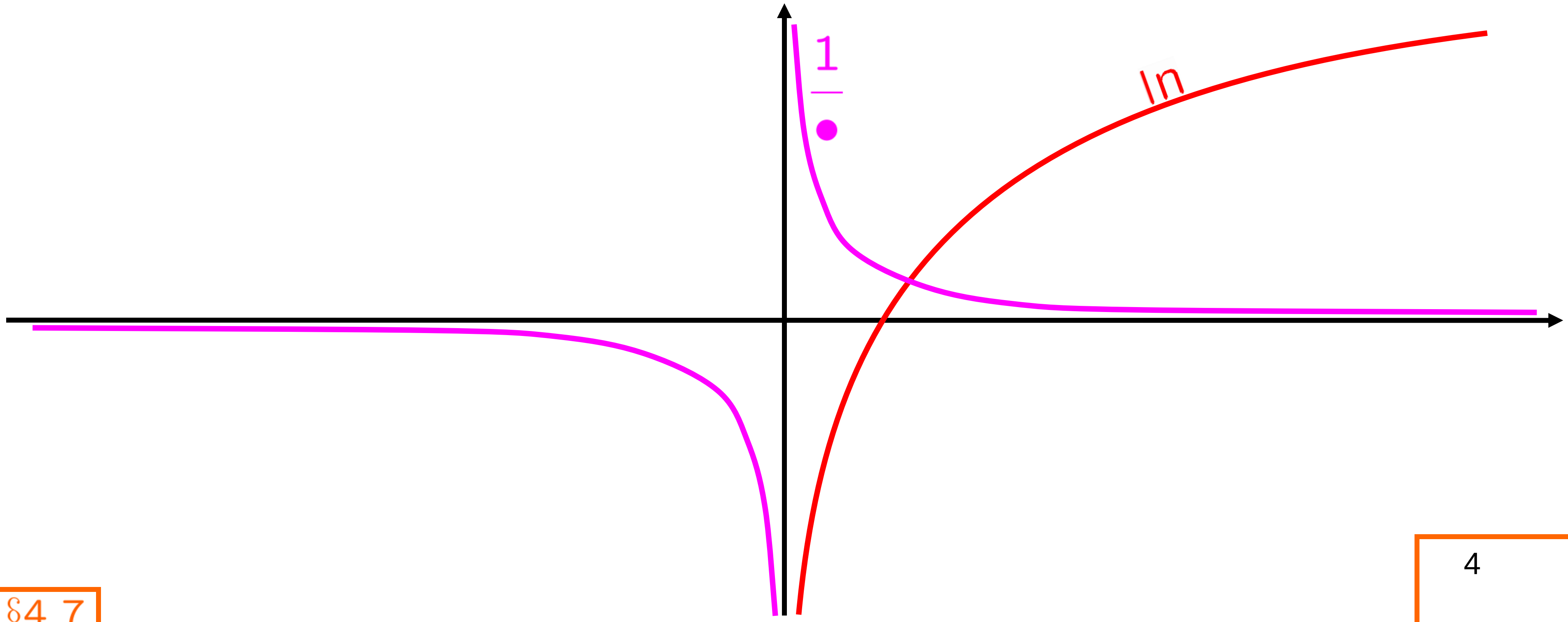
Solution: $\frac{d}{dx}[\ln x] \overset{x>0}{=} \frac{1}{x}$

$$\frac{d}{dx}[\ln x] \qquad \overset{x>0}{=} \frac{1}{x}$$

Goal: $\dfrac{d}{dx}[\ln x]$

Solution: $\dfrac{d}{dx}[\ln x] \overset{x>0}{=} \dfrac{1}{x}$

$\ln' \overset{?}{=} \dfrac{1}{\bullet}$

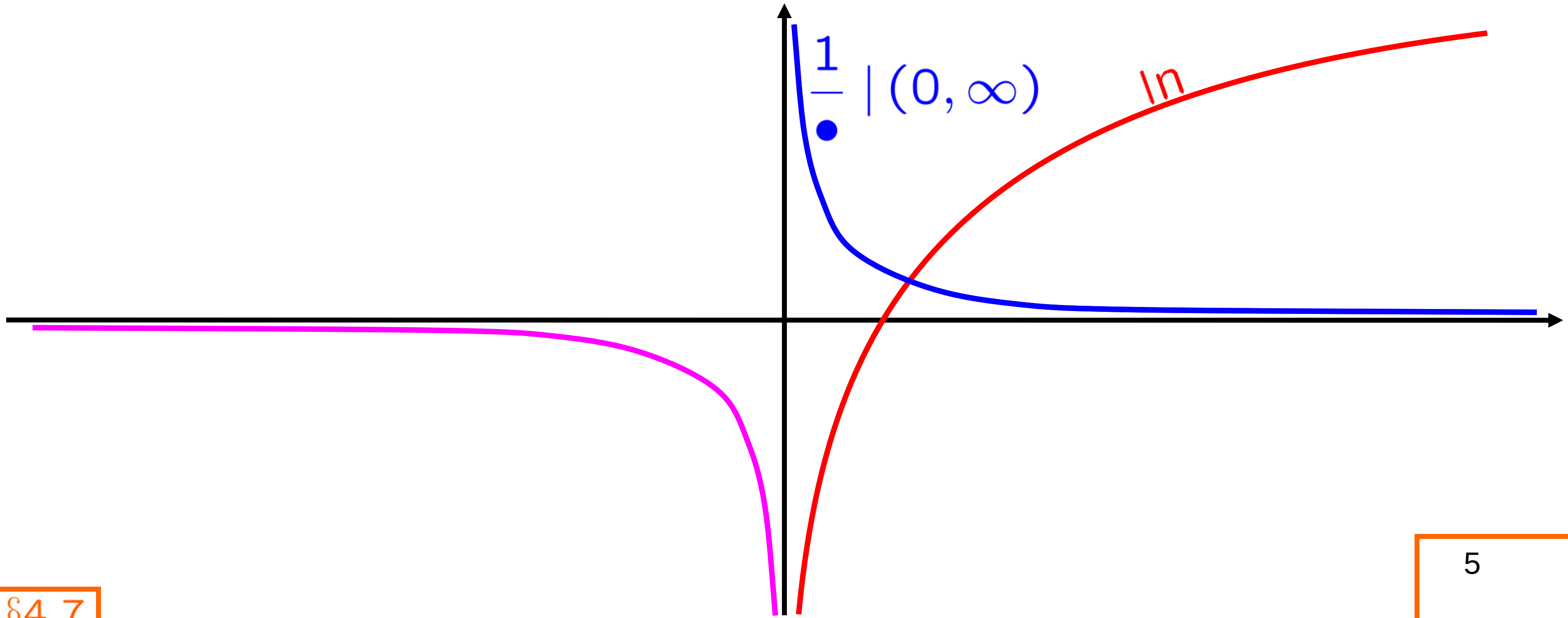


§4.7

Goal: $\frac{d}{dx}[\ln x]$

Solution: $\frac{d}{dx}[\ln x] \overset{x>0}{=} \frac{1}{x}$

$\ln' \overset{?}{=} \frac{1}{\bullet} \mid (0, \infty)$



Goal: $\frac{d}{dx}[\ln x]$

Solution: $\frac{d}{dx}[\ln x] \overset{x>0}{=} \frac{1}{x}$

$\ln' = \frac{1}{\bullet} \mid (0,\infty)$

UNNEEDED

$\frac{d}{dx}[\ln(-x)] \overset{x<0}{=} \overset{\text{RESTR}}{\left[\frac{1}{+x}\right]}[+1] = \frac{1}{x}$

$\ln|x| = \begin{cases} \ln x, & \text{if } x>0 \\ \ln(-x), & \text{if } x<0 \end{cases}$

$\frac{d}{dx}[\text{la}\, x] = \frac{d}{dx}[\ln|x|] = \frac{1}{x}$

$|x| = \begin{cases} x, & \text{if } x>0 \\ -x, & \text{if } x<0 \end{cases}$

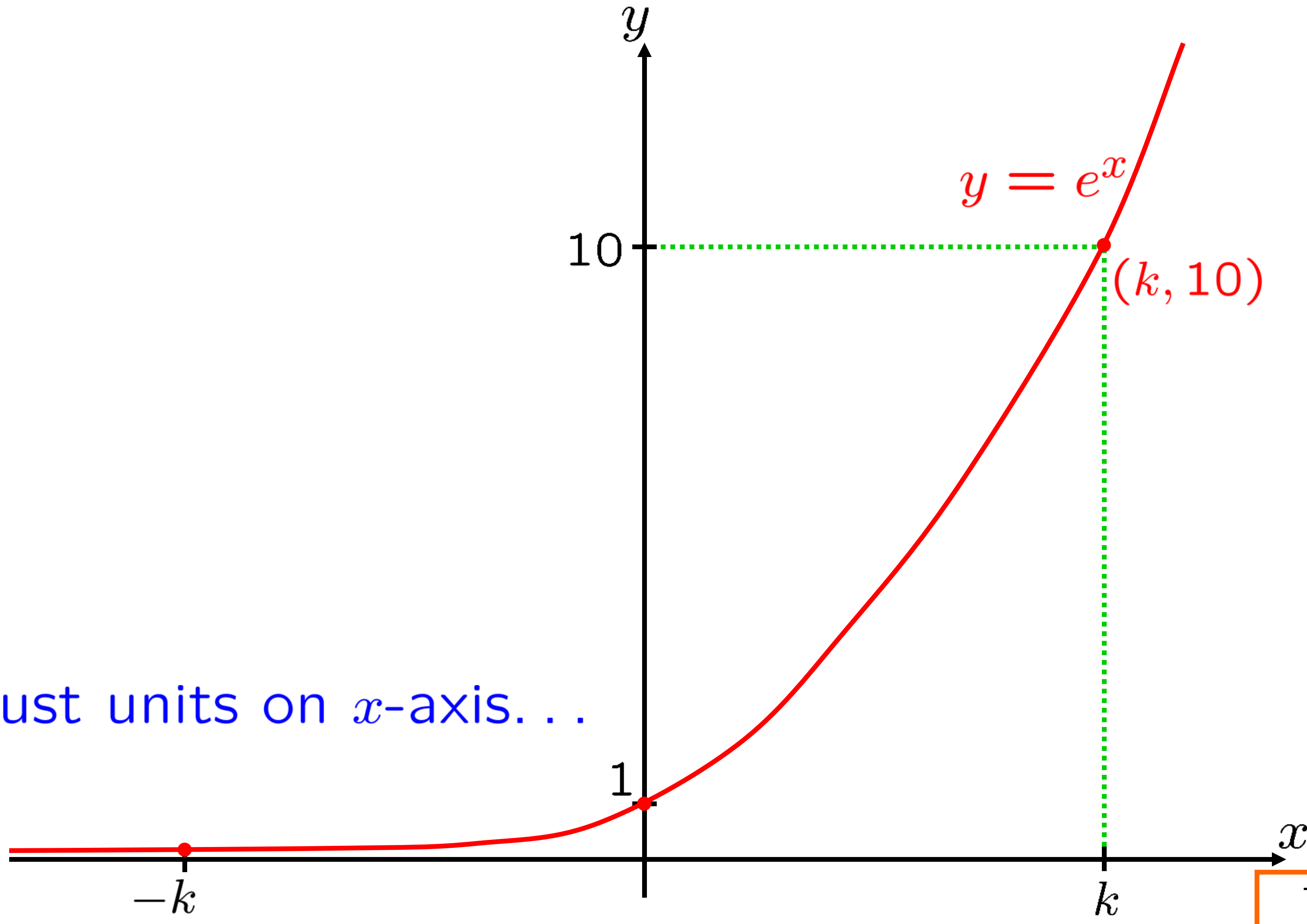
Definition: $\text{la}\, x := \ln|x|$

Let's see this in graphs. . .

$e^{\bullet}$ expr. of x: e^x

$k := 2.302585093$

$e^k = 10$



adjust units on x-axis...

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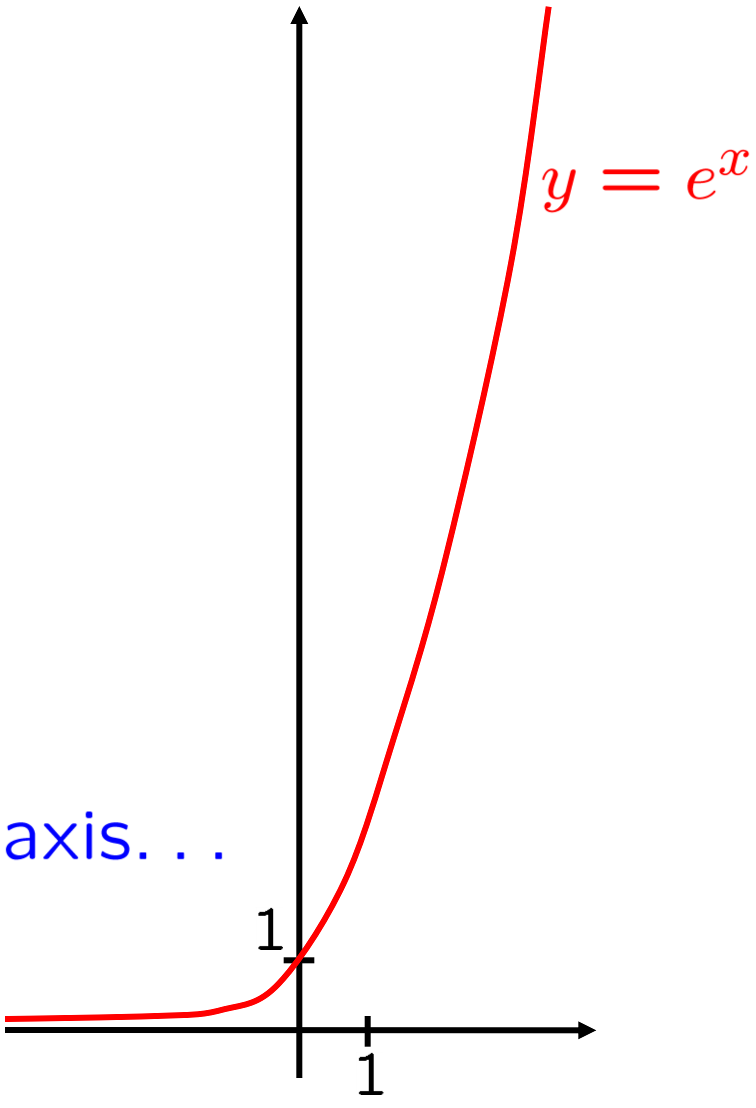
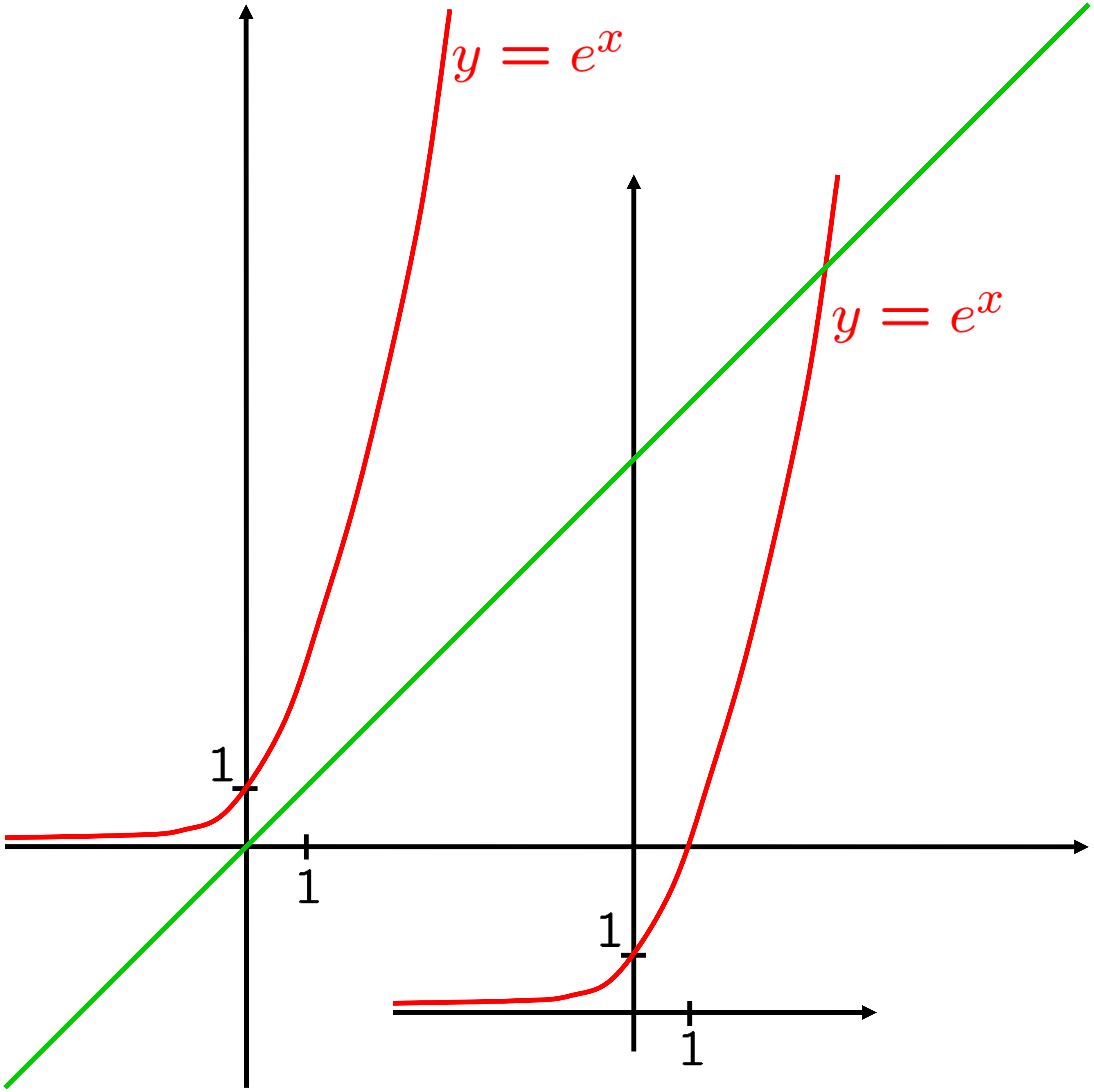
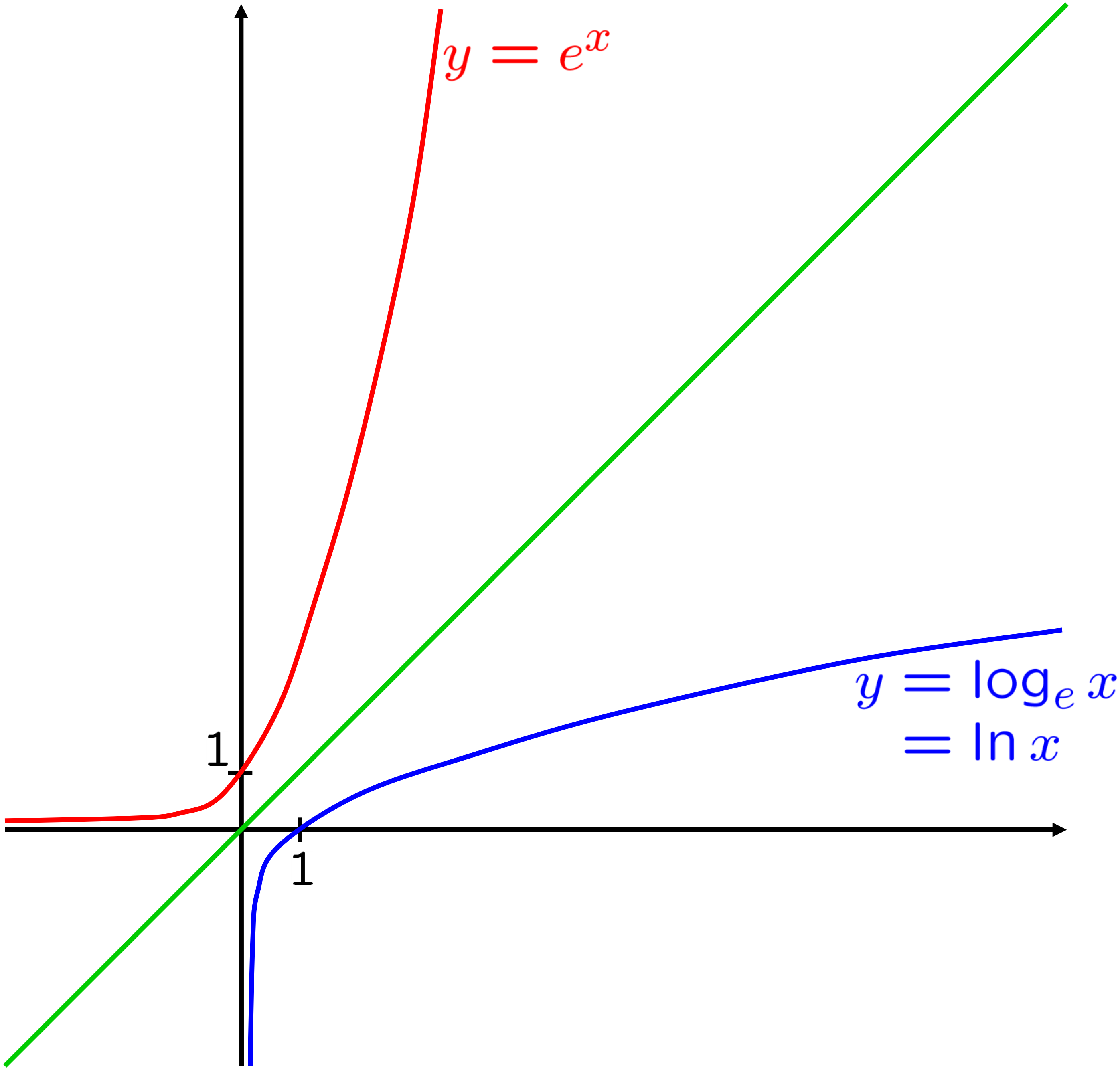
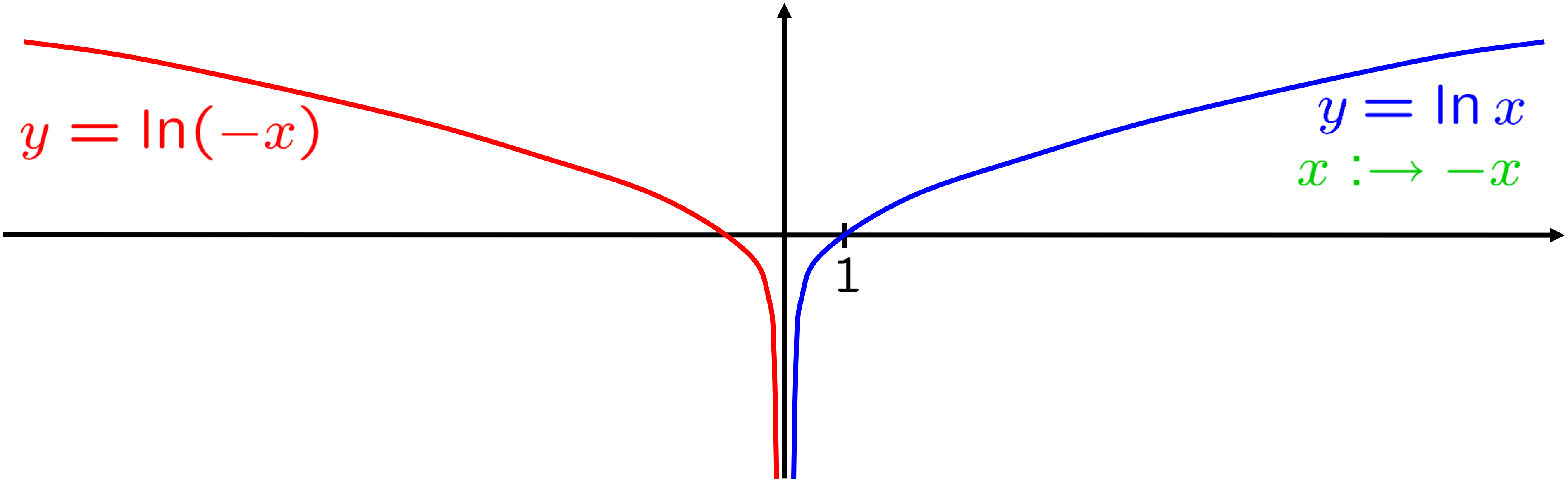
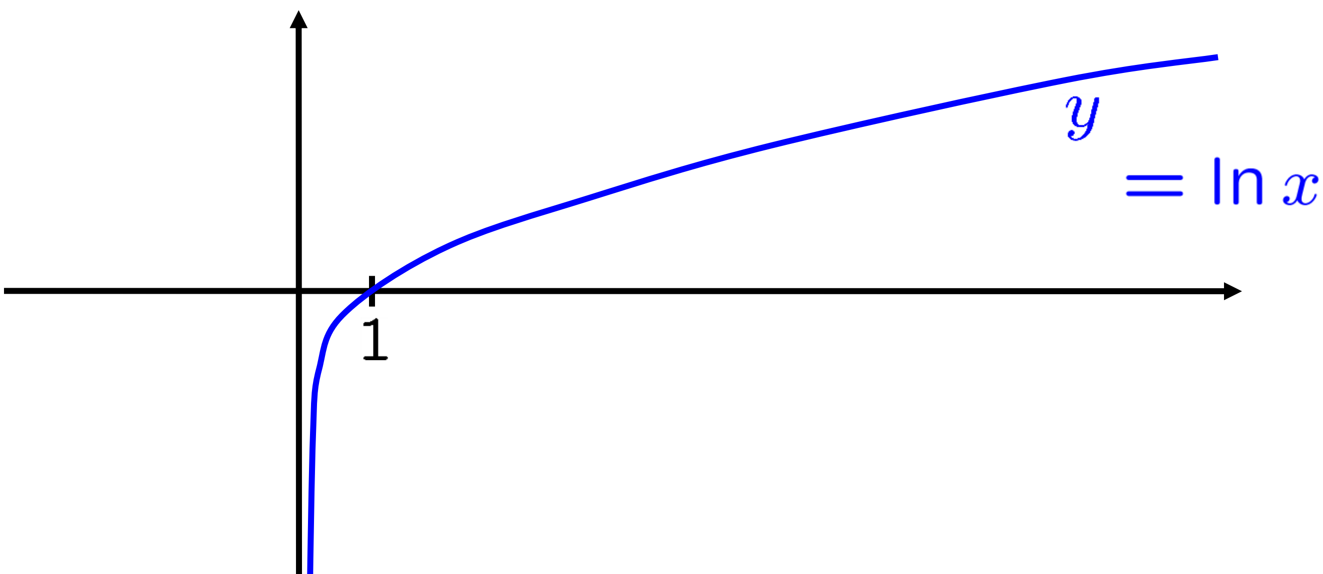


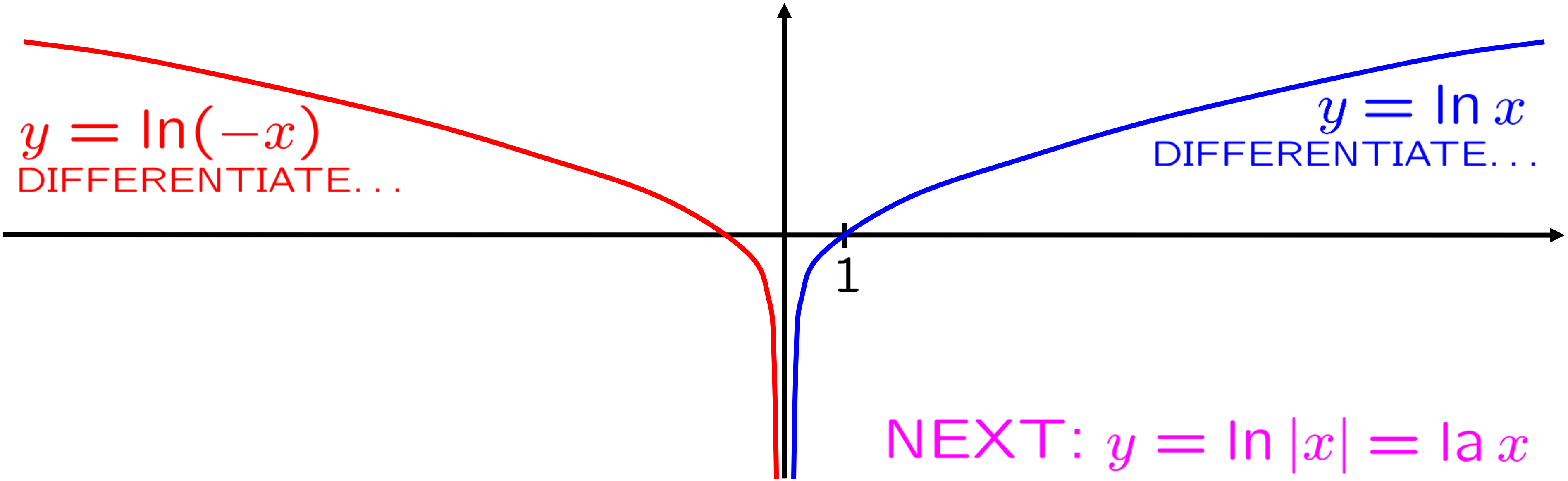


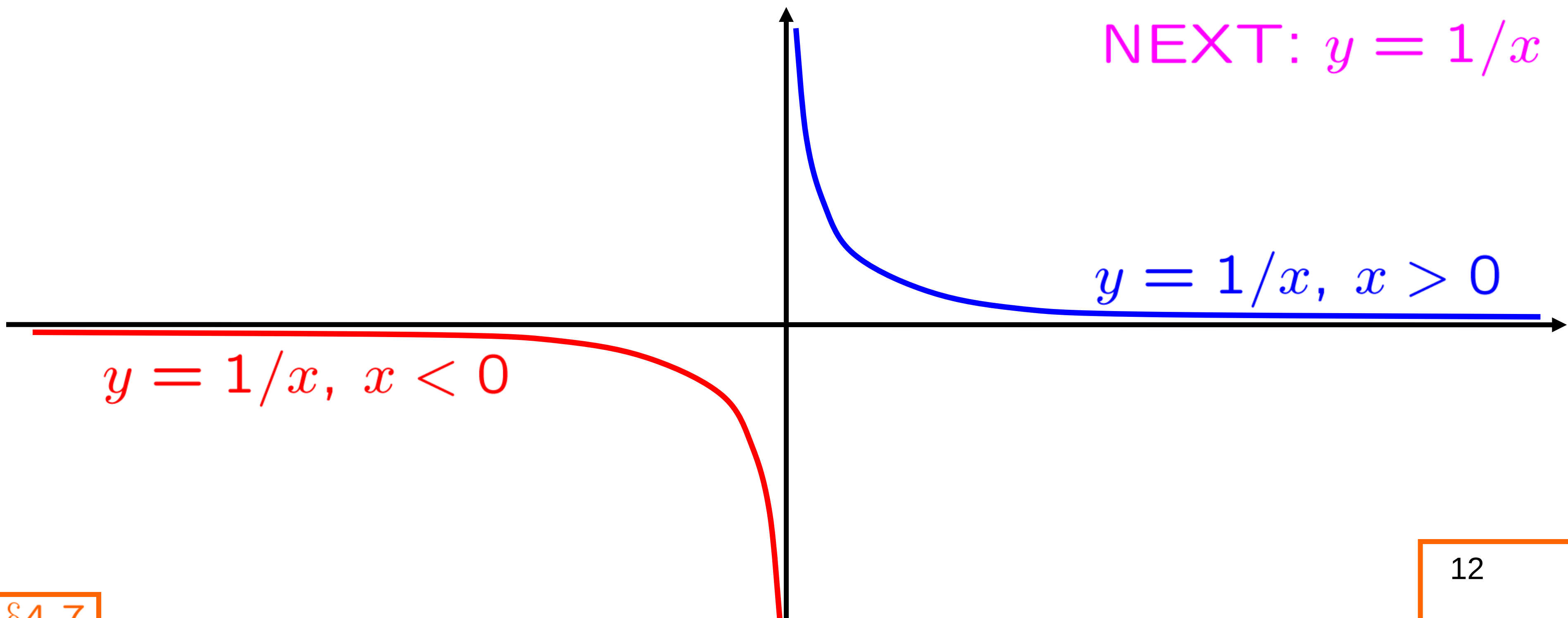
$y = e^x$
$y = e^x$
1
1
1
1

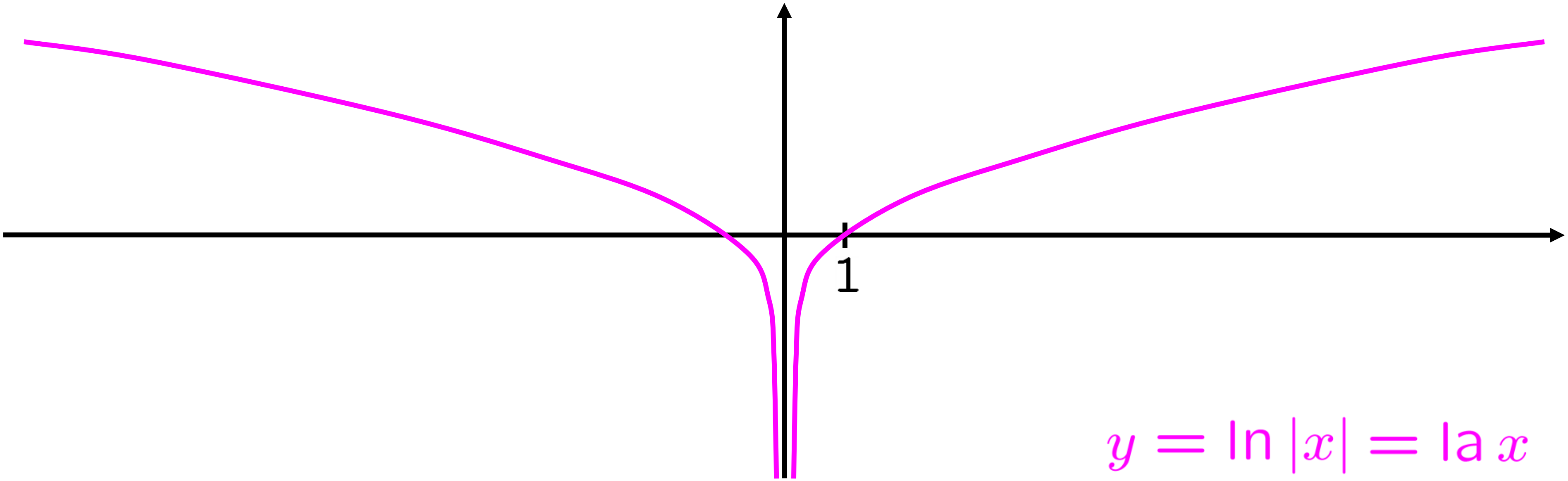

$y = e^x$
1
1
$y = \log_e x$
$= \ln x$

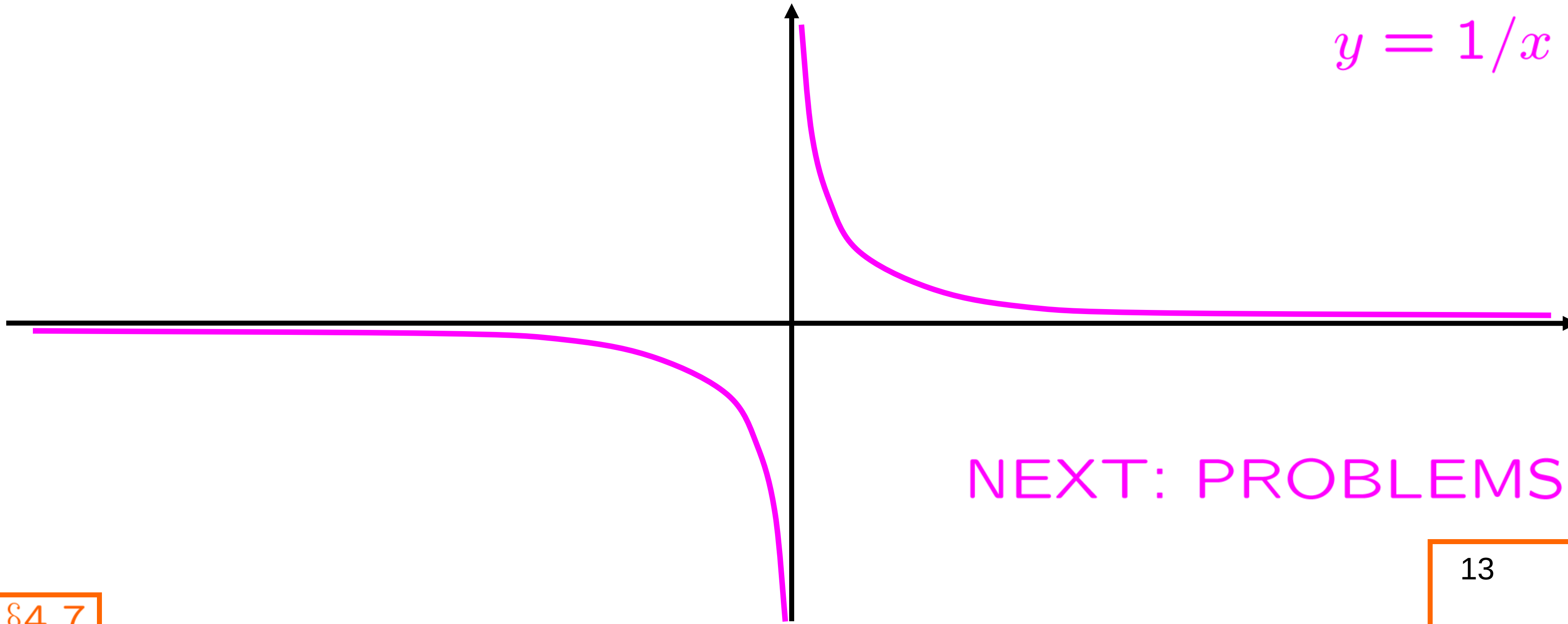

$y = \ln(-x)$
$y = \ln x$
$x :\to -x$
1


y
$= \ln x$
1


$y = \ln(-x)$
DIFFERENTIATE...
$y = \ln x$
DIFFERENTIATE...
1
NEXT: $y = \ln|x| = \text{la}\, x$


NEXT: $y = 1/x$
$y = 1/x,\ x > 0$
$y = 1/x,\ x < 0$





NEXT: PROBLEMS

EXAMPLE: Differentiate $y = \sqrt[3]{\ln x}$.

$$\forall x > 0, \quad \frac{dy}{dx} = \frac{1}{3}(\ln x)^{-2/3}\frac{1}{x}$$

$$= \frac{1}{3x\sqrt[3]{(\ln x)^2}} \qquad \blacksquare$$

WE CAN SIMPLIFY A LITTLE . . .

Example: $\dfrac{d}{dx}[\ln(x^7 + x^4 + 5x)]$

$x^7 + x^4 + 5x > 0$

UNNEEDED

RESTR

Sol'n: $\dfrac{d}{dx}[\ln(x^7 + x^4 + 5x)] = \left[\dfrac{1}{x^7 + x^4 + 5x}\right][7x^6 + 4x^3 + 5]$

$$= \frac{7x^6 + 4x^3 + 5}{x^7 + x^4 + 5x}$$

NOTE: $\dfrac{d}{dx}[\text{la}(x^7 + x^4 + 5x)] = \dfrac{7x^6 + 4x^3 + 5}{x^7 + x^4 + 5x}$

$f(x) > 0$

General Fact: $\dfrac{d}{dx}[\ln(f(x))] = \dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}[\text{la}(f(x))]$

EXAMPLE: Differentiate $y = \ln(x^4 + 2)$.

$$\frac{dy}{dx} = \frac{4x^3}{x^4 + 2} \quad \blacksquare$$

$x^4 + 2$ is always > 0.

$f(x) > 0$

General Fact: $\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)} = \frac{d}{dx}[\text{la}(f(x))]$

EXAMPLE: Differentiate the function.

$$f(x) = \ln(x^3 + 2x - 5)$$

$$f'(x) = \frac{3x^2 + 2}{x^3 + 2x - 5} \quad \blacksquare$$

$x^3 + 2x - 5 > 0$

NOTE: $\frac{d}{dx}[\text{la}(x^3 + 2x - 5)] = \frac{3x^2 + 2}{x^3 + 2x - 5}$

$f(x) > 0$

General Fact: $\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)} = \frac{d}{dx}[\text{la}(f(x))]$

EXAMPLE: Differentiate the function.

$$h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

Domain: $x \geq 1$
NOT differentiable at $x = 1$

Assuming $x > 1$.

$$h'(x) = \frac{1 + \frac{1}{2}(x^2 - 1)^{-1/2}(2x)}{x + (x^2 - 1)^{1/2}} \left[\frac{(x^2 - 1)^{1/2}}{(x^2 - 1)^{1/2}}\right]$$

$$= \frac{(x^2 - 1)^{1/2} + x \quad 1}{[x + (x^2 - 1)^{1/2}][(x^2 - 1)^{1/2}]}$$

$$h'(x) \overset{x \geq 1}{=} \frac{1}{\sqrt{x^2 - 1}}$$ ■

General Fact: $\frac{d}{dx}[\ln(f(x))] \overset{f(x) > 0}{=} \frac{f'(x)}{f(x)} = \frac{d}{dx}[\ln(f(x))]$

§4.7

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$$h'(x) \overset{x \geq 1}{=} \frac{1}{\sqrt{x^2 - 1}} \quad \blacksquare$$

NOTE: $\frac{d}{dx}\left[\text{la}\left(x + \sqrt{x^2 - 1}\right)\right] = \frac{1}{\sqrt{x^2 - 1}}$

EXAMPLE: Let $y := \dfrac{\ln x}{x^3}$. Find dy/dx and d^2y/dx^2.

Assuming $x > 0$.

$$\frac{dy}{dx} = \frac{(x^{3})(\;1/x\;) - (\ln x)(\;3x^2\;)}{x^6} = \frac{x^2(1 - 3\ln x)}{x^{6}} = \frac{1 - 3\ln x}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{(x^{4})(-3/x) - (1 - 3\ln x)(\;4x^3\;)}{x^8}$$

$$= \frac{x^3[-3 - (1 - 3\ln x)(4)]}{x^{8}}$$

WE CAN SIMPLIFY A LITTLE . . .

$$= \frac{-3 - (4 - 12\ln x)}{x^5} = \frac{-7 + 12\ln x}{x^5}$$ ■

STOP