CALCULUS
Maxima and minima

cf. §6.1, p. 105 **DEFINITION**: Let $f : D \to \mathbb{R}$ be a function. $D \subseteq \mathbb{R}$

We say f **has a global maximum (or absolute maximum) at** c if $f(c) \geq f(x)$, $\forall x \in D$,

in which case the number $f(c)$ is called **the maximum value of** f.

Understood: $c \in D$

Similarly, we say f **has a global minimum (or absolute minimum) at** c if $f(c) \leq f(x)$, $\forall x \in D$,

in which case the number $f(c)$ is called **the minimum value of** f.

Understood: $c \in D$

"**extremum**" = "maximum or minimum"

Plurals: minima, maxima, extrema

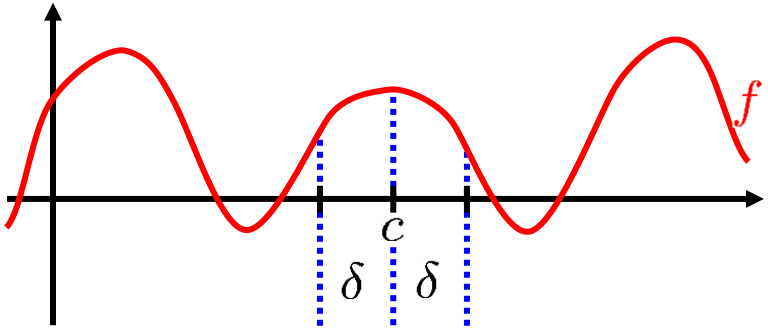
Next: **Local** extrema. . .

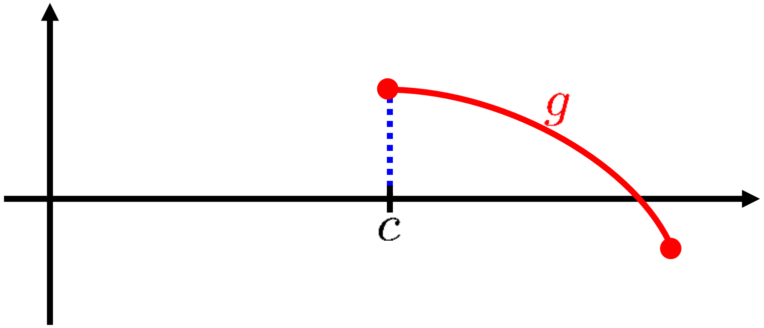
cf. §5.1, p. 93 **DEFINITION**: Let $f : D \to \mathbb{R}$ be a function. $D \subseteq \mathbb{R}$

We say f **has a local maximum (or relative maximum) at** c

if $\exists \delta > 0$ s.t.: $f(c) \geq f(x)$, $\forall x \in (c - \delta, c + \delta)$.

Understood: $(c - \delta, c + \delta) \subseteq D$





g does NOT have a local max at c,
but g does have a global max at c.

cf. §5.1, p. 93 **DEFINITION**: Let $f : D \to \mathbb{R}$ be a function. $D \subseteq \mathbb{R}$

We say f **has a local maximum (or relative maximum) at** c

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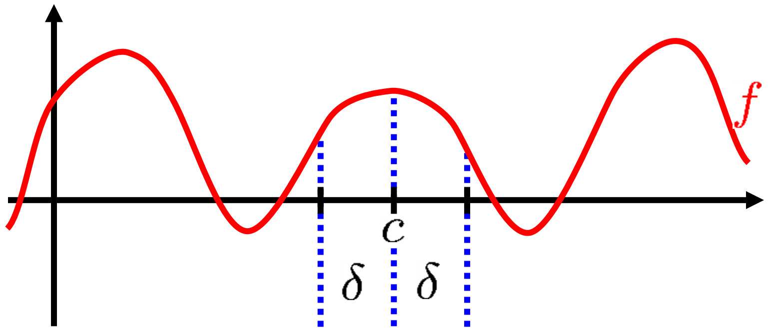

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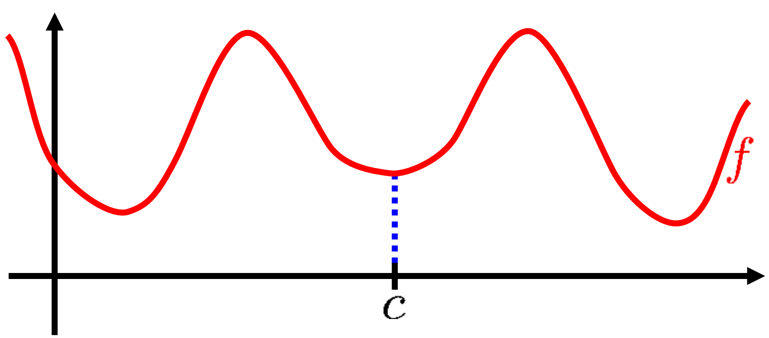


cf. §5.1, p. 94 DEFINITION: Let $f : D \to \mathbb{R}$ be a function. $D \subseteq \mathbb{R}$

A number c is called a **critical point of f** if

either $[f'(c) = 0]$ or $[f'(c)$ does not exist$]$.

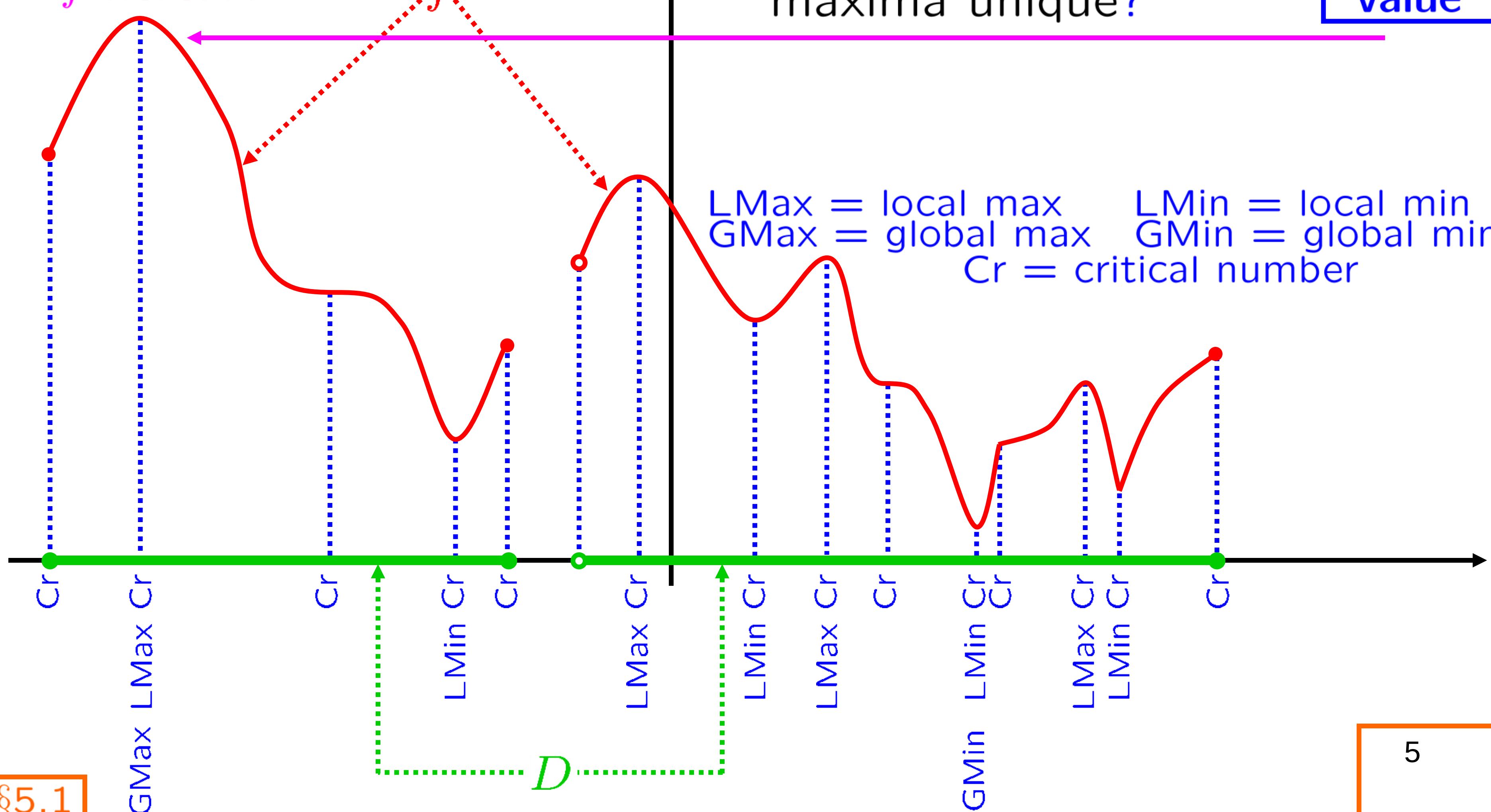
Understood: $c \in D$

"critical number"

Let's change f here. . .

Are global maxima unique?

"critical value"



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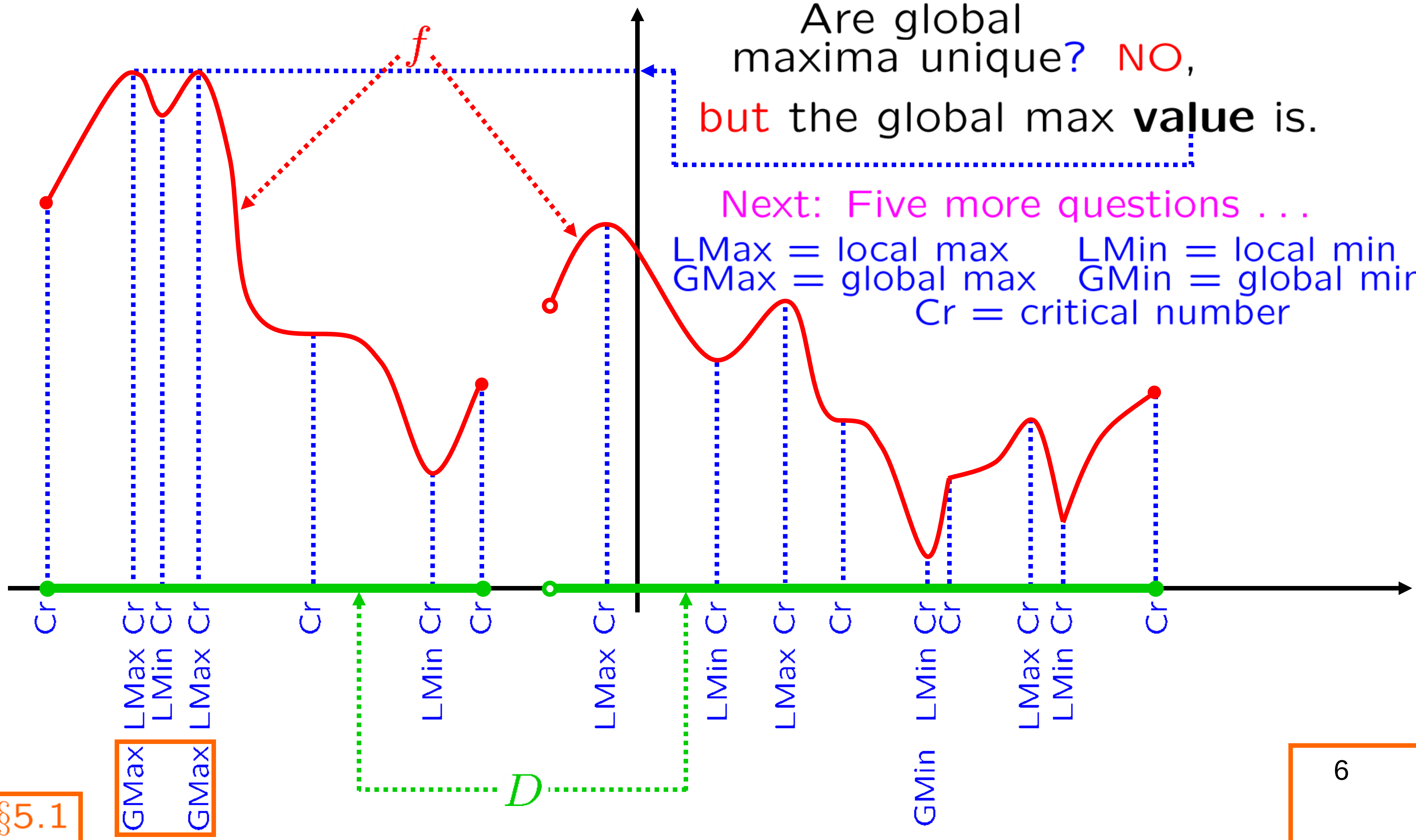

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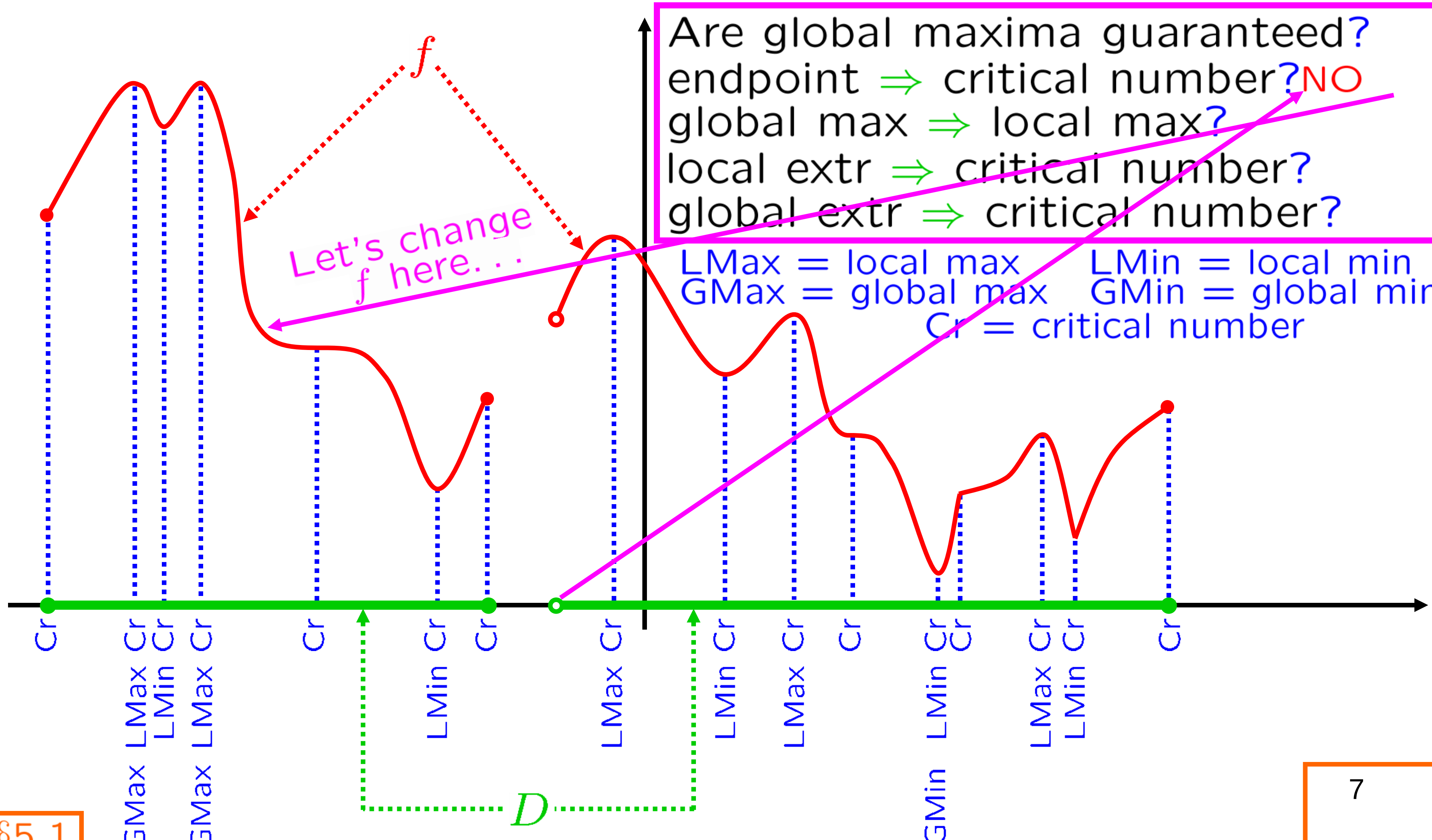
Understood: $c \in D$

"critical number"

Are global maxima guaranteed?
endpoint $\Rightarrow$ critical number? NO
global max $\Rightarrow$ local max?
local extr $\Rightarrow$ critical number?
global extr $\Rightarrow$ critical number?



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Understood: $c \in D$

"critical number"

Are global maxima guaranteed? NO

endpoint $\Rightarrow$ critical number? NO

global max $\Rightarrow$ local max?

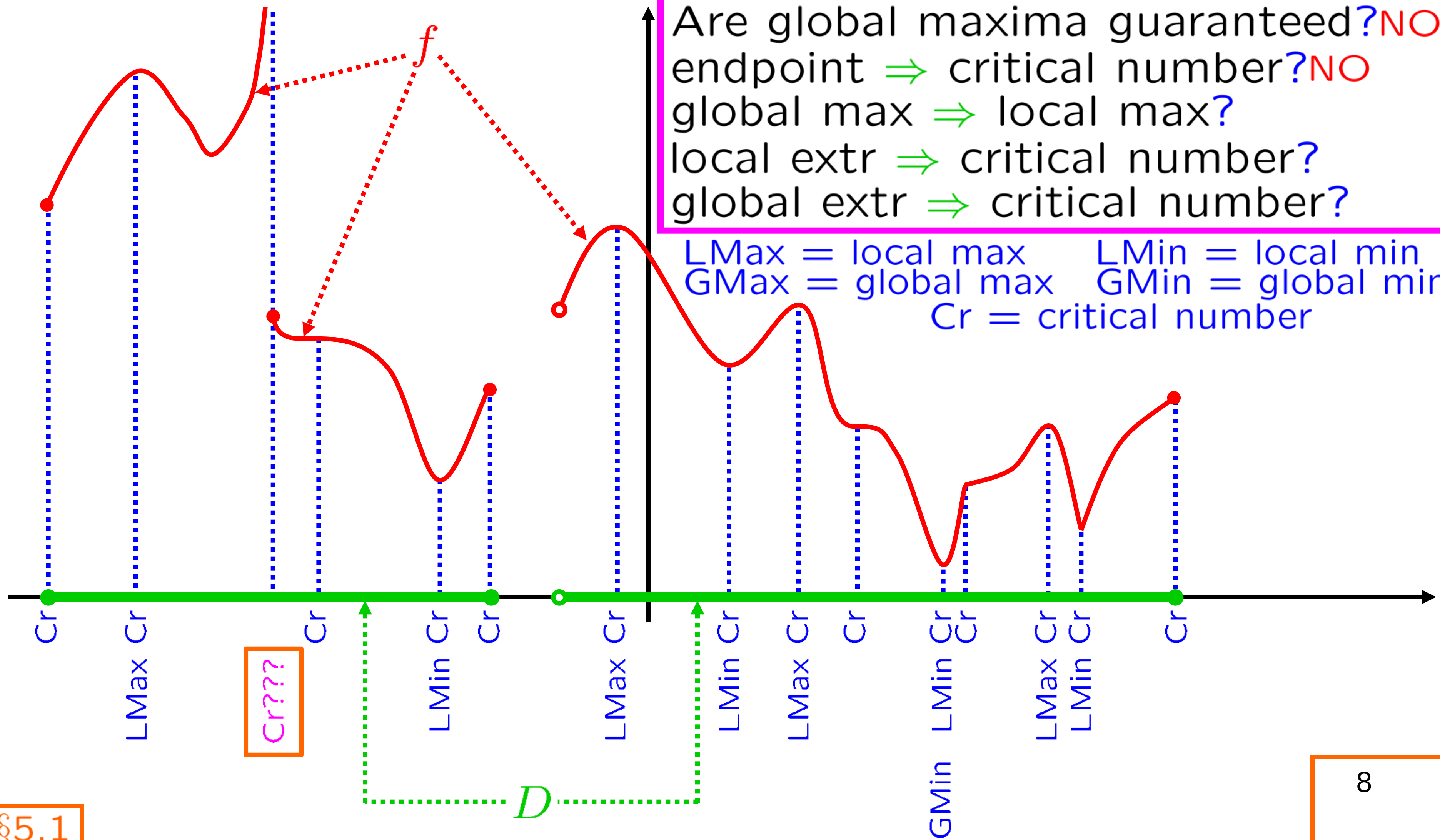
local extr $\Rightarrow$ critical number?

global extr $\Rightarrow$ critical number?

LMax = local max LMin = local min

GMax = global max GMin = global min

Cr = critical number



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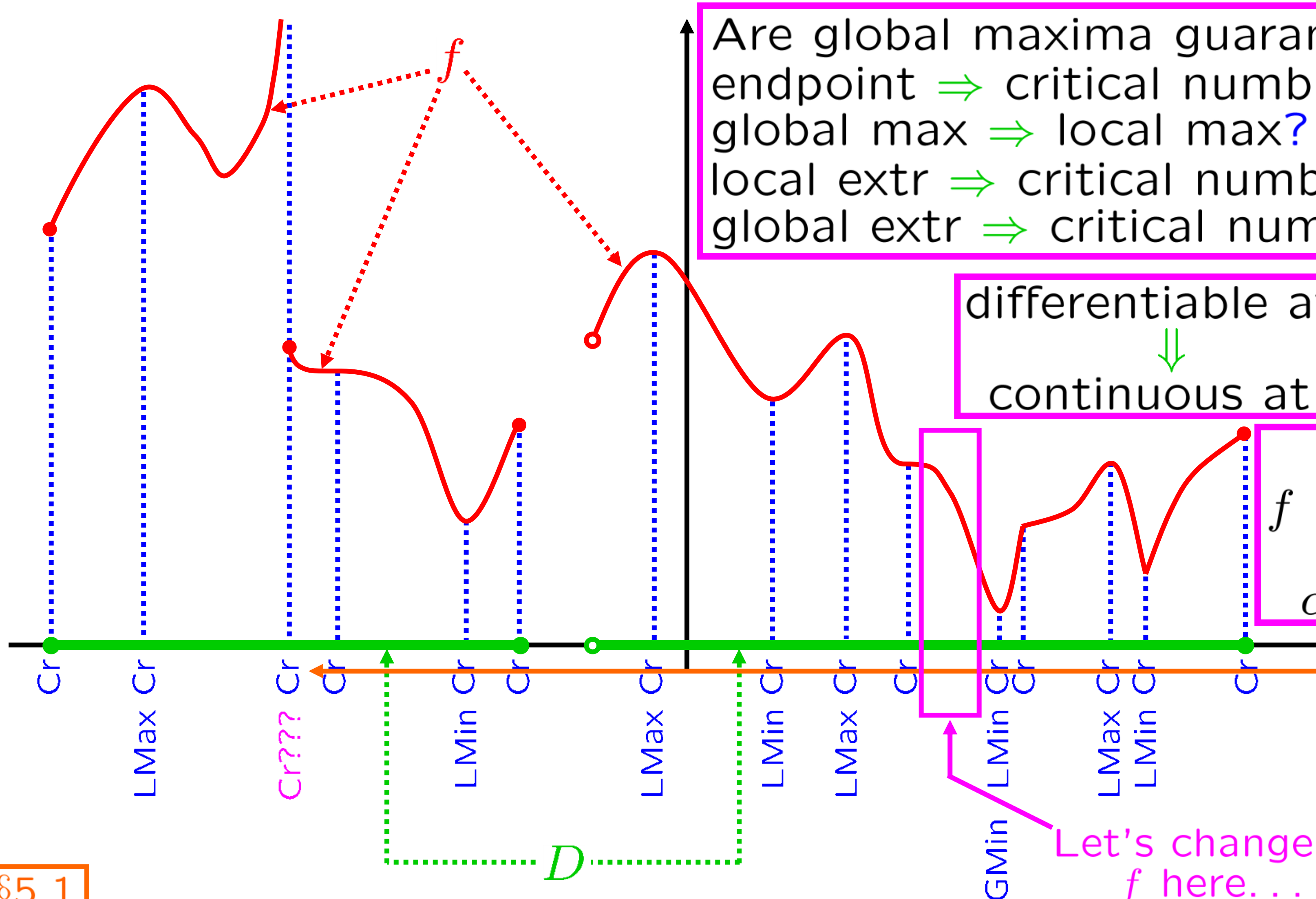
Understood: $c \in D$

"critical number"

Are global maxima guaranteed? NO
endpoint $\Rightarrow$ critical number? NO
global max $\Rightarrow$ local max?
local extr $\Rightarrow$ critical number?
global extr $\Rightarrow$ critical number?

differentiable at c
$\Downarrow$
continuous at c

$\forall c \in D$,
f disc. at c
$\Downarrow$
c critical



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$\Downarrow$

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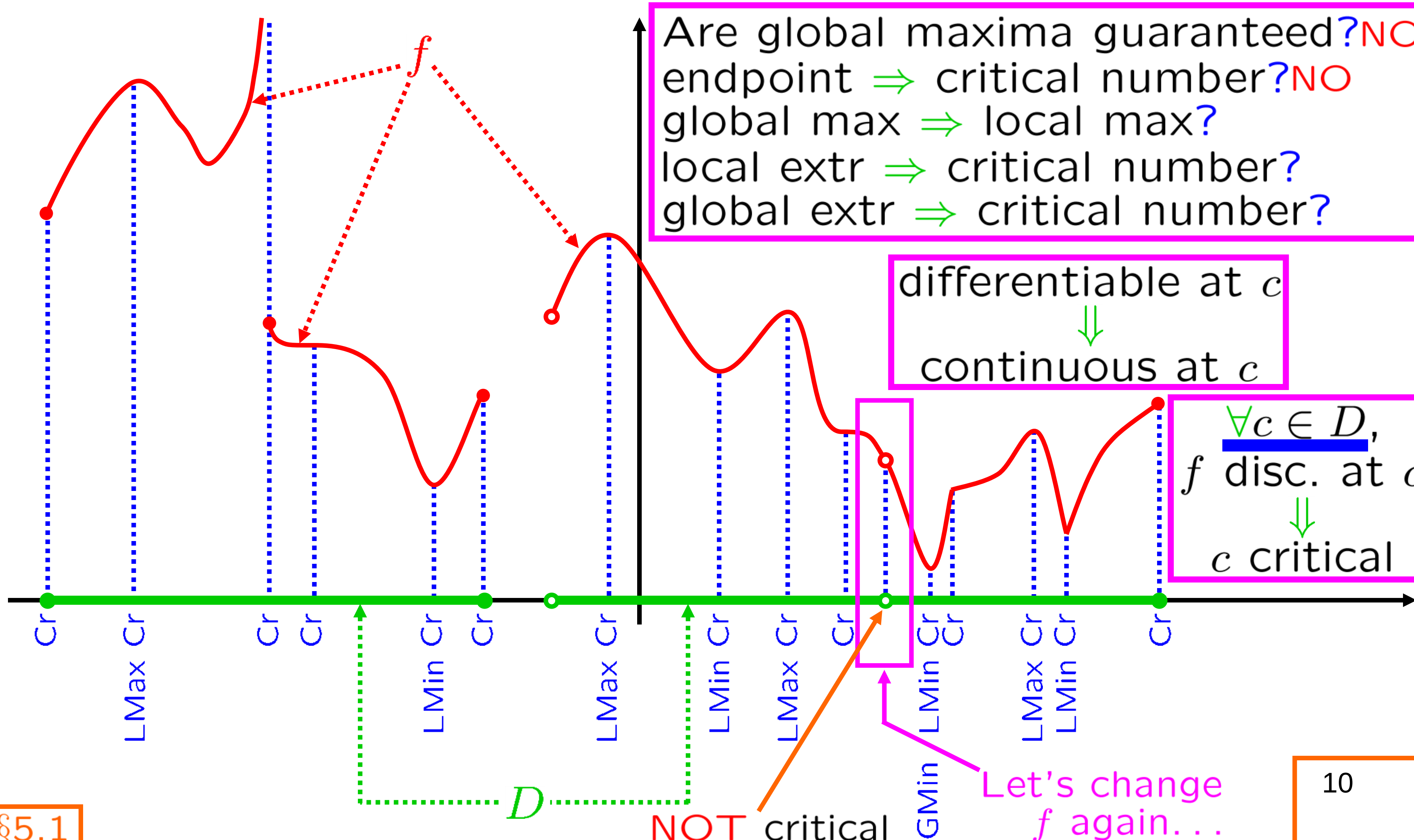

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Understood: $c \in D$

"critical number"

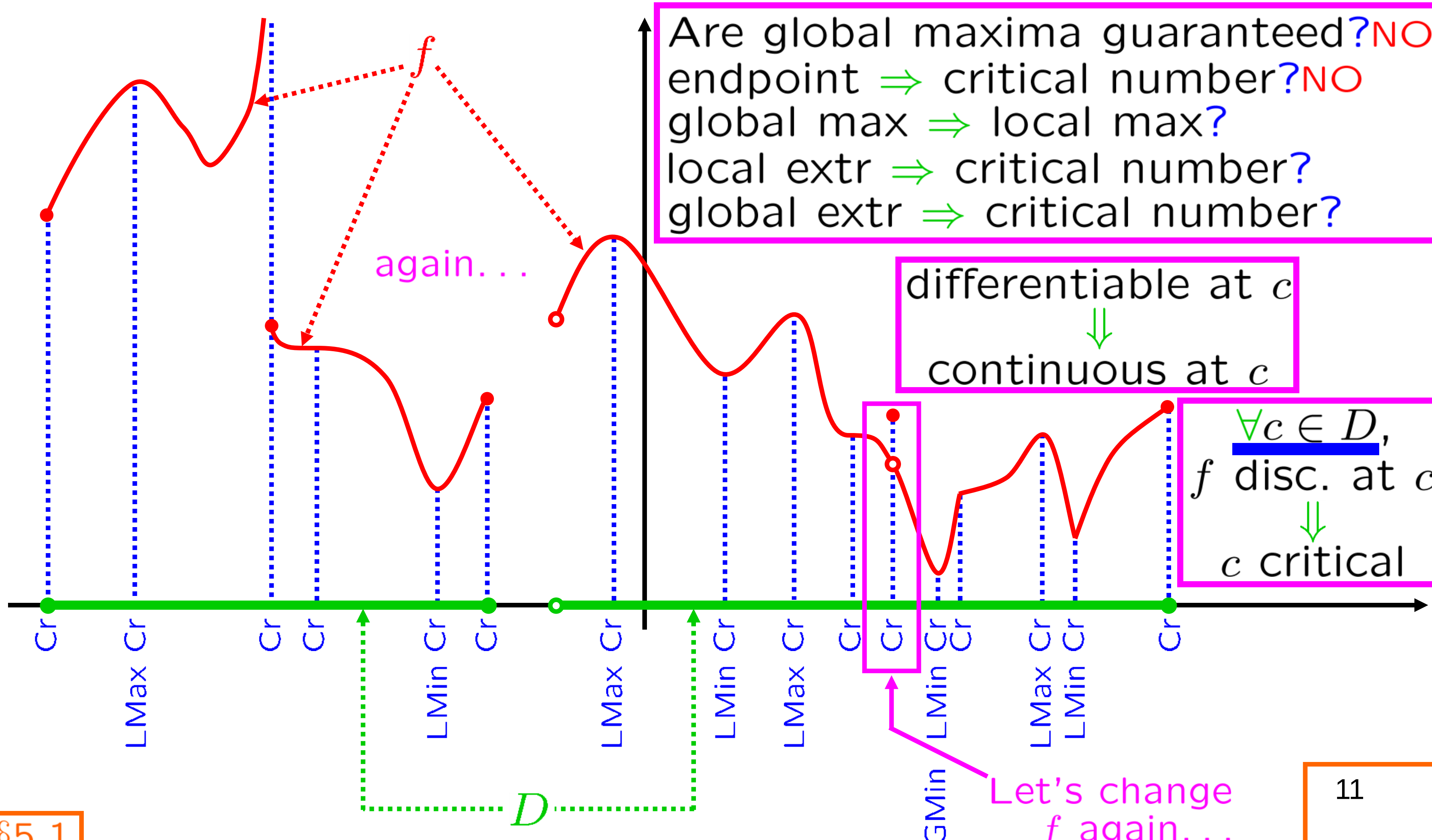
Are global maxima guaranteed? NO

endpoint $\Rightarrow$ critical number? NO

global max $\Rightarrow$ local max?

local extr $\Rightarrow$ critical number?

global extr $\Rightarrow$ critical number?



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Understood: $c \in D$

"critical number"

Are global maxima guaranteed? NO

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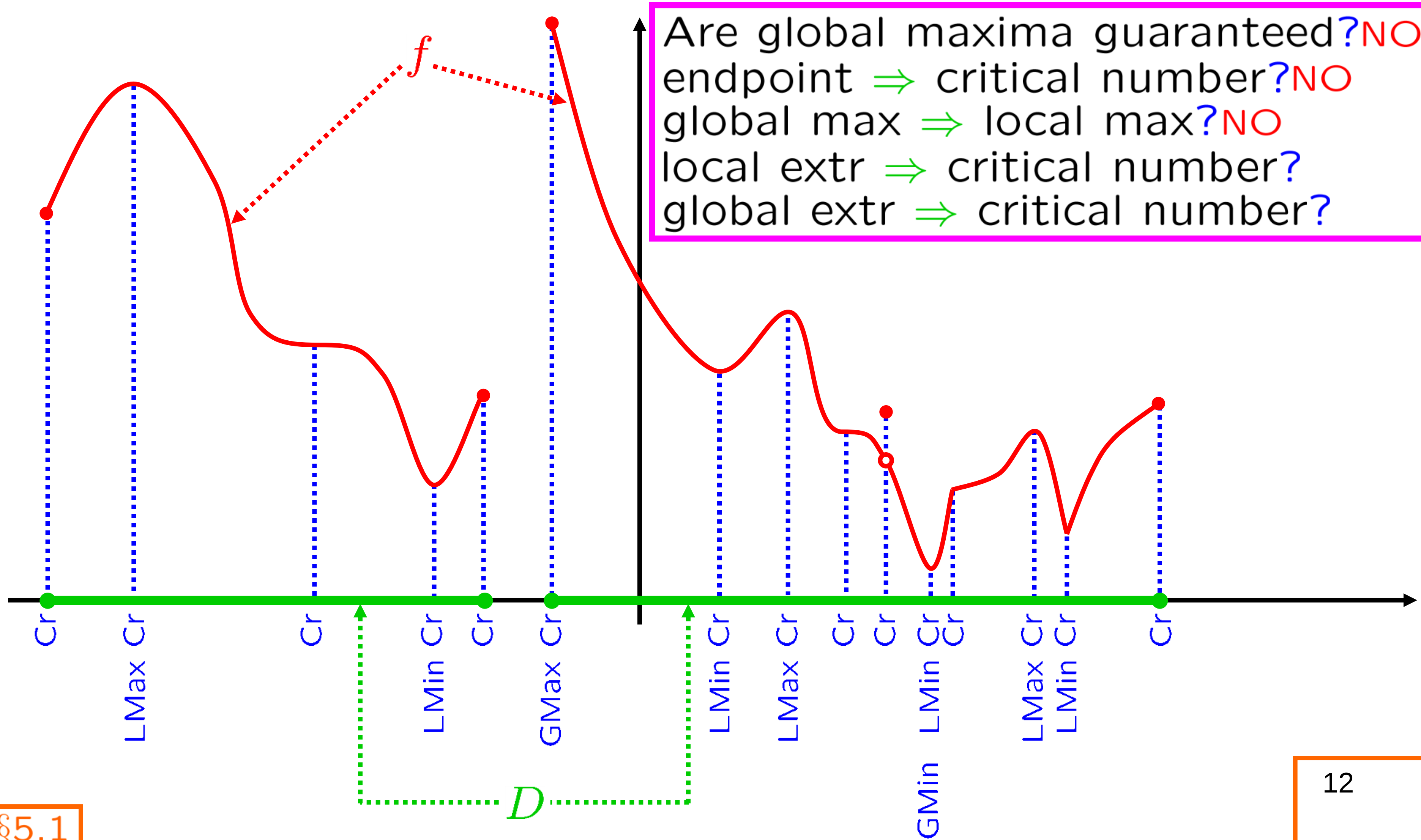


cf. §5.1, p. 94 TH'M 5.1 (FERMAT'S TH'M):

Any local extremum is at a critical point.

Any global extremum is at a critical point.

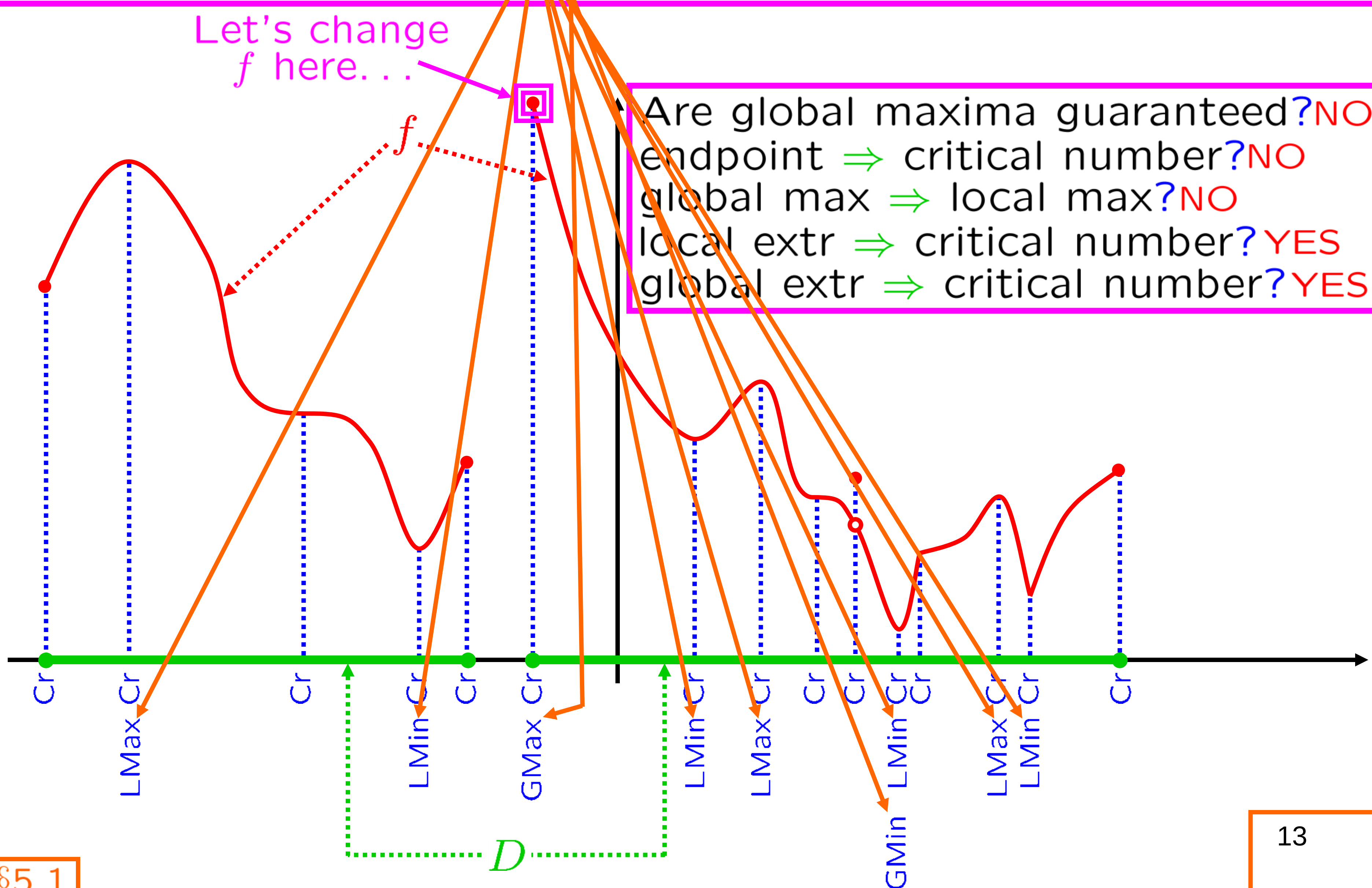
Are global maxima guaranteed? NO

endpoint ⇒ critical number? NO

global max ⇒ local max? NO

local extr ⇒ critical number? YES

global extr ⇒ critical number? YES



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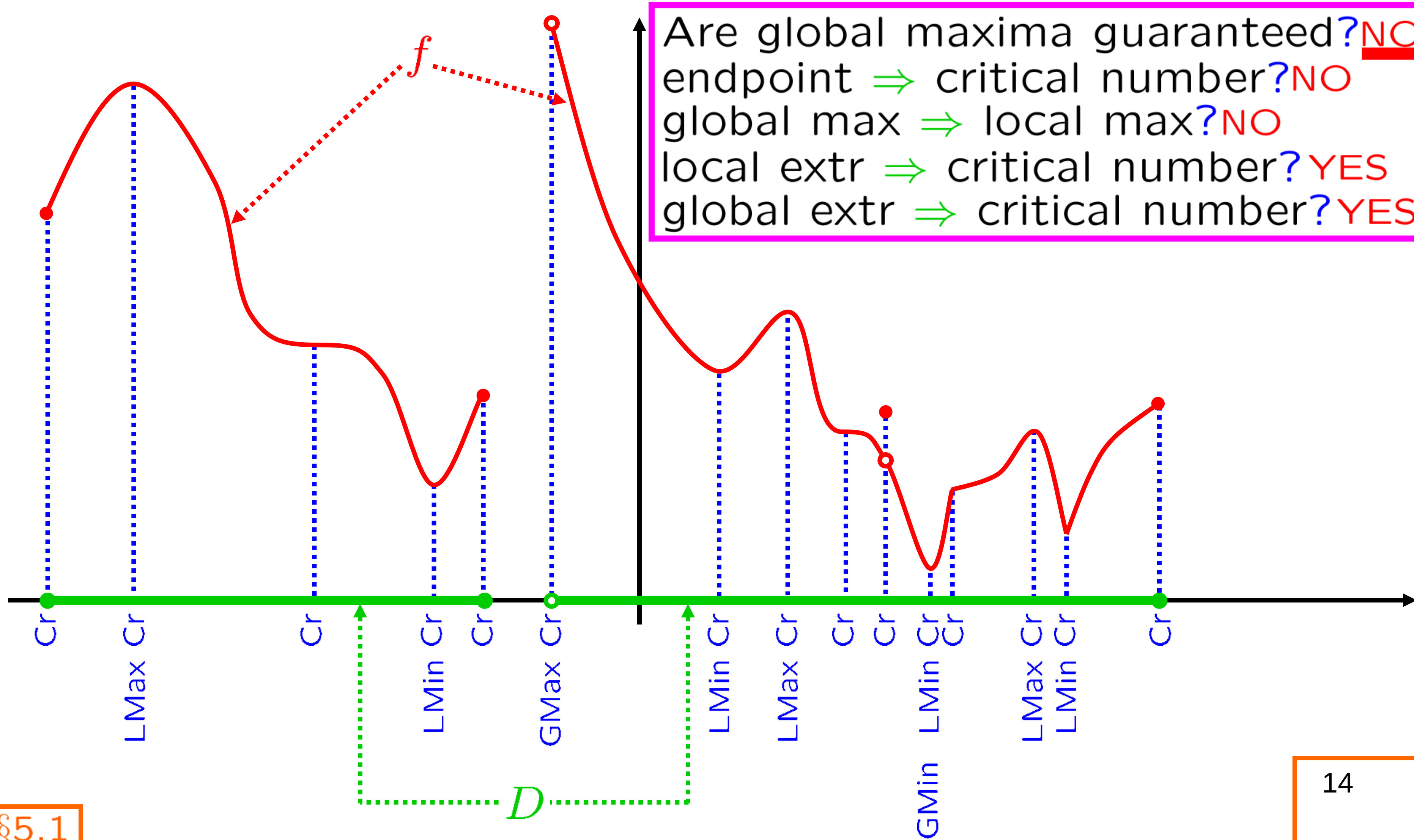
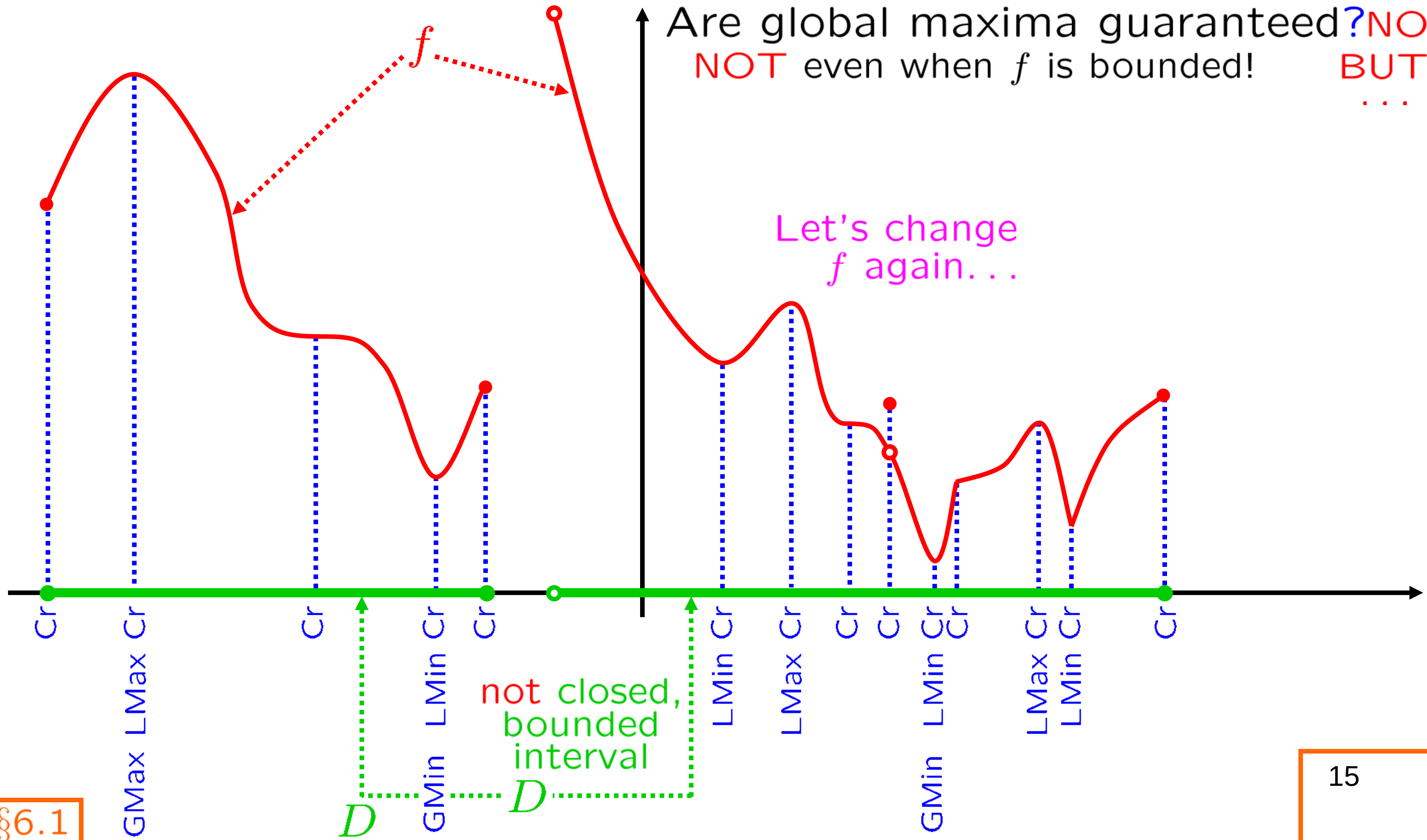


Are global maxima guaranteed? NO
endpoint ⇒ critical number? NO
global max ⇒ local max? NO
local extr ⇒ critical number? YES
global extr ⇒ critical number? YES

cf. §6.1, p. 105 TH'M 6.2 (EXTREME VALUE TH'M):
If D is a compact (*i.e.*, closed, bounded) interval,
and if $f : D \to \mathbb{R}$ is continuous on D,
then f has a global max and a global min.



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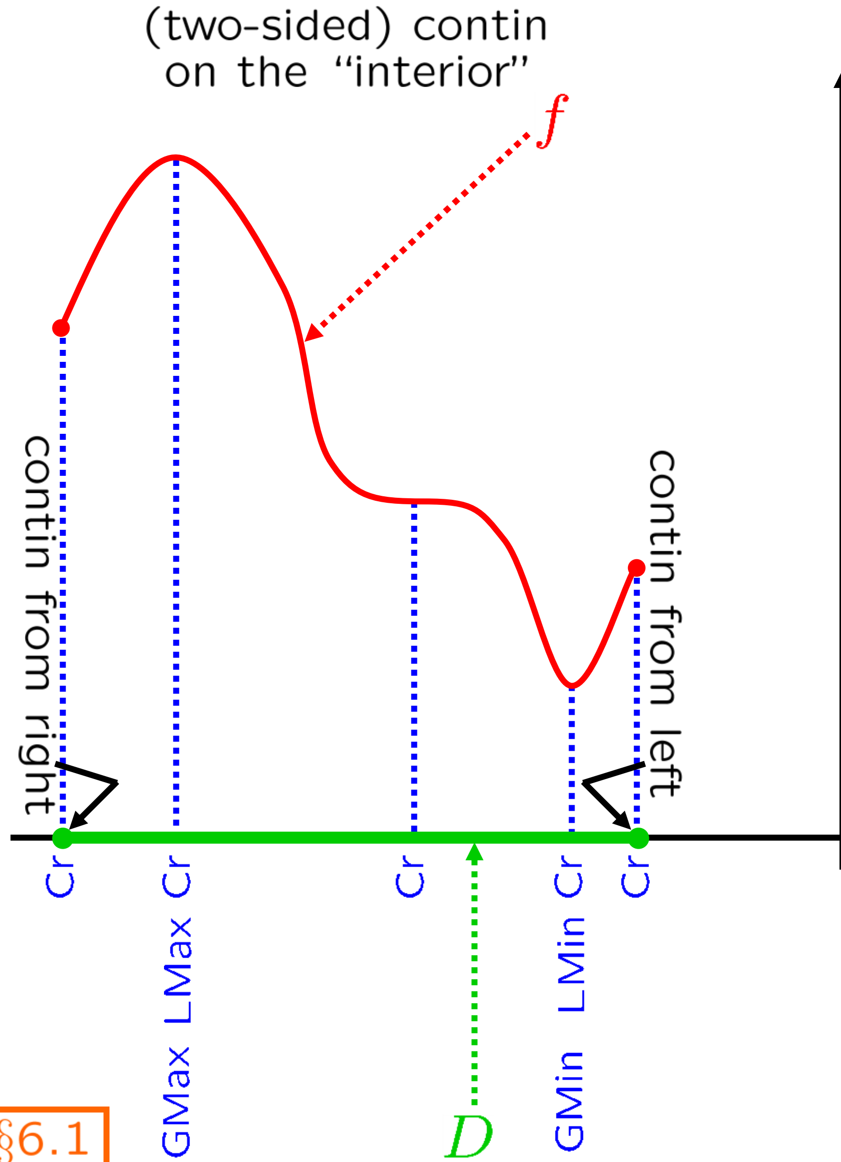
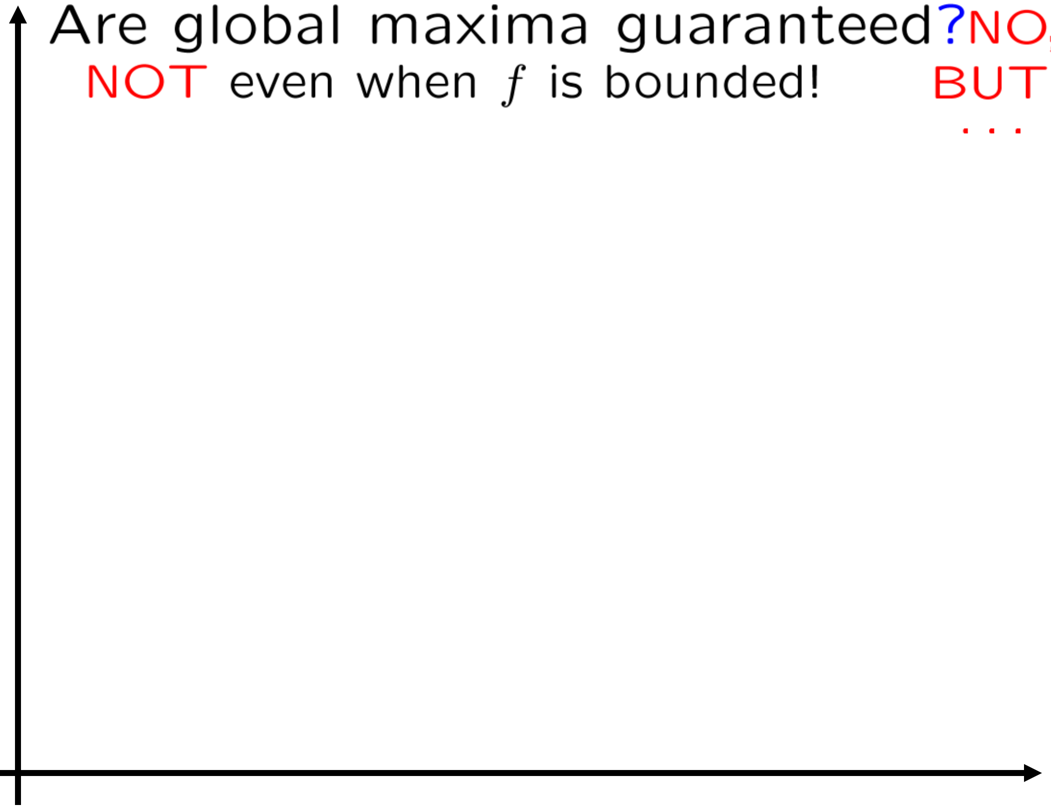


Are global maxima guaranteed? NO,
NOT even when f is bounded! BUT
. . .



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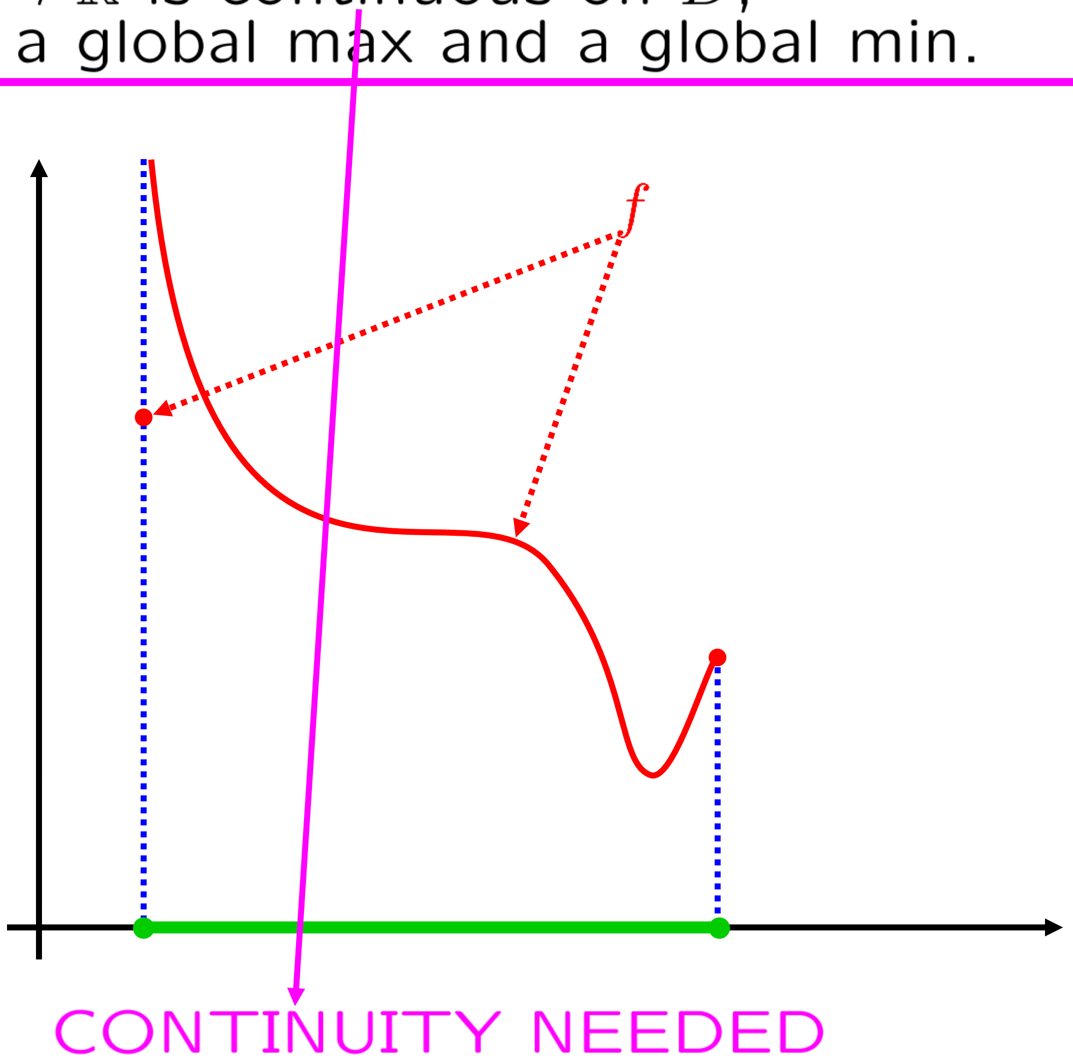

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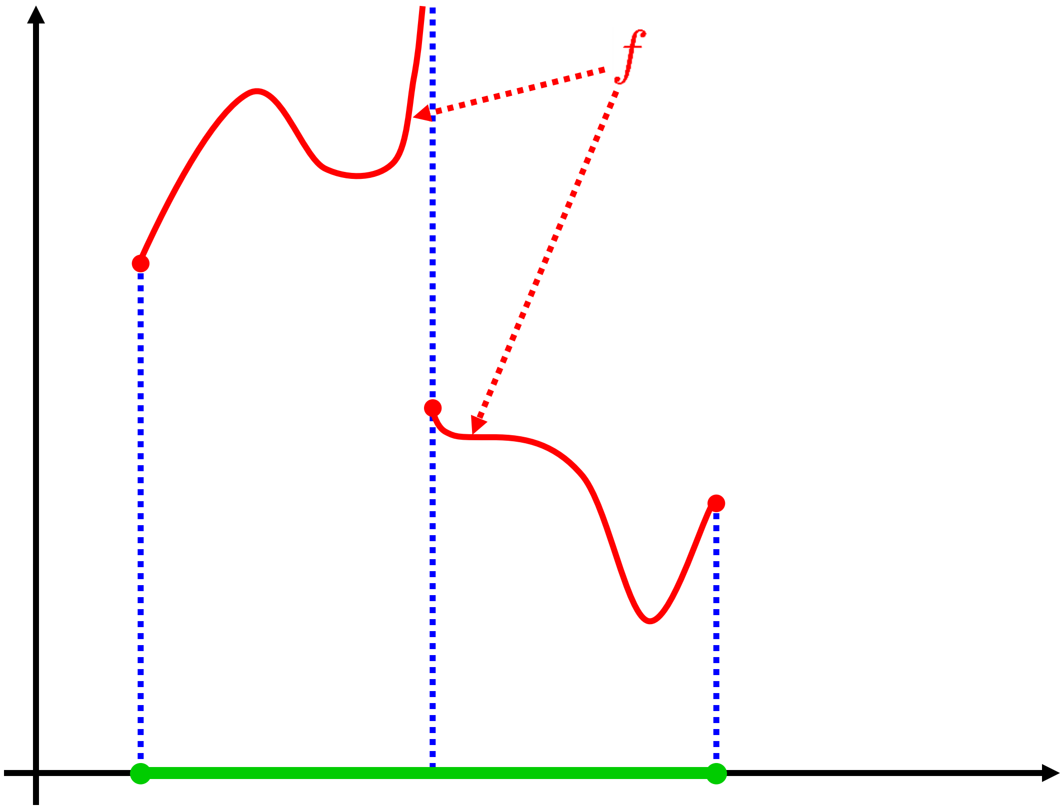

CONTINUITY NEEDED

cf. §6.1, p. 105 TH'M 6.2 (EXTREME VALUE TH'M):

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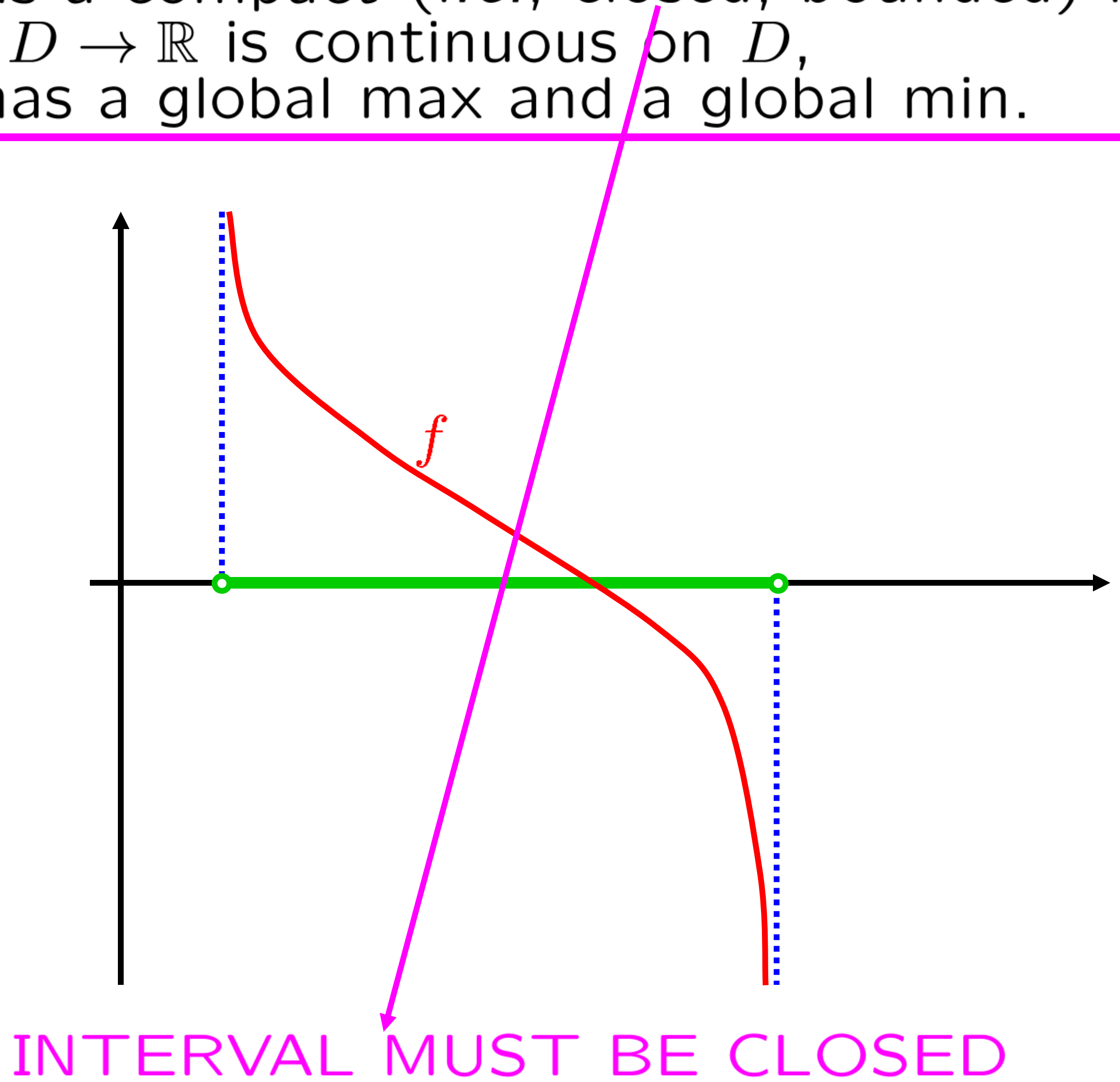

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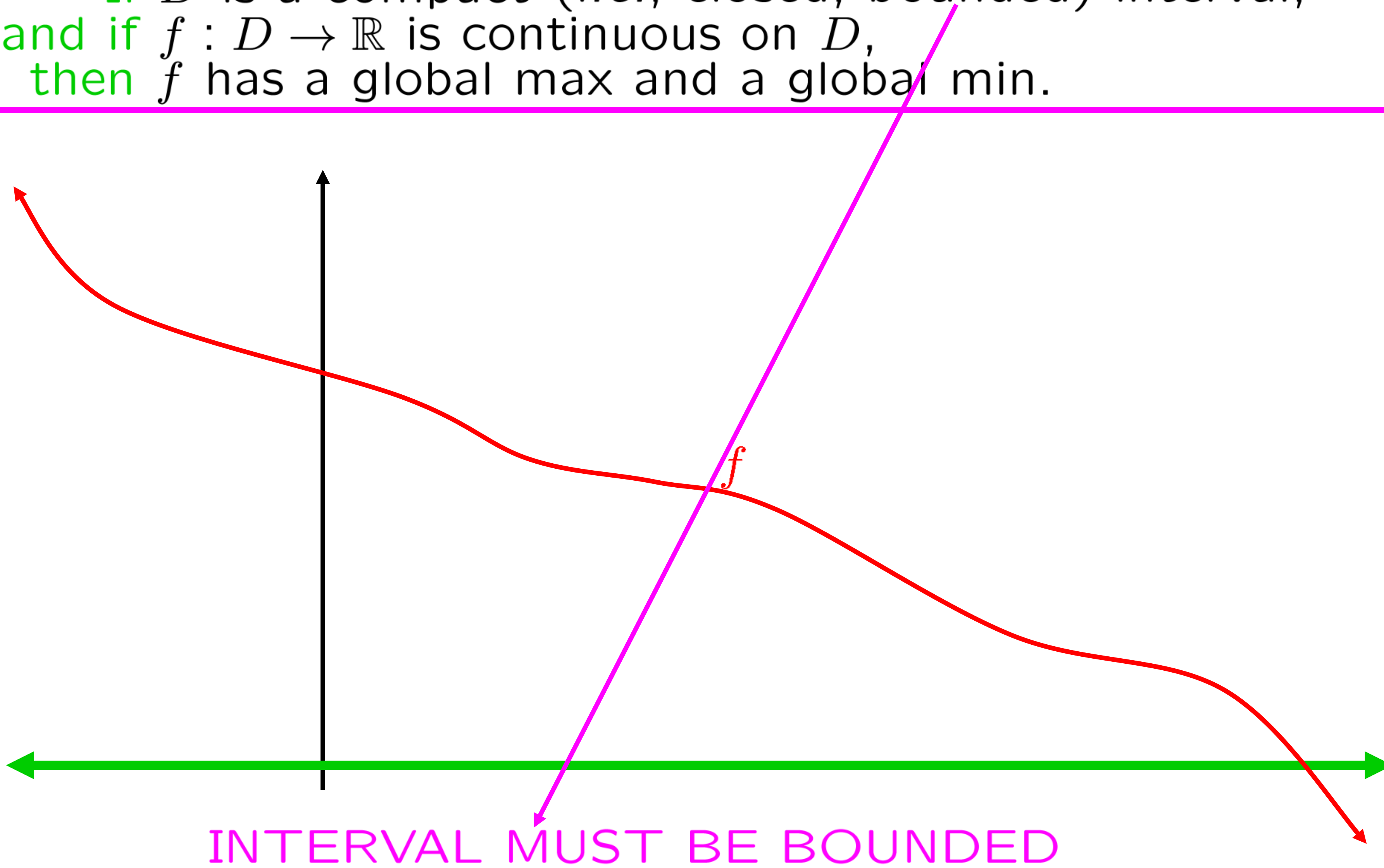

cf. §6.1, p. 105 TH'M 6.2 (EXTREME VALUE TH'M):

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cf. §5.1, p. 94 TH'M 5.1 (FERMAT'S TH'M):

Any local extremum is at a critical point.
Any global extremum is at a critical point.

FINDING GLOBAL EXTREMA:

If you know a function has a global max (resp. min), then you can find it:
compute the values at critical points and find the largest (resp. the smallest).

EXAMPLE: Find the global max and min values of

$f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = x^4 - 8x^2 + 8 \qquad$ on $\qquad -\frac{1}{2} \le x \le 4.$

$0 = f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2)$

or DNE

Critical points of $f|[-\frac{1}{2}, 4]$: $-\frac{1}{2}$, 0, ~~-2,~~ 2, 4

$f(-\frac{1}{2}) = 6 + (1/16)$

$f(0) = 8$

$f(2) = -8$ Global min value is -8, attained at 2.

§6.1 $f(4) = 136$ Global max value is 136, attained at 4.

SKILL
global max-min

EXAMPLE: Find the global maximum and minimum values of $f(x) = x^3 - 12x + 7$ on $[0, 5]$.

SKILL
global max-min

$0 = f'(x) = 3x^2 - 12$ ← DIVIDE BY 3

or DNE

$0 = x^2 - 4 = (x + 2)(x - 2)$

Critical points of $f|[0, 5]$: 0, ~~-2~~, 2, 5

$f(0) = 0 + 0 + 7 = 7$

$f(2) = 8 - 24 + 7 = -9$ global minimum value

$f(5) = 125 - 60 + 7 = 72$ global maximum value ■

EXAMPLE: Find the global maximum and minimum values

SKILL
global max-min

of $f(x) = \dfrac{x^2 - 7}{x^2 + 7}$ on the interval $[-5, 5]$.

$$0 = f'(x) = \frac{(x^2+7)(2x) - (x^2-7)(2x)}{(x^2+7)^2}$$

or DNE

$$= \frac{(2x^3 + 14x) + (2x^3 + 14x)}{(x^2+7)^2}$$

$$0 = \frac{28x}{(x^2+7)^2} \quad \Leftrightarrow \quad x = 0$$

Critical points of $f|[-5,5]$: $-5,\ 0,\ 5$

not unique

$$f(\pm 5) = \frac{25-7}{25+7} = \frac{18}{32} = \frac{9}{16}$$ global maximum value

$$f(0) = \frac{-7}{7} = -1$$ global minimum value

§6.1

EXAMPLE: Find the global maximum and minimum values

SKILL
global max-min

of $f(x) = \frac{x^2+7}{x^2-7}$ on the interval $[-5, 5]$.

$f(x)$ undefined at $x = \pm\sqrt{7}$

The question doesn't make sense. ■

EXAMPLE: Find the global maximum and minimum values

SKILL
global max-min

of $f(x) = \frac{x^2+7}{x^2-7}$ on $[-5, 5]\backslash\{-\sqrt{7}, \sqrt{7}\}$.

$$\lim_{x\to-\sqrt{7}+}\left(\frac{x^2+7}{x^2-7}\right) = \infty \qquad \lim_{x\to\sqrt{7}+}\left(\frac{x^2+7}{x^2-7}\right) = \infty$$

$$\lim_{x\to-\sqrt{7}-}\left(\frac{x^2+7}{x^2-7}\right) = -\infty \qquad \lim_{x\to\sqrt{7}-}\left(\frac{x^2+7}{x^2-7}\right) = -\infty$$

There is no global maximum.
There is no global minimum. ■

EXAMPLE: Find the global maximum and minimum values

SKILL
global max-min

of $f(x) = \dfrac{x^2+7}{x^2-7}$ on the interval $[-1, 1]$.

$f'(x)$ undefined for $x = \pm\sqrt{7}$

$$0 = f'(x) = \frac{(x^2-7)(2x) - (x^2+7)(2x)}{(x^2-7)^2}$$

or DNE

$f(x)$ undefined at $x = \pm\sqrt{7}$

$$= \frac{(2x^3 - 14x) - (2x^3 + 14x)}{(x^2-7)^2}$$

$$0 = \frac{-28x}{(x^2-7)^2} \quad \Leftrightarrow \quad x = 0$$

Critical points of $f|[-1,1]$: $-1,\ 0,\ 1$

not unique

$$f(\pm 1) = \frac{1+7}{1-7} = \frac{8}{-6} = -\frac{4}{3}$$ global minimum value

$$f(0) = \frac{7}{-7} = -1$$ global maximum value

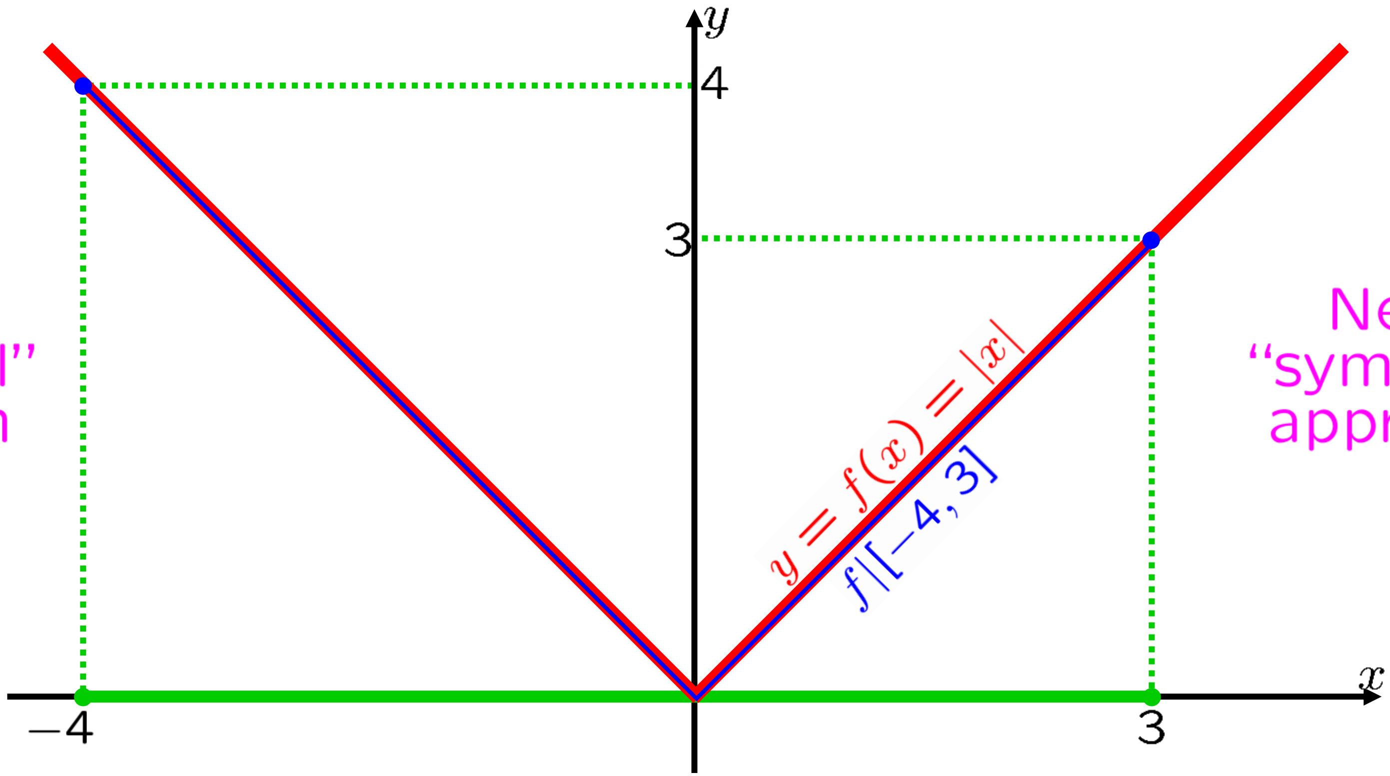
EXAMPLE: Find the global maximum and minimum values of $f(x) = |x|$ on the interval $[-4, 3]$.

SKILL
global max-min

$$f'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$$

global max value $= 4$

First: "graphical" approach



Next: "symbolic" approach

global min value $= 0$

EXAMPLE: Find the global maximum and minimum values of $f(x) = |x|$ on the interval $[-4, 3]$.

SKILL
global max-min

$f(x)$ is defined at $x = 0$

$$0 \neq f'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$$

or DNE
$x = 0$

Critical points of $f|[-4, 3]$: $-4,\ 0,\ 3$

$f(-4) = 4$ global maximum value

$f(0) = 0$ global minimum value

$f(3) = 3$

EXAMPLE: Find the global maximum and minimum values

SKILL

global max-min

of $f(x) = 2x - (\ln x)$ on the interval $[1, 7]$.

$f'(x)$ undefined for $x \leq 0$

$$0 = f'(x) \underset{x>0}{=} 2 - \frac{1}{x} \quad \Leftrightarrow \quad x = \frac{1}{2}$$

or DNE

$f(x)$ undefined for $x \leq 0$

Critical points of $f|[1,7]$: ~~$\frac{1}{2}$~~ 1, 7

$f(1) = 2 - (\ln 1) = 2$ global minimum value

$f(7) = 14 - (\ln 7) \doteq 12.05$ global maximum value

cf. STEWART, §4.1, p. 276 EXAMPLE 10: A model for the distance traveled by the shuttle *Discovery* during a mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

$$p(t) = 0.0003255t^4 - 0.03010t^3 + 11.80t^2 - 3.083t$$

(in feet). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

$$v(t) = 0.001302t^3 - 0.09030t^2 + 23.60t - 3.083$$

$$a(t) = 0.003906t^2 - 0.1806t + 23.60$$

$$a'(t) = 0.007812t - 0.1806$$

$$0 = 0.007812t - 0.1806$$

or DNE

$$0.007812t = 0.1806$$

$$t = \frac{0.1806}{0.007812} \doteq 23.12$$

cf. STEWART, §4.1, p. 276 EXAMPLE 10: A model for the distance traveled by the shuttle *Discovery* during a mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

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$$a(t) = 0.003906t^2 - 0.1806t + 23.60$$
$$a'(t) = 0.007812t - 0.1806$$

critical pts for a:
23.12

23.12

cf. STEWART, §4.1, p. 276 EXAMPLE 10: A model for the distance traveled by the shuttle *Discovery* during a mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

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$$v(t) = 0.001302t^3 - 0.09030t^2 + 23.60t - 3.083$$
$$a(t) = 0.003906t^2 - 0.1806t + 23.60$$
$$a'(t) = 0.007812t - 0.1806$$

critical pts for a: 23.12

critical pts for $a|[0,126]$: $0, 23.12, 126$

$a(0) = 23.60$

global min value: $a(23.12) \doteq 21.51$

global max value: $a(126) \doteq 62.86$ ■

SKILL
applied max-min

$$\frac{a(126)}{32} \doteq 1.96$$

1.96 "g"s is OK.

§6.1

EXAMPLE: Find the critical points of the function

$$f(x) = 2x^3 - x^2 + x - 5.$$

$$0 = f'(x) = 6x^2 - 2x + 1 \neq 0, \ \forall x \in \mathbb{R}$$

or DNE

The "discriminant": $(-2)^2 - 4(6)(1) = 4 - 24 < 0$

$(b^2 - 4ac)$

No critical points. ■

SKILL
critical pts

Quadratic formula: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

EXAMPLE: Find the critical points of the function

$$f(x) = (x^{-3})(\ln x). \longleftarrow \text{PRODUCT RULE}$$

$f'(x)$ undefined for $x \leq 0$

$$0 = f'(x) = (-3x^{-4})(\ln x) + (x^{-3})(1/x)$$

or DNE

$f(x)$ undefined for $x \leq 0$

EXAMPLE: Find the critical points of the function

$$f(x) = (x^{-3})(\ln x).$$

$f'(x)$ undefined for $x \leq 0$

$$0 = f'(x) = (-3x^{-4})(\ln x) + (x^{-3})(1/x)$$

$x^4 \times$ $$(3x^{-4})(\ln x) = (x^{-3})(1/x)$$

$(1/3)\times$ → $$3(\ln x) = \underbrace{x^4(x^{-3})(1/x)}_{1}$$

EXPONENTIATE → $$\ln x = 1/3$$

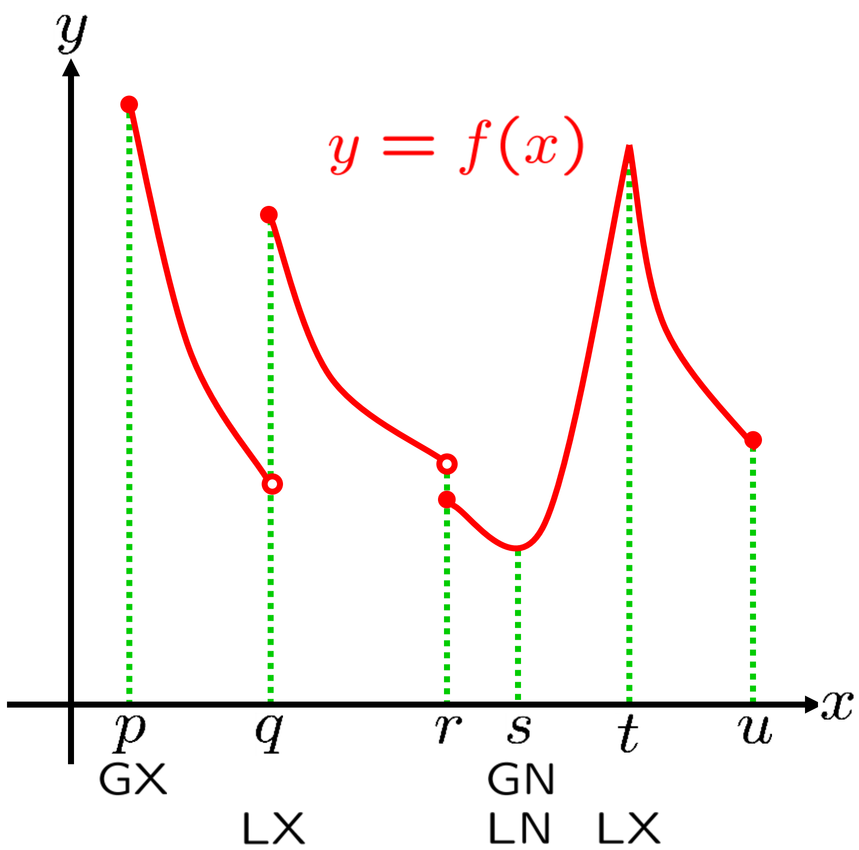
$$x = e^{1/3}$$

$e^{1/3}$ is the only critical point.

SKILL
critical pts

EXAMPLE: a. For each of the numbers p, q, r, s, t and u, state whether the function f has a global maximum or minimum at that number.

b. For each of the numbers p, q, r, s, t and u, state whether the function f has a local maximum or minimum at that number.



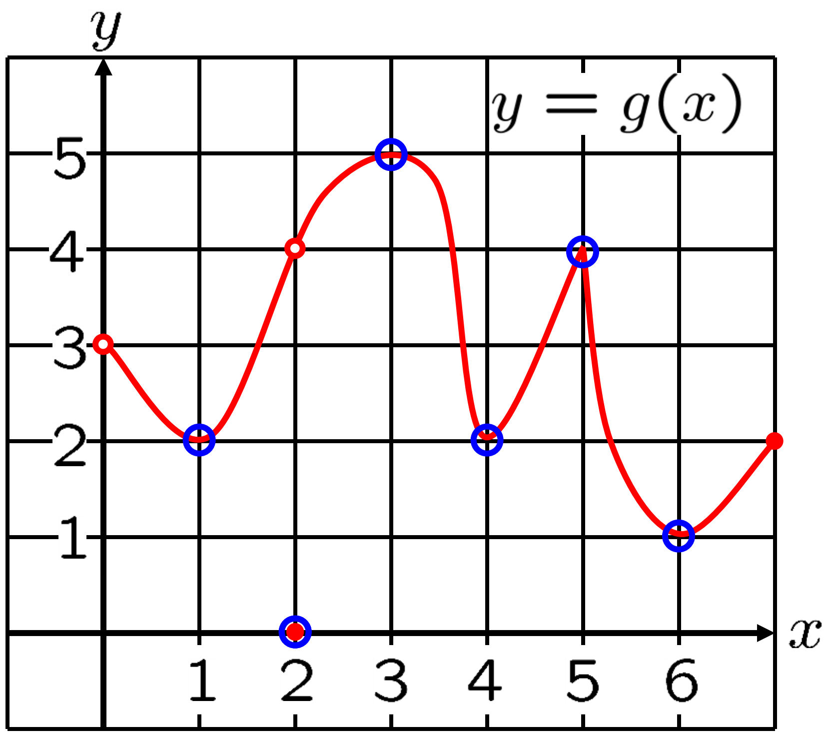
GX = global max
GN = global min

LX = local max
LN = local min

SKILL
max-min from gph

EXAMPLE: a. Use the graph to state the global maximum and minimum values of g.

b. Use the graph to state the local maximum and minimum values of g.



global max value: 5 at 3

global min value: 0 at 2

local max values: 5 at 3

4 at 5

local min values: 2 at 1

0 at 2

2 at 4

1 at 6

SKILL

max-min from gph

SKILL
loc extr
Whitman problems
§5.1, p. 97, #1-12

SKILL
misc loc extr critical
Whitman problems
§5.1, p. 97, #13-14

SKILL
loc extr critical in families
Whitman problems
§5.1, p. 97, #15-18

SKILL
find global extr values
Whitman problems
§6.1, p. 115, #1