CALCULUS
Antidifferentiation problems

EXAMPLE: Find the set of all antiderivatives of

$$f(x) = 6x^2 - 4x + 7.$$

Check your answer by differentiation. ☺

$$6\left[\frac{x^3}{3}\right] - 4\left[\frac{x^2}{2}\right] + 7x + C$$

SKILL
all antiderivs

$$\| $$

$$\frac{d}{dx}\left[2x^3 - 2x^2 + 7x + C\right] = 6x^2 - 4x + 7 \quad ■$$

EXAMPLE: Find the set of all antiderivatives of

$$f(x) = \frac{8 + 4x^3 - 3x^6}{x^4}.$$

Check your answer by differentiation.

$$f(x) = 8x^{-4} + 4\left[\frac{1}{x}\right] - 3x^2$$

ANTIDIFF

$$8\left[\frac{x^{-3}}{-3}\right] + 4[\ln x] - \cancel{3}\left[\frac{x^3}{\cancel{3}}\right] + C$$

$$\|$$

$$-\frac{8}{3x^3} + 4[\ln x] - x^3 + C$$

EXAMPLE: Find the set of all antiderivatives of

$$f(x) = \frac{8 + 4x^3 - 3x^6}{x^4}.$$

Check your answer by differentiation.

$$\frac{d}{dx}\left[-\frac{8}{3x^3} + 4\,[\ln x] - x^3 + C\right]$$

$$-\frac{8}{3x^3} + 4\,[\ln x] - x^3 + C$$

EXAMPLE: Find the set of all antiderivatives of

$$f(x) = \frac{8 + 4x^3 - 3x^6}{x^4}.$$

Domain: $x \neq 0$

Check your answer by differentiation. ☺

$$\frac{d}{dx}\left[-\frac{8}{3x^3} + 4\,[\ln x] - x^3 + C\right]$$

Domain: $x > 0$

$$= -\frac{8}{3}\left[\frac{d}{dx}\left(x^{-3}\right)\right] + 4\left[\frac{d}{dx}(\ln x)\right] - \left[\frac{d}{dx}\left(x^3\right)\right]$$

$$= +\frac{8}{\cancel{3}}\left[+\cancel{3}x^{-4}\right] + 4\left[\frac{1}{x}\right] - 3x^2$$

$$= 8x^{-4} + \frac{4}{x} - 3x^2$$

$$= \frac{8}{x^4} + \frac{4x^3}{x^4} - \frac{3x^6}{x^4} = \frac{8 + 4x^3 - 3x^6}{x^4}$$

EXAMPLE: Find the set of all antiderivatives of

$$f(x) = \frac{8 + 4x^3 - 3x^6}{x^4}. \qquad \text{Domain: } x \neq 0$$

Check your answer by differentiation.

Wrong: $-\dfrac{8}{3x^3} + 4\,[\ln x] - x^3 + C$ Domain: $x > 0$

Acceptable: $-\dfrac{8}{3x^3} + 4\,[\ln(|x|)] - x^3 + C$ Domain: $x \neq 0$

Perfect (better than necessary, for this class):

$$\left\{ \begin{pmatrix} -\dfrac{8}{3x^3} + 4\,[\ln x] - x^3 + A, & \text{if } x > 0 \\ -\dfrac{8}{3x^3} + 4\,[\ln(-x)] - x^3 + B, & \text{if } x < 0 \end{pmatrix} \;\middle|\; A, B \in \mathbb{R} \right\}$$

EXAMPLE: Find the set of all antiderivatives of

$$f(x) = \frac{8 + 4x^3 - 3x^6}{x^4}.$$

Check your answer by differentiation.

Wrong: $-\frac{8}{3x^3} + 4\,[\ln x] - x^3 + C$ Think about the domains.

Acceptable: $-\frac{8}{3x^3} + 4\,[\ln(|x|)] - x^3 + C$

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Perfect (better than necessary, for this class):

$$\left\{ \begin{pmatrix} -\frac{8}{3x^3} + 4\,[\ln x] - x^3 + A, & \text{if } x > 0 \\ -\frac{8}{3x^3} + 4\,[\ln(-x)] - x^3 + B, & \text{if } x < 0 \end{pmatrix} \middle| \; A, B \in \mathbb{R} \right\}$$

EXAMPLE: Find the set of all antiderivatives of

$$f(x) = \frac{3 - x^2}{1 + x^2}.$$

$$\begin{array}{r} -1 \\ x^2 + 1 \overline{\smash{\big)}\, -x^2 + 3} \\ \underline{-x^2 - 1} \\ 4 \end{array}$$

$$f(x) = \frac{3 - x^2}{1 + x^2} = -1 + \frac{4}{1 + x^2}$$

EXAMPLE: Find the set of all antiderivatives of

$$f(x) = \frac{3 - x^2}{1 + x^2}.$$

Check your answer by differentiation. ☺

$$\begin{array}{r} -1 \\ x^2 + 1 \overline{\smash{\big)} -x^2 + 3} \\ \underline{-x^2 - 1} \\ 4 \end{array}$$

$$f(x) = \frac{3 - x^2}{1 + x^2} = -1 + \frac{4}{1 + x^2}$$

$$= -1 + 4\left(\frac{1}{1 + x^2}\right)$$

ANTIDIFF $-x + 4(\arctan(x)) + C$

$$\frac{d}{dx}[-x + 4(\arctan(x)) + C] = -1 + 4\left(\frac{1}{1 + x^2}\right)$$

$$= \frac{-(1 + x^2)}{1 + x^2} + \frac{4}{1 + x^2}$$

$$= \frac{3 - x^2}{1 + x^2} \blacksquare$$

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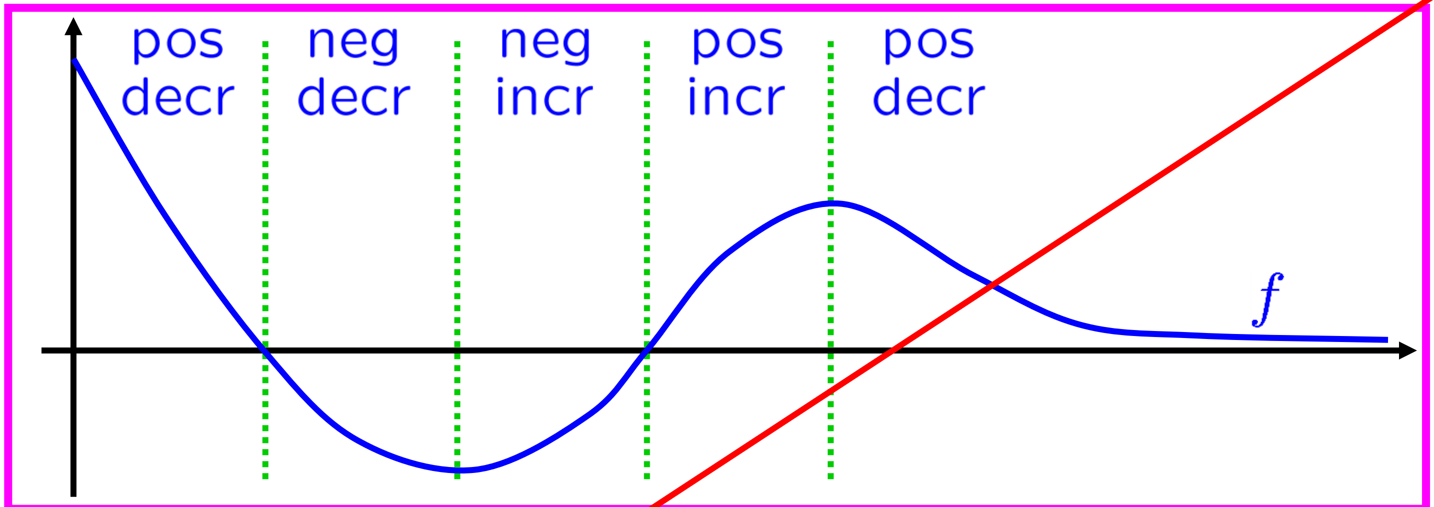
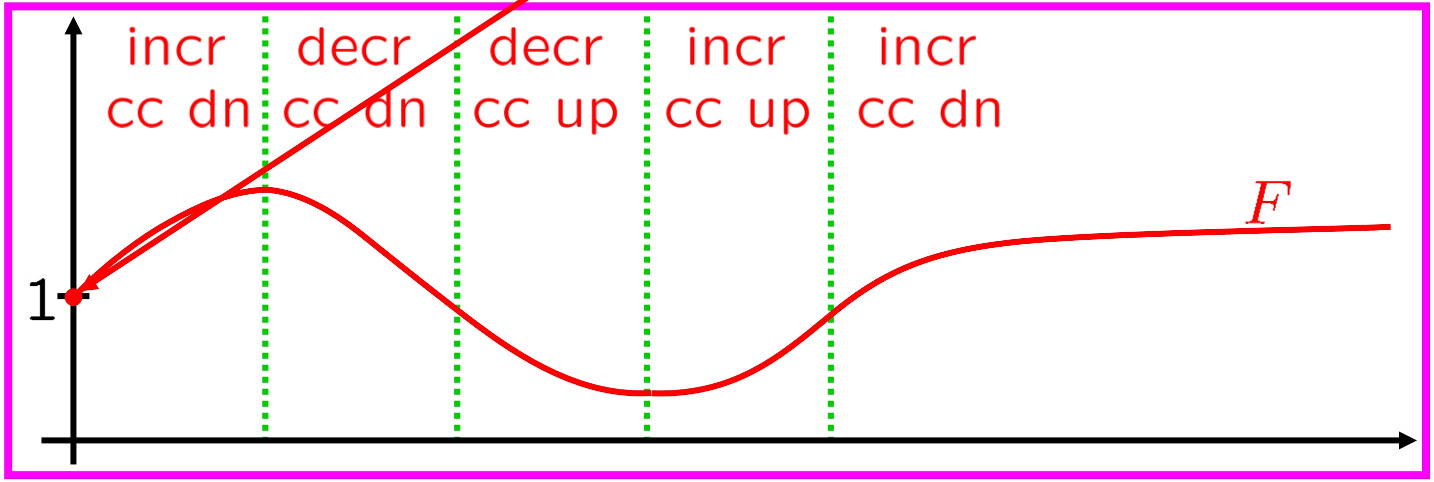
all antiderivs

§8.1

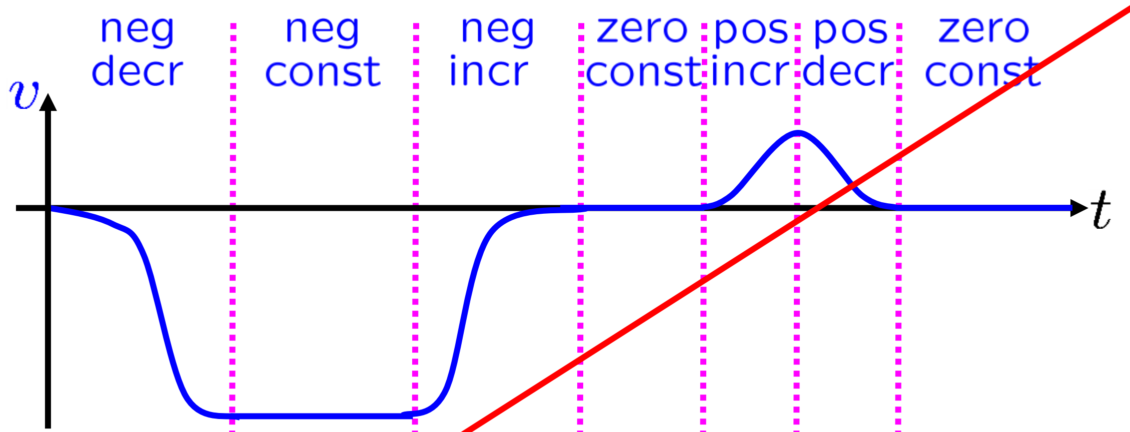
EXAMPLE: The graph of a function f is given below. Make a rough sketch of the antiderivative F of f on $[0, \infty)$ such that $F(0) = 1$.



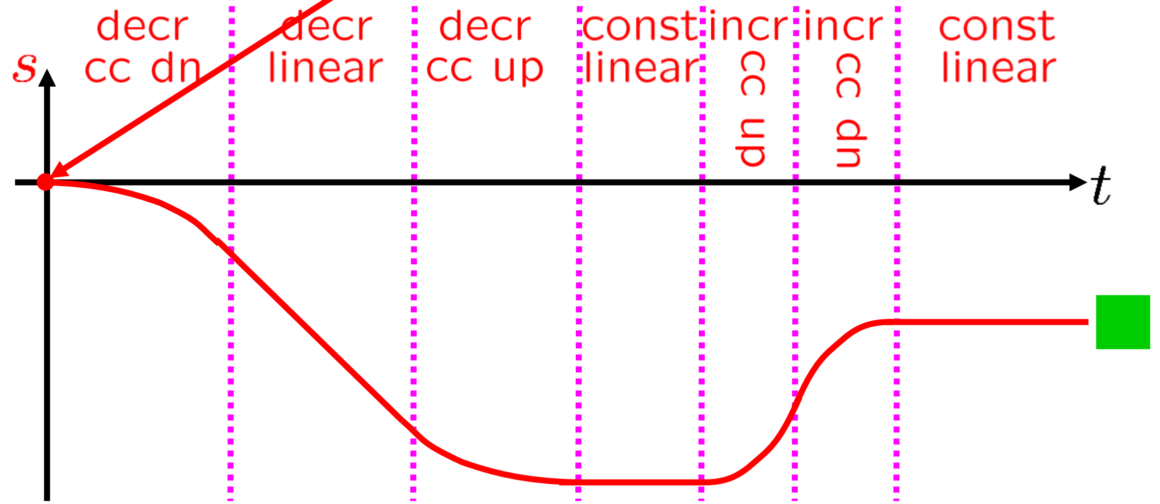


SKILL antideriv from gph

EXAMPLE: A particle is moving on a number line. The graph of its velocity function is shown below. Sketch the graph of its position function, assuming its position at time 0 is 0.



$s(t) :=$ position at time t $\qquad (s(t))^\bullet = v(t)$



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antideriv from gph

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EXAMPLE: Find the antiderivative $F(x)$ of

$$f(x) = 5 + 2(1 + x^2)^{-1}$$

that satisfies $F(1) = -1$. Check your answer by comparing the graphs of f and F.

$$F(x) = 5x + 2(\arctan(x)) - 6 - \frac{\pi}{2}$$

ANTIDIFF

$$\left[F(x) = 5x + 2(\arctan(x)) + C\right]_{x:\to 1}$$

$$-1 = F(1) = 5 + 2\left(\frac{\pi}{4}\right) + C$$

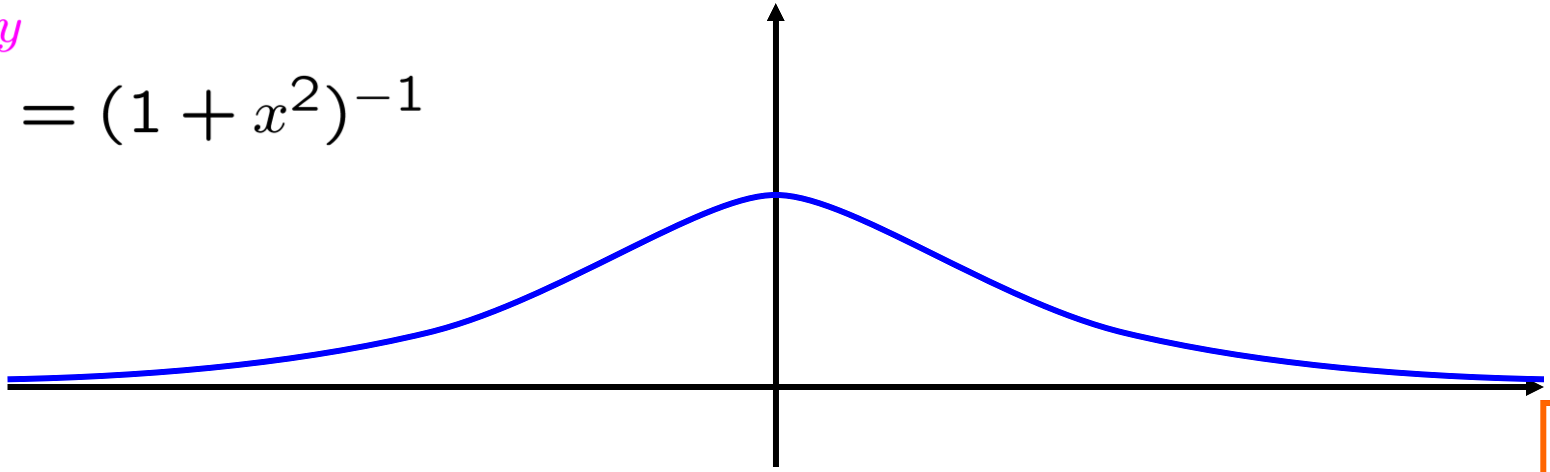
$$-6 - \frac{\pi}{2} = C$$

Exercise with hints:
Graph $y = f(x)$ . . .

$$y = (1 + x^2)^{-1}$$

$y :\to \frac{1}{2}y$

$$\frac{1}{2}y = (1 + x^2)^{-1}$$



EXAMPLE: Find the antiderivative $F(x)$ of

$$f(x) = 5 + 2(1 + x^2)^{-1}$$

that satisfies $F(1) = -1$. Check your answer by comparing the graphs of f and F.

$$F(x) = 5x + 2(\text{arctan}(x)) - 6 - \frac{\pi}{2}$$

$$F(x) = 5x + 2(\text{arctan}(x)) + C$$

$$-1 = F(1) = 5 + 2\left(\frac{\pi}{4}\right) + C$$

$$-6 - \frac{\pi}{2} = C$$

Exercise with hints: Graph $y = f(x)$ . . .

$$y = (1 + x^2)^{-1}$$

$y :\to \frac{1}{2}y$

$$\frac{1}{2}y = (1 + x^2)^{-1} \quad \Leftrightarrow \quad y = 2(1 + x^2)^{-1}$$

$y :\to y - 5$

$$y - 5 = 2(1 + x^2)^{-1}$$

$$\Updownarrow$$

$$y = 5 + 2(1 + x^2)^{-1}$$

EXAMPLE: Find the antiderivative $F(x)$ of

$$f(x) = 5 + 2(1 + x^2)^{-1}$$

that satisfies $F(1) = -1$. Check your answer by comparing the graphs of f and F.

$$F(x) = 5x + 2(\arctan(x)) - 6 - \frac{\pi}{2}$$

$$F(x) = 5x + 2(\arctan(x)) + C$$

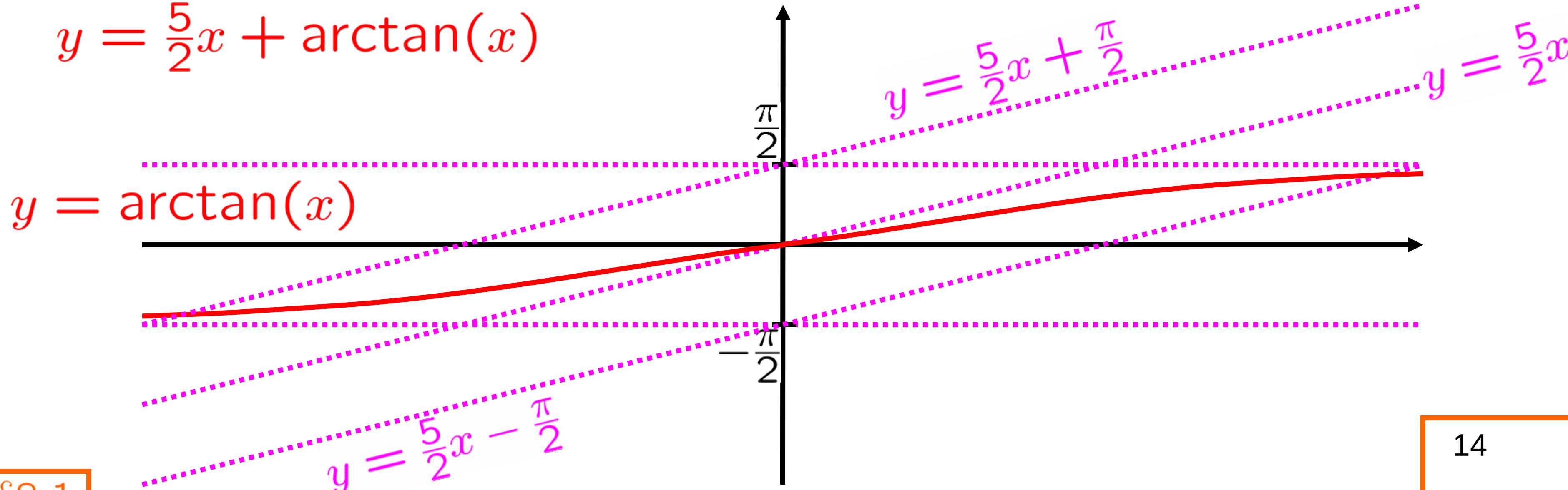
$$-1 = F(1) = 5 + 2\left(\frac{\pi}{4}\right) + C$$

$$-6 - \frac{\pi}{2} = C$$

Exercise with hints: Graph $y = F(x)$ ...

Next:

$y = \frac{5}{2}x + \arctan(x)$



EXAMPLE: Find the antiderivative $F(x)$ of

$$f(x) = 5 + 2(1 + x^2)^{-1}$$

that satisfies $F(1) = -1$. Check your answer by comparing the graphs of f and F.

$$F(x) = 5x + 2(\text{arctan}(x)) - 6 - \tfrac{\pi}{2}$$

$$F(x) = 5x + 2(\text{arctan}(x)) + C$$

$$-1 = F(1) = 5 + 2\left(\tfrac{\pi}{4}\right) + C$$

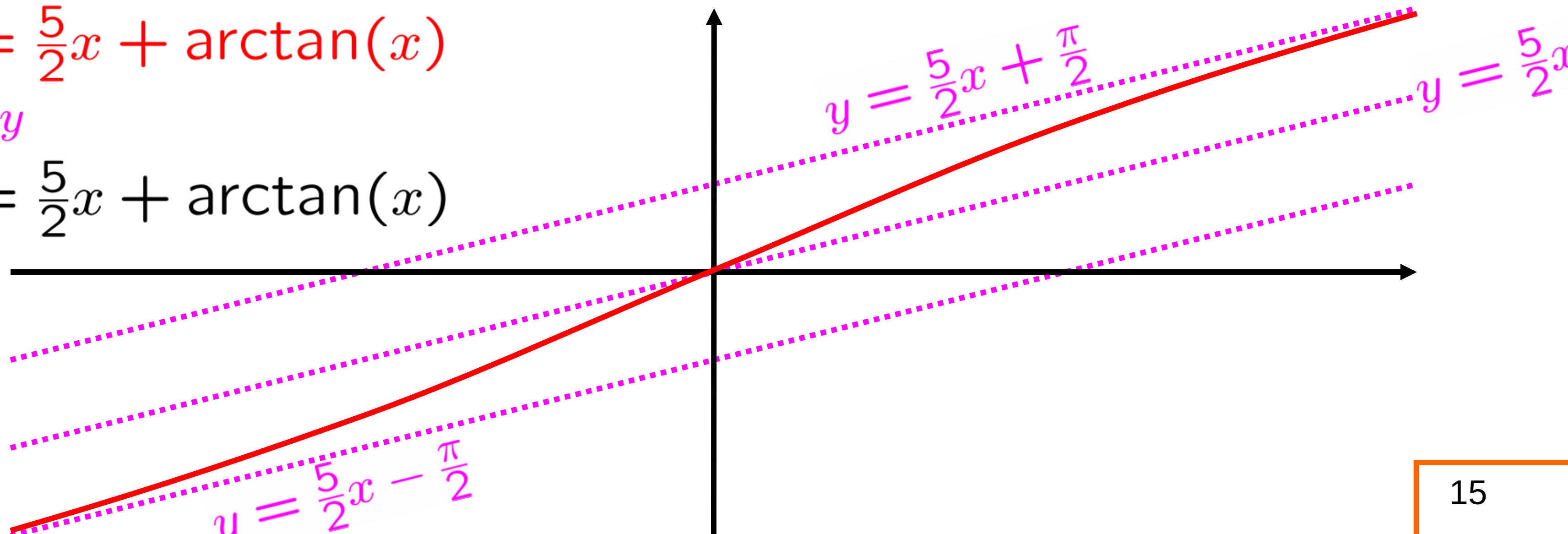
$$-6 - \tfrac{\pi}{2} = C$$

Exercise with hints:
Graph $y = F(x)$ . . .

$y = \frac{5}{2}x + \text{arctan}(x)$

$y :\to \frac{1}{2}y$

$\frac{1}{2}y = \frac{5}{2}x + \text{arctan}(x)$



EXAMPLE: Find the antiderivative $F(x)$ of

$$f(x) = 5 + 2(1 + x^2)^{-1}$$

that satisfies $F(1) = -1$. Check your answer by comparing the graphs of f and F.

$$F(x) = 5x + 2(\arctan(x)) - 6 - \tfrac{\pi}{2}$$

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one antideriv

$$F(x) = 5x + 2(\arctan(x)) + C$$

$$-1 = F(1) = 5 + 2\left(\tfrac{\pi}{4}\right) + C$$

$$-6 - \tfrac{\pi}{2} = C$$

Exercise with hints:
Graph $y = F(x)$ . . .

$$y = \tfrac{5}{2}x + \arctan(x)$$

$y :\to \tfrac{1}{2}y$

$$\tfrac{1}{2}y = \tfrac{5}{2}x + \arctan(x) \Leftrightarrow y = 5x + 2(\arctan(x))$$

$y :\to y + 6 + \tfrac{\pi}{2}$

$$y + 6 + \tfrac{\pi}{2} = 5x + 2(\arctan(x))$$

$$\Updownarrow$$

$$y = 5x + 2(\arctan(x)) - 6 - \tfrac{\pi}{2}$$

SKILL
graphing

EXAMPLE: Find the set of all functions f s.t.

$$f''(x) = 2 - 40x^3 + 56x^6.$$

$$f''(x) = 56x^6 - 40x^3 + 2$$

ANTIDIFF

$$f'(x) = 56\left[\frac{x^7}{7}\right] - 40\left[\frac{x^4}{4}\right] + 2x + C$$

$$= 8x^7 - 10x^4 + 2x + C$$

ANTIDIFF

$$f(x) = 8\left[\frac{x^8}{8}\right] - 10\left[\frac{x^5}{5}\right] + 2\left[\frac{x^2}{2}\right] + Cx + D$$

$$= x^8 - 2x^5 + x^2 + Cx + D \blacksquare$$

$$f(x) \in \{x^8 - 2x^5 + x^2 + Cx + D \,|\, C, D \in \mathbb{R}\}$$

SKILL
all double antiderivs

EXAMPLE: Find the set of all functions f s.t.

$$f''(t) = 12t - (15/4)\sqrt{t}.$$

$$f''(t) = 12t - (15/4)t^{1/2}$$

ANTIDIFF
$$f'(t) = 12\left[\frac{t^2}{2}\right] - (15/4)\left[\frac{t^{3/2}}{3/2}\right] + C$$
$$= 6t^2 - (5/2)t^{3/2} + C$$

ANTIDIFF
$$f(t) = 6\left[\frac{t^3}{3}\right] - \cancel{(5/2)}\left[\frac{t^{5/2}}{\cancel{5/2}}\right] + Ct + D$$
$$= 2t^3 - t^{5/2} + Ct + D$$
$$= 2t^3 - \sqrt{t^5} + Ct + D \quad \blacksquare$$

$$f(t) \in \left\{2t^3 - \sqrt{t^5} + Ct + D \,\middle|\, C, D \in \mathbb{R}\right\}$$

SKILL all double antiderivs

EXAMPLE: Find the function f s.t.

$$f''(x) = 2 - 6x + 80x^3, \quad f(0) = -4 \quad \text{and} \quad f'(0) = 4.$$

$$f''(x) = 80x^3 - 6x + 2$$

ANTIDIFF

$$f'(x) = 20x^4 - 3x^2 + 2x + 4$$

ANTIDIFF

$$f(x) = 4x^5 - x^3 + x^2 + 4x - 4 \blacksquare$$

SKILL
one double antideriv

EXAMPLE: Find the function f s.t.

$$f''(t) = 3e^t + 4\cos(2t), \quad f(0) = 3 \quad \text{and} \quad f'(\pi) = 0.$$

ANTIDIFF

$$f'(t) = 3e^t + 4\left[\frac{\sin(2t)}{2}\right] + C$$

$$= 3e^t + 2\sin(2t) + C$$

$t{:}\to\pi$

$$= 3e^t + 2\sin(2t) - 3e^\pi$$

$$-3e^\pi = C$$

$$0 = 3e^\pi + 2\sin(2\pi) + C$$

$$= 3e^\pi + C$$

EXAMPLE: Find the function f s.t.

$$f''(t) = 3e^t + 4\cos(2t), \quad f(0) = 3 \quad \text{and} \quad f'(\pi) = 0.$$

$$f'(t) = 3e^t + 2\sin(2t) - 3e^\pi$$

ANTIDIFF

e^t

$$= 3e^t + 2\sin(2t) - 3e^\pi$$

EXAMPLE: Find the function f s.t.

$$f''(t) = 3e^t + 4\cos(2t), \quad f(0) = 3 \quad \text{and} \quad f'(\pi) = 0.$$

$$f'(t) = 3e^t + 2\sin(2t) - 3e^\pi$$

ANTIDIFF

$$f(t) = 3e^t + 2\left[\frac{-\cos(2t)}{2}\right] - 3e^\pi t + D$$

$$= 3e^t - \cos(2t) - 3e^\pi t + D$$

$t{:}\to 0$

$$= 3e^t - \cos(2t) - 3e^\pi t + 1$$

SKILL
one double antideriv

$$1 = D$$

STOP

$$3 = 3e^0 - \cos(2\cdot 0) - 3e^\pi \cdot 0 + D$$
$$= 3 - 1 - 0 + D$$
$$= 2 + D$$