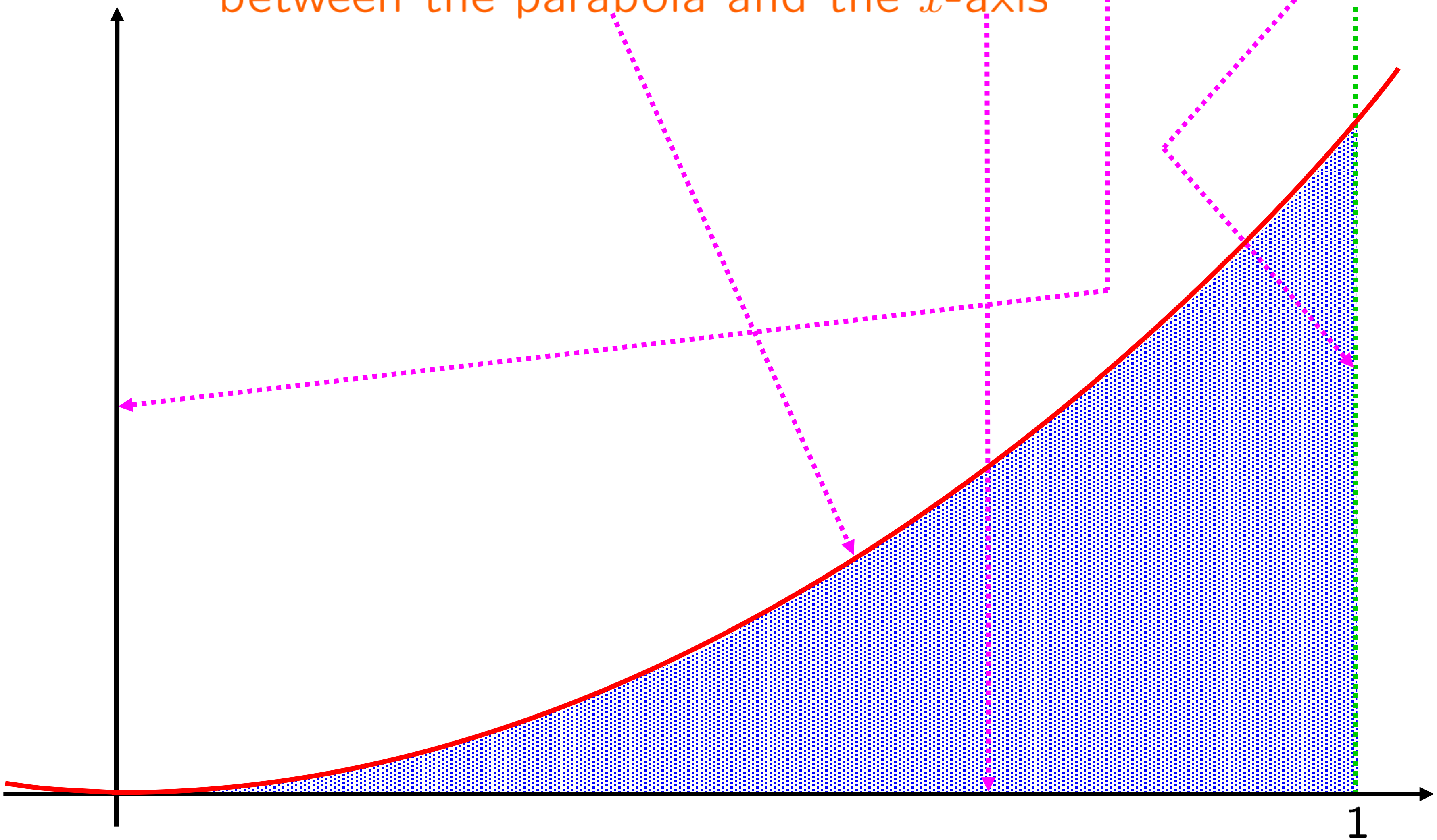
CALCULUS
Riemann sums and the definition of the definite integral

EXAMPLE: Use rectangles to estimate the area "under" the parabola $y = x^2$ from $x = 0$ to $x = 1$.

between the parabola and the x-axis



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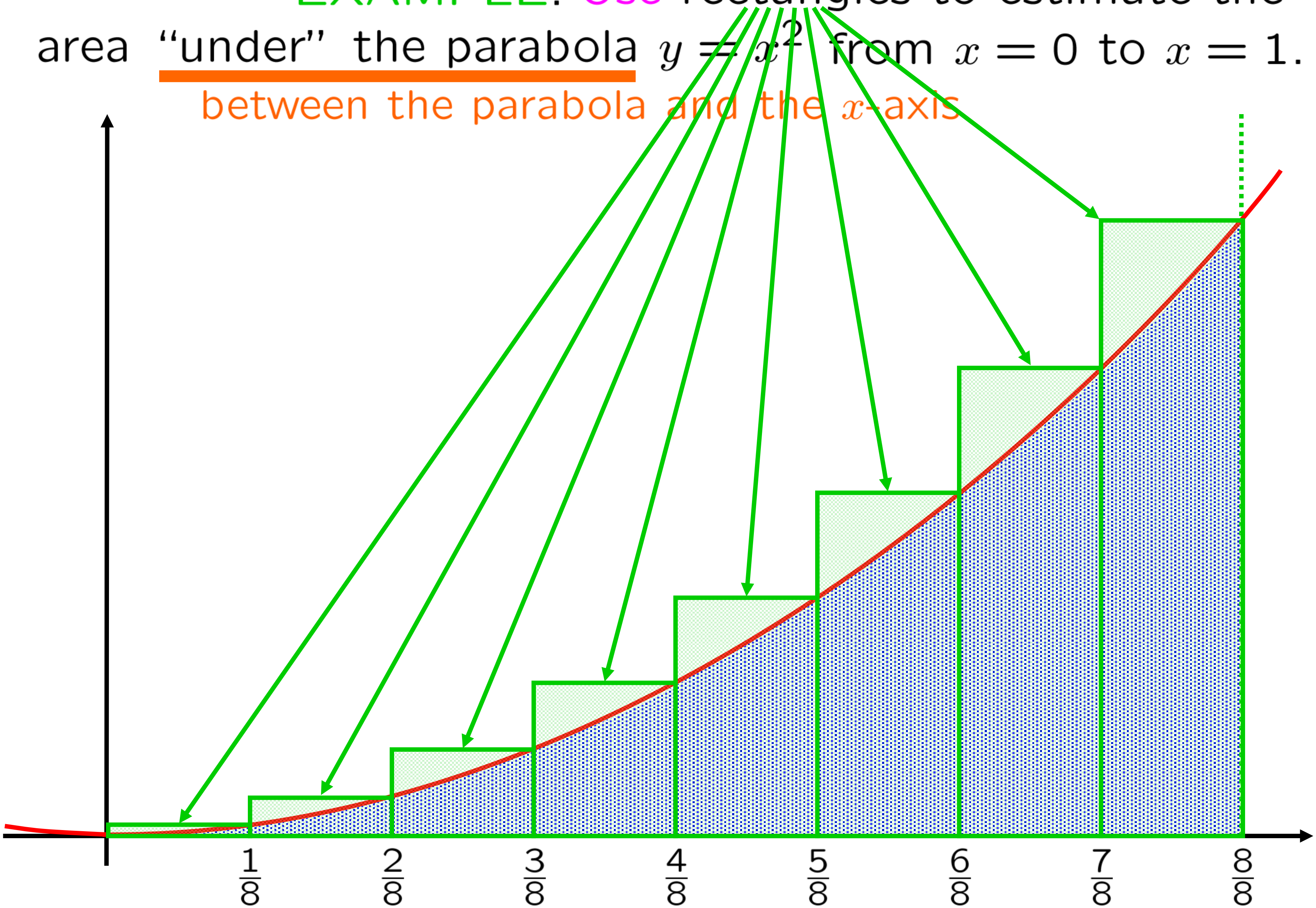


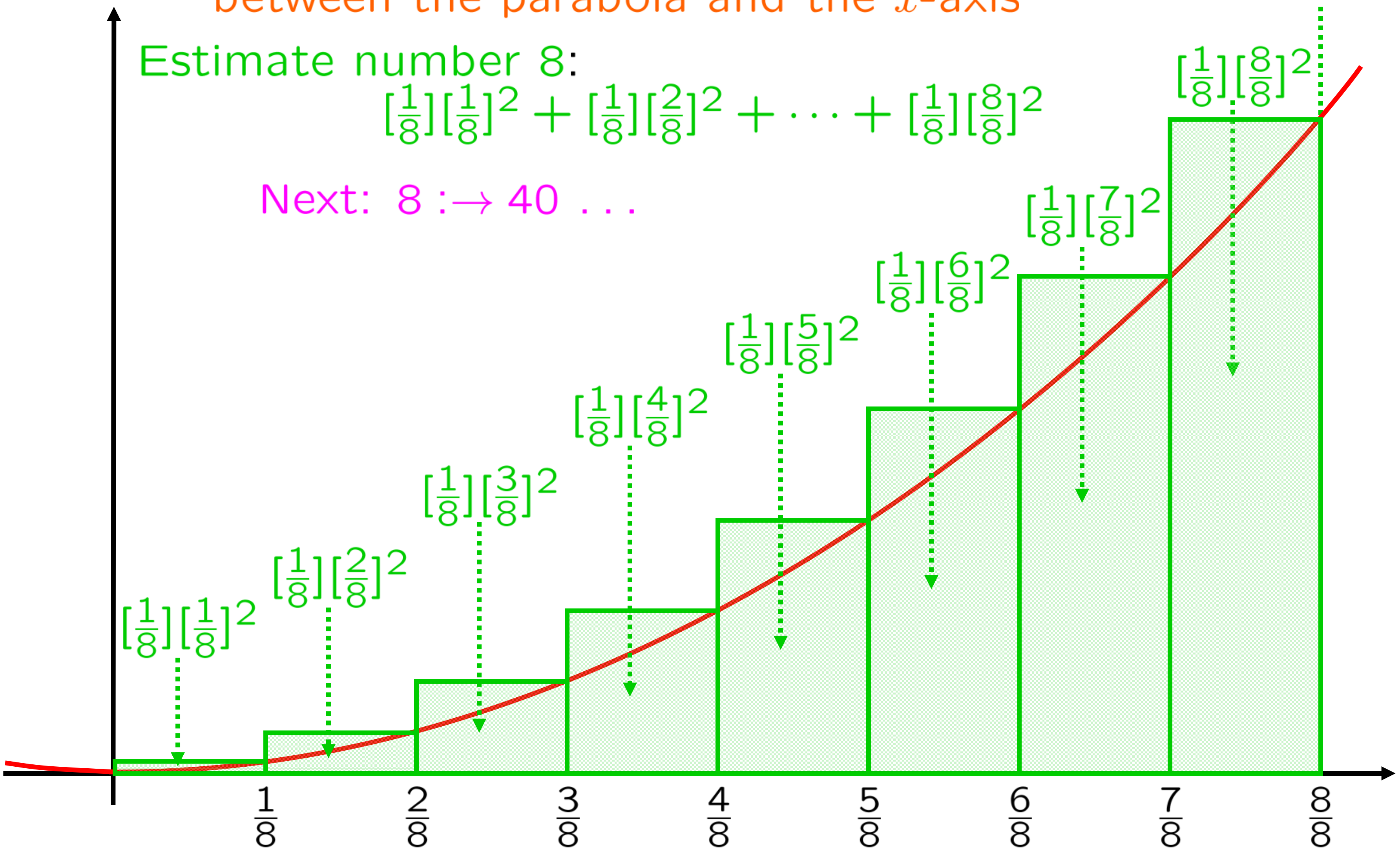
EXAMPLE: Use rectangles to estimate the area "under" the parabola $y = x^2$ from $x = 0$ to $x = 1$.

between the parabola and the x-axis

Estimate number 8:

$$[\tfrac{1}{8}][\tfrac{1}{8}]^2 + [\tfrac{1}{8}][\tfrac{2}{8}]^2 + \cdots + [\tfrac{1}{8}][\tfrac{8}{8}]^2$$

Next: $8 :\to 40 \ldots$



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Estimate number 40:

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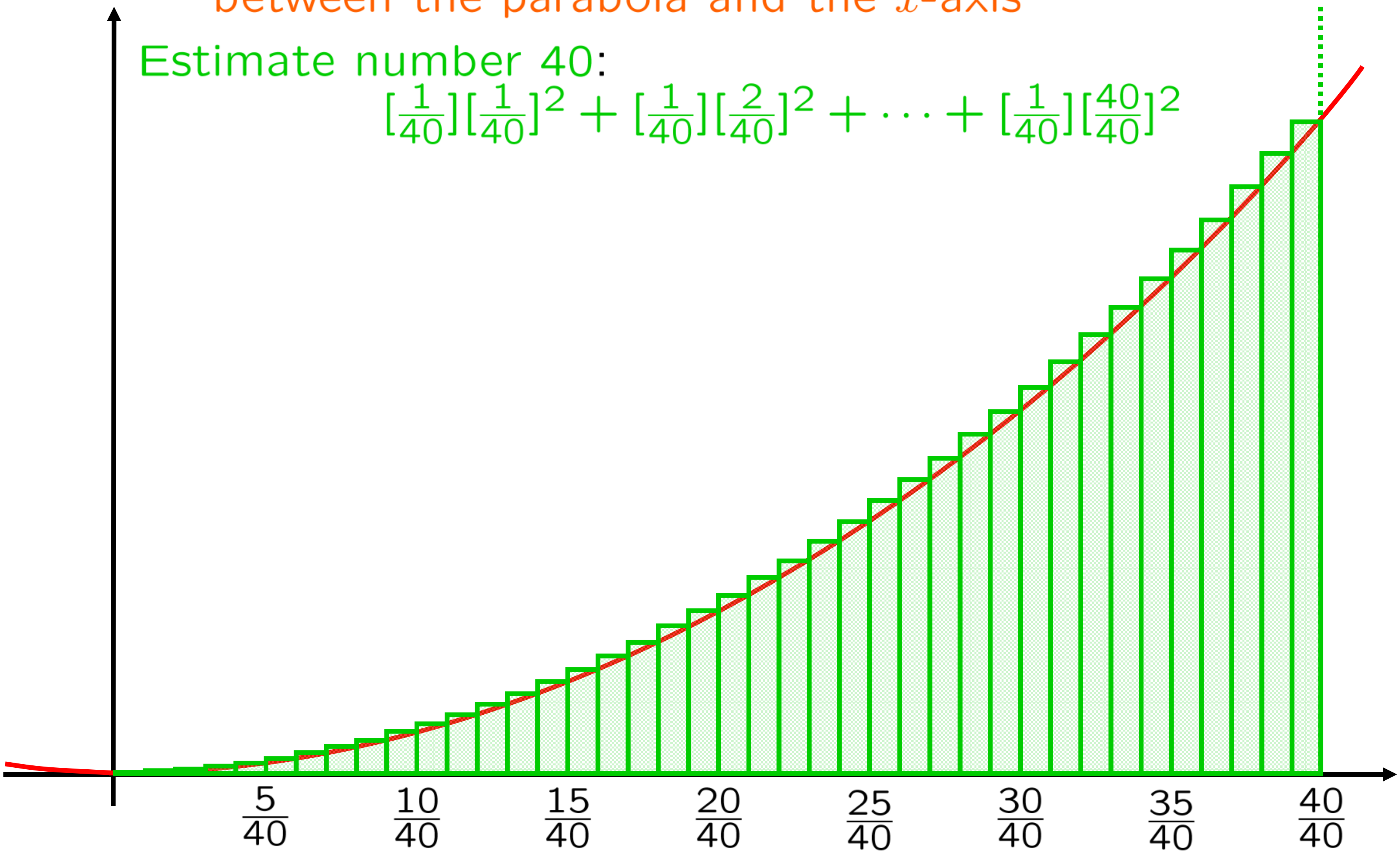

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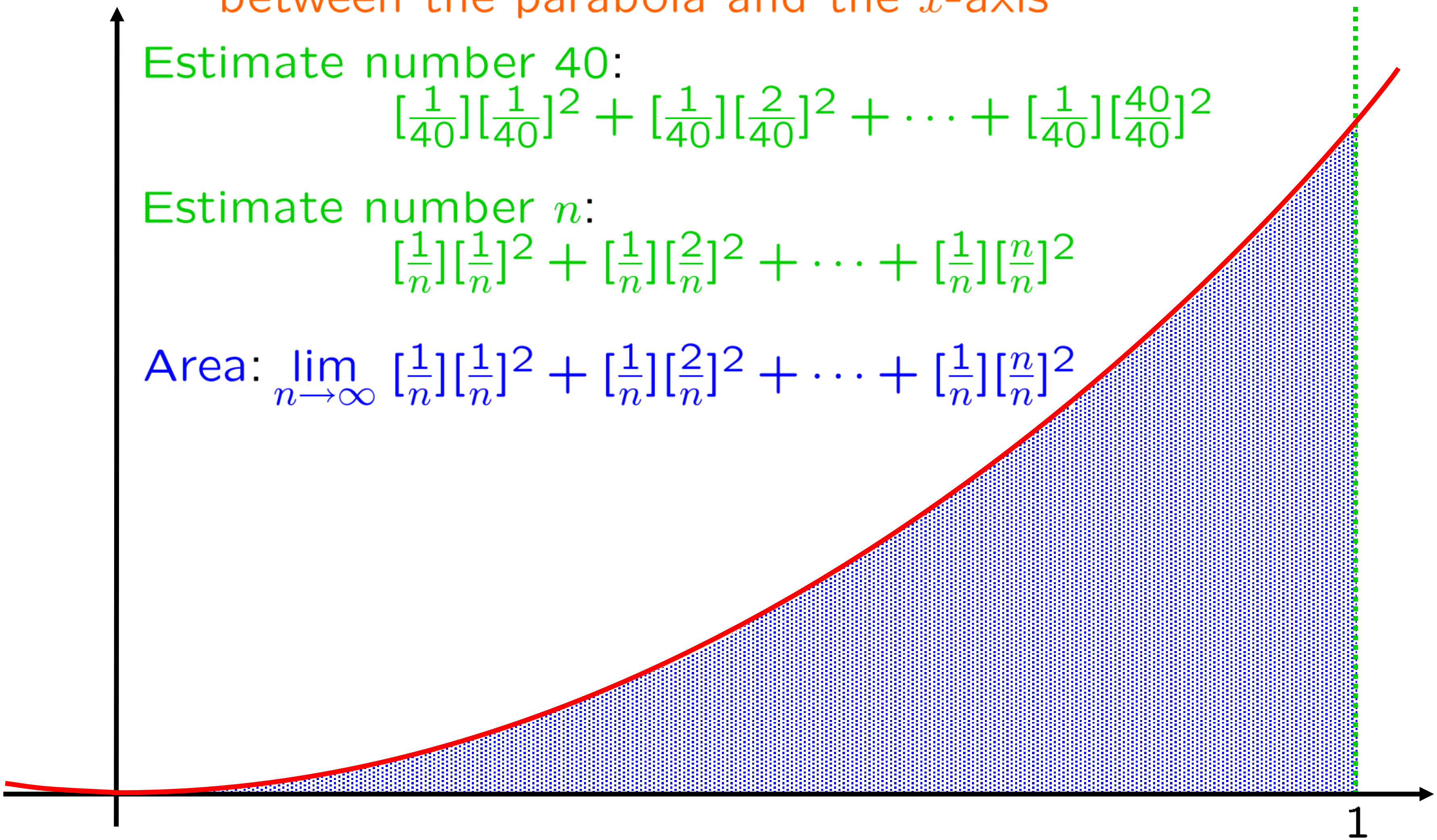
Estimate number 40:

$$[\tfrac{1}{40}][\tfrac{1}{40}]^2 + [\tfrac{1}{40}][\tfrac{2}{40}]^2 + \cdots + [\tfrac{1}{40}][\tfrac{40}{40}]^2$$

Estimate number n:

$$[\tfrac{1}{n}][\tfrac{1}{n}]^2 + [\tfrac{1}{n}][\tfrac{2}{n}]^2 + \cdots + [\tfrac{1}{n}][\tfrac{n}{n}]^2$$

Area: $\lim_{n\to\infty} [\tfrac{1}{n}][\tfrac{1}{n}]^2 + [\tfrac{1}{n}][\tfrac{2}{n}]^2 + \cdots + [\tfrac{1}{n}][\tfrac{n}{n}]^2$



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$$= \lim_{n\to\infty} [\tfrac{1}{n}][\tfrac{1^2}{n^2}] + [\tfrac{1}{n}][\tfrac{2^2}{n^2}] + \cdots + [\tfrac{1}{n}][\tfrac{n^2}{n^2}]$$

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$= \lim_{n\to\infty} [\frac{1}{n}][\frac{1^2}{n^2}] + [\frac{1}{n}][\frac{2^2}{n^2}] + \cdots + [\frac{1}{n}][\frac{n^2}{n^2}]$

$= \lim_{n\to\infty} \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{n^2}{n^3}$

asymptotics

SAME DEGREE

$= \lim_{n\to\infty} \frac{1^2 + \cdots + n^2}{n^3} = \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{2}{6} = \frac{1}{3}$ ■

COMMON DENOMINATOR

Kinda hard. . .

$$1^2 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

EXAMPLE: Use rectangles to estimate the area "under" the parabola $y = x^2$ from $x = 0$ to $x = 1$.

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Kinda hard. . .

IOU: An easier approach, via the Fundamental Theorem of Calculus (Later topic.)

Next: General discussion of "area under a curve" . . .

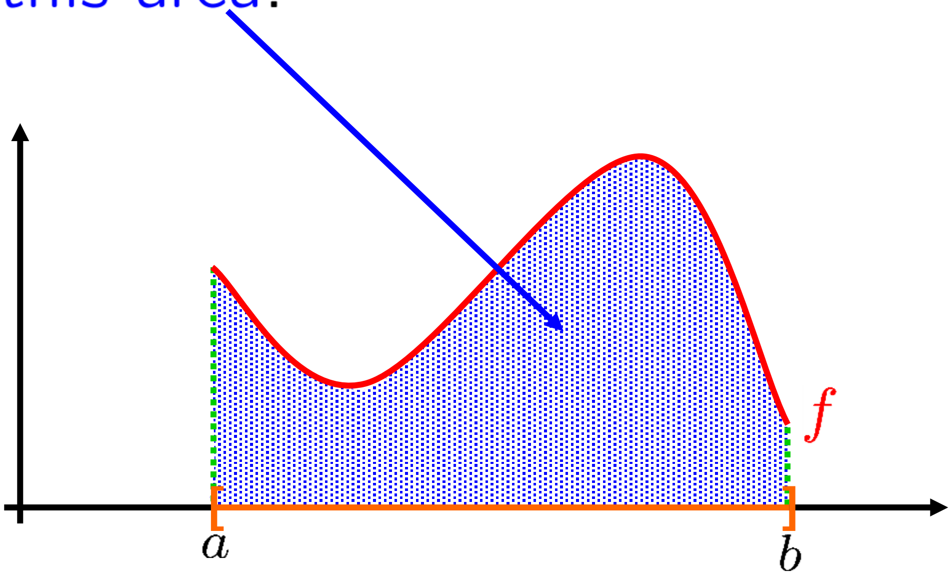
DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.

Let f be a function. Assume that f is continuous on $[a, b]$.

Next: Estimate with three rectangles . . .

3rd partition: Partition "the big interval" $[a, b]$
into three "subintervals" . . .
some terminology . . .

Goal: Find this area.



$$h_3 = \frac{b - a}{3}$$

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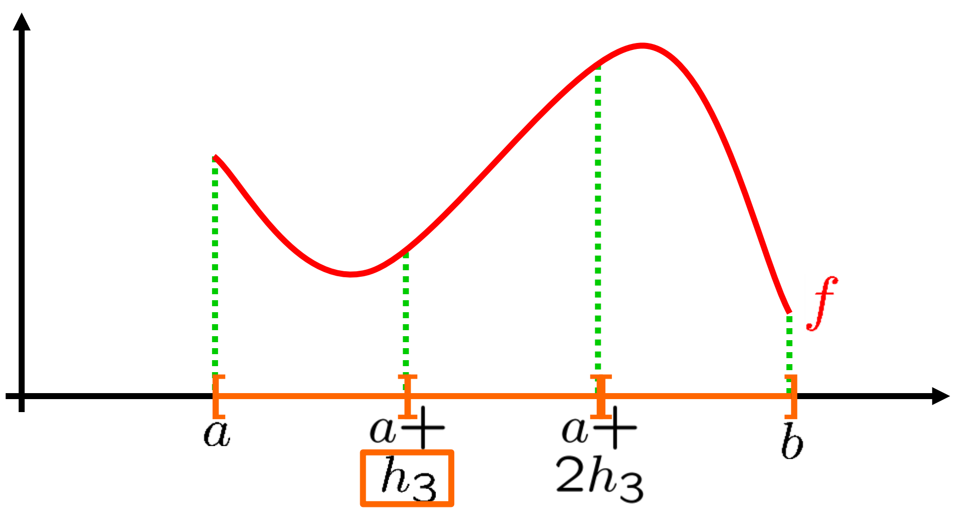

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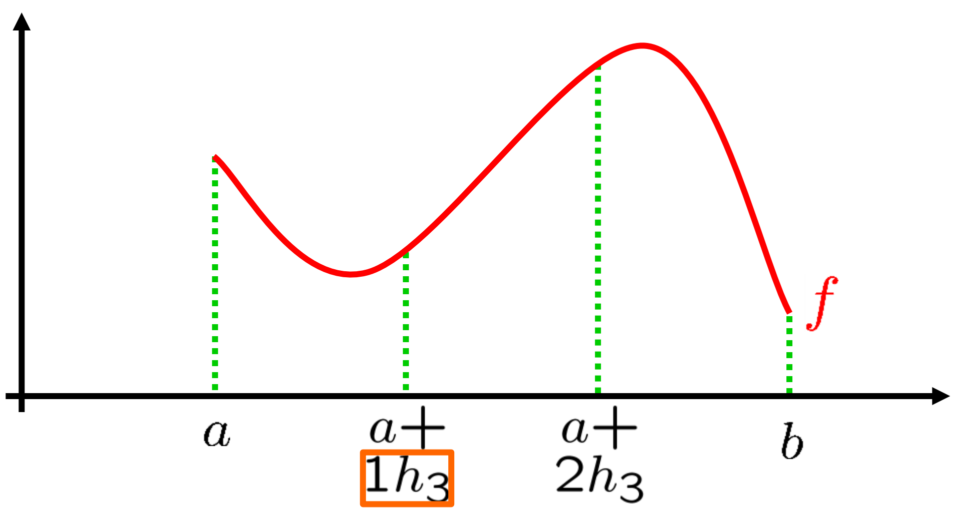

$$3h_3 = b - a$$
$$a + 3h_3 = b$$

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some terminology . . .

partition of $[a, b]$
into three subintervals
all of length h_3

3rd subinterval
3rd partition
1 convention

left endpoint

midpoint

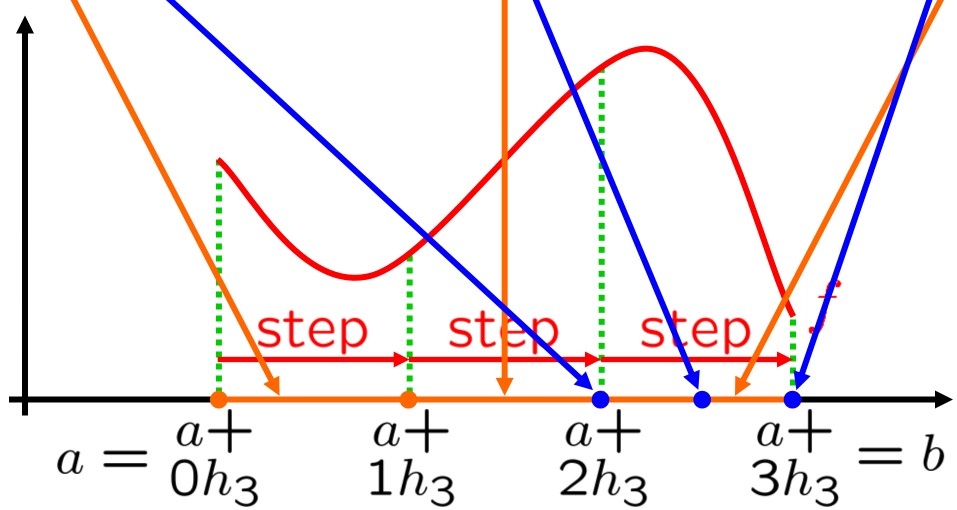
right endpoint

first subinterval

second subinterval

third subinterval

The "1 convention"



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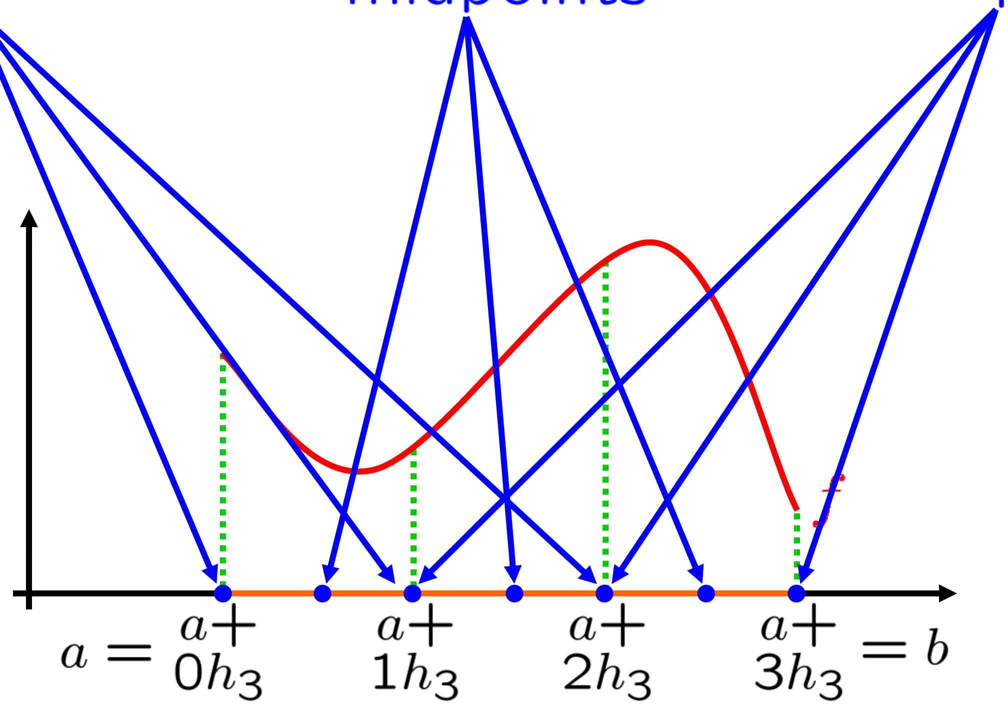
left endpoints midpoints right endpoints



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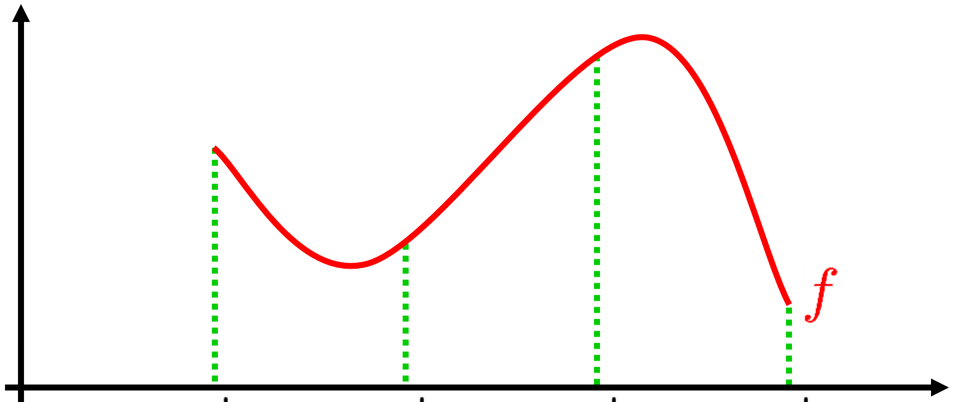
Next: Estimate with three rectangles ...

3rd partition: Partition "the big interval" $[a, b]$
into three "subintervals" ...

Next: 10th partition of $[a, b]$...

3rd partition of $[a, b]$



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DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.
Let f be a function. Assume that f is continuous on $[a, b]$.
$\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

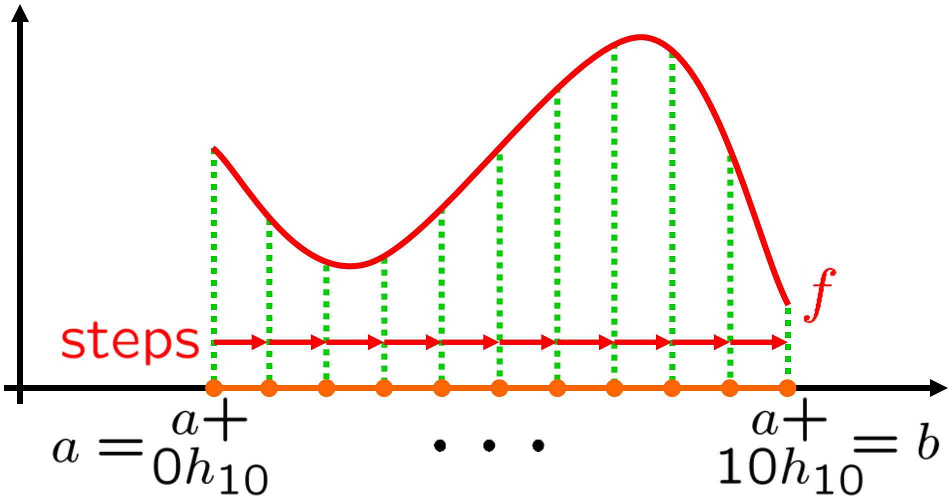
width of the subintervals in the nth partition

WARNING: h is for "horizontal", not "height"

10th partition of $[a, b]$

width of the subintervals in the 10th partition

$$h_{10} = \frac{b-a}{10}$$



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Alternate notation: $\triangle x$ instead of h_n

width of the subintervals in the nth partition

WARNING: h is for "horizontal", not "height"

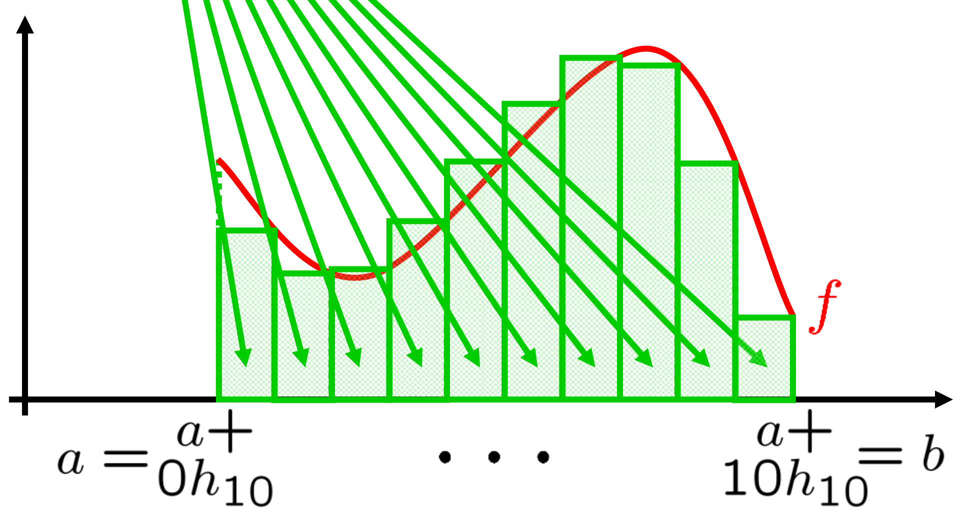
Back to the 3rd partition. . .

These rectangles have width h_{10}, not height.

10th partition of $[a, b]$

width of the subintervals in the 10th partition

$$h_{10} = \frac{b - a}{10}$$



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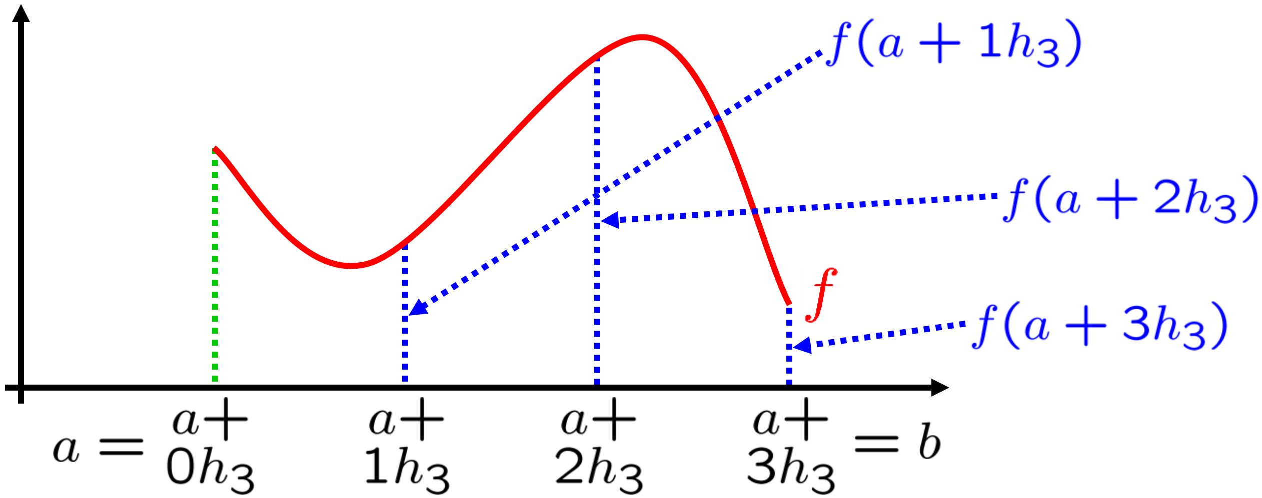

Values of f at the right endpts

$$h_3 = \frac{b - a}{3}$$

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$\forall$integers $n \geq 1$, let $h_n := (b-a)/n$,

DOES NOT DEPEND ON j

if desired

let $$\boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)]$$

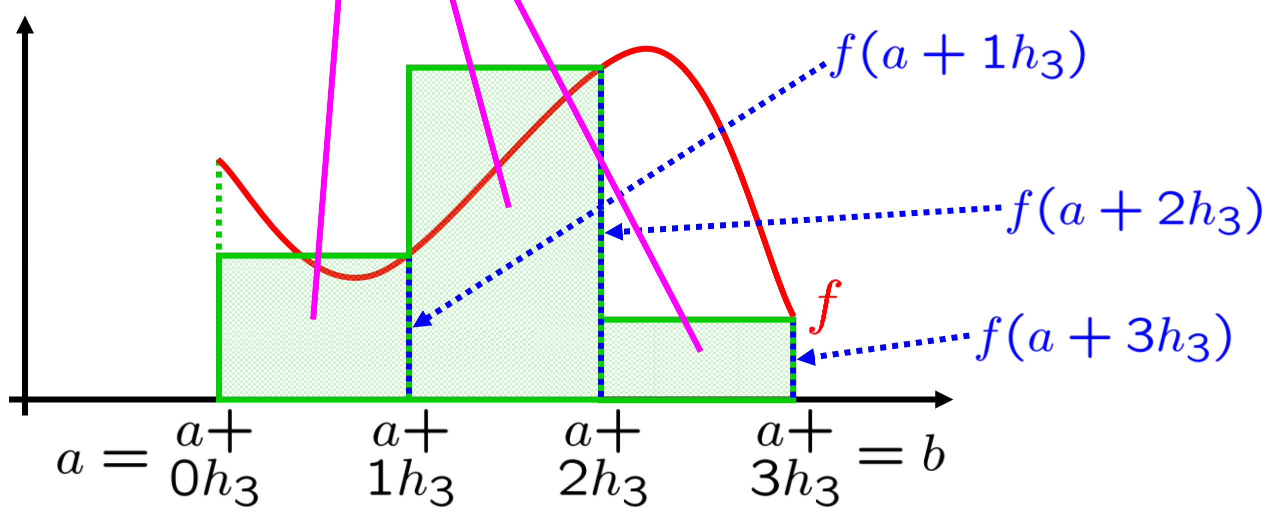
Right 3rd Riemann Sum from a to b of f

DOES NOT DEPEND ON j

if desired

$$R_3 S_a^b f = \text{total shaded area} = \left\{ \begin{array}{c} [h_3][f(a+1h_3)] \\ + \\ [h_3][f(a+2h_3)] \\ + \\ [h_3][f(a+3h_3)] \end{array} \right\} = \sum_{j=1}^{3} [h_3][f(a+jh_3)]$$

COMMON FACTOR



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DOES NOT DEPEND ON j

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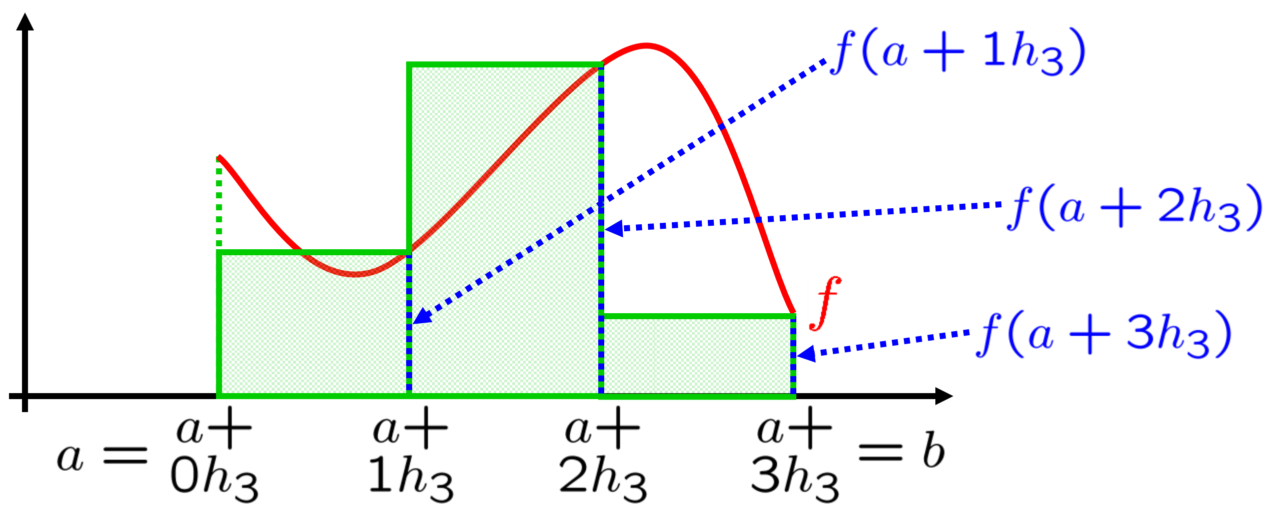
let $\boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)]$

$$= h_n \sum_{j=1}^{n} f(a + jh_n)$$

Next: Midpoint Riemann sums . . .



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$$\| $$

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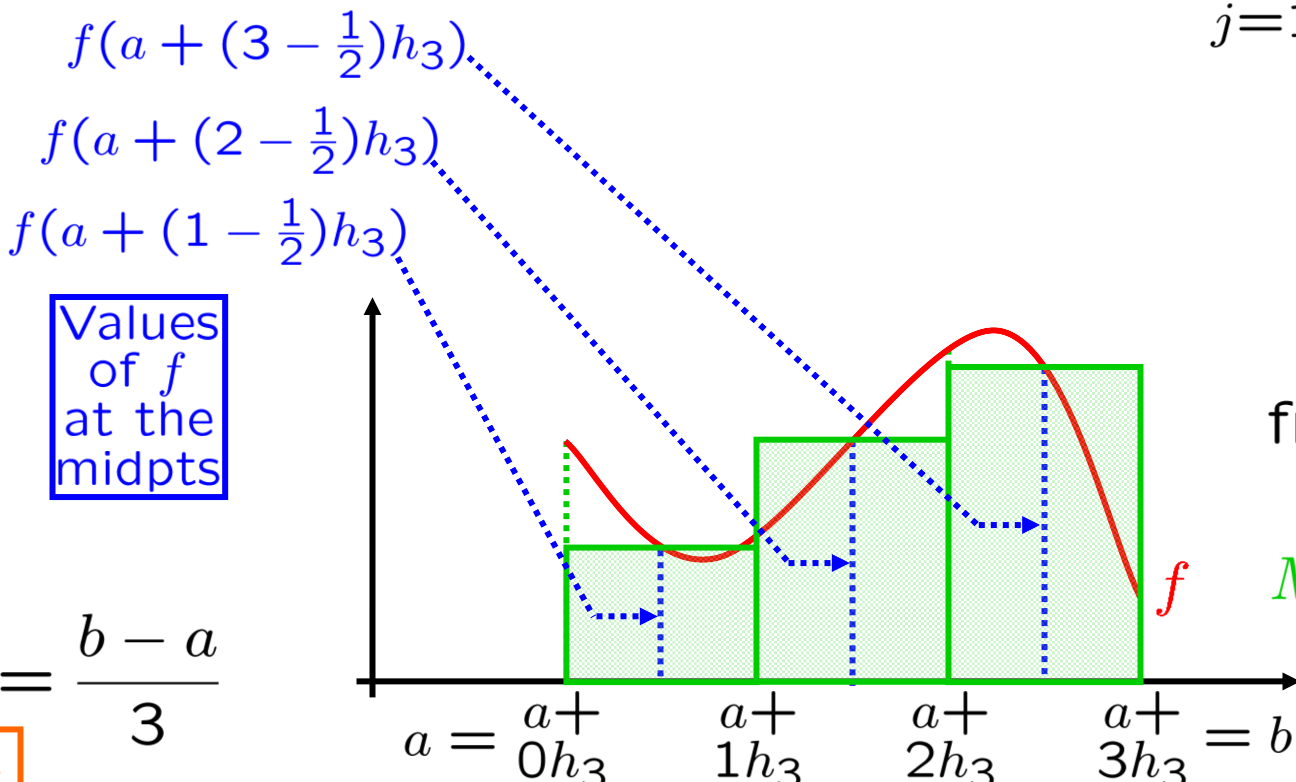

Midpoint 3rd Riemann Sum from a to b of f

$M_3 S_a^b f =$ total shaded area

$$h_3 = \frac{b - a}{3}$$

§7.1

DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.
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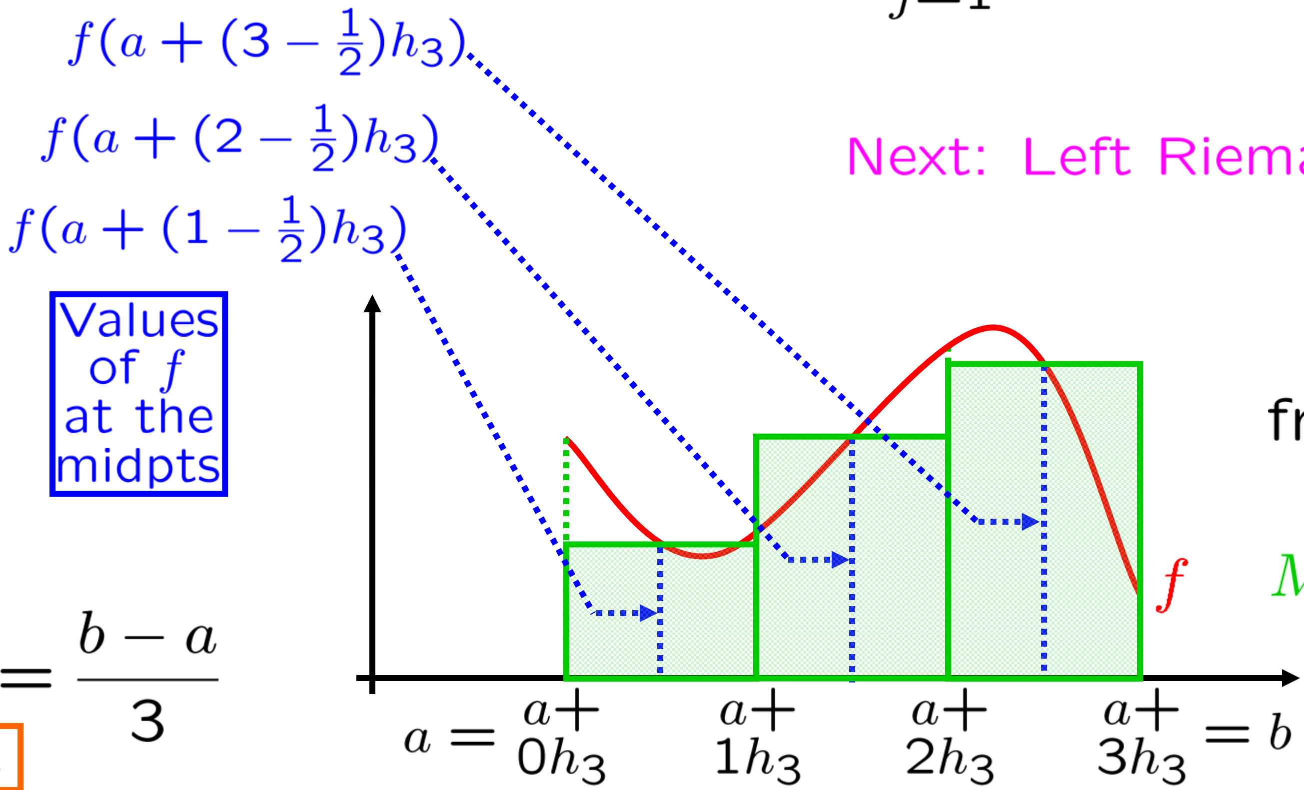
let $\boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \frac{1}{2})h_n)]$

Next: Left Riemann sums . . .



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$M_3 S_a^b f =$ total shaded area

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$\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

$$\text{let } \boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)],$$

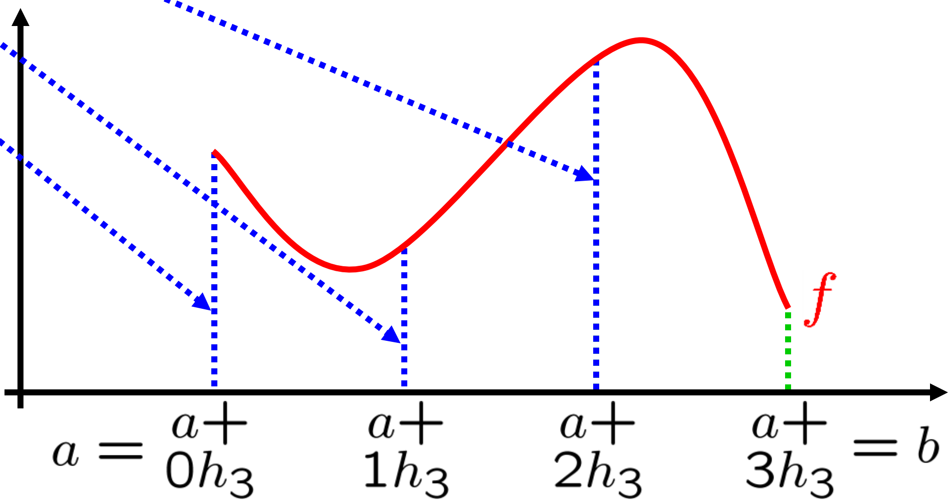
$$\text{let } \boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \tfrac{1}{2})h_n)]$$

Values of f at the left endpts

$f(a + (3 - 1)h_3)$

$f(a + (2 - 1)h_3)$

$f(a + (1 - 1)h_3)$



$$h_3 = \frac{b - a}{3}$$

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DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.

Let f be a function. Assume that f is continuous on $[a, b]$.

$\forall$integers $n \geq 1$, let $h_n := (b-a)/n$,

let $\boxed{R_n S_a^b f} := \sum_{j=1}^{n} \boxed{[h_n]}[f(a + jh_n)]$, (if desired)

let $\boxed{M_n S_a^b f} := \sum_{j=1}^{n} \boxed{[h_n]}[f(a + (j - \frac{1}{2})h_n)]$ (if desired)

DOES NOT DEPEND ON j

& let $\boxed{L_n S_a^b f} := \sum_{j=1}^{n} \boxed{[h_n]}[f(a + (j-1)h_n)]$. (if desired)

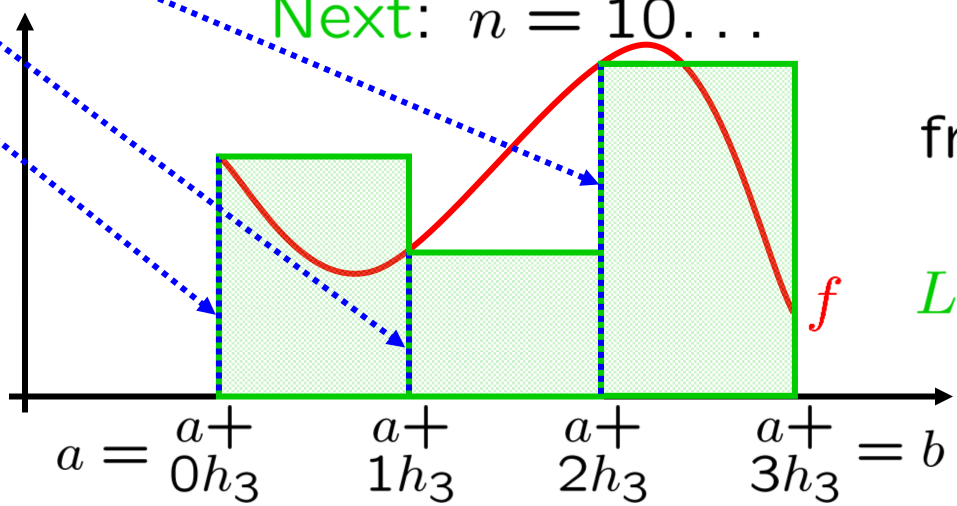
Values of f at the left endpts

$f(a + (3-1)h_3)$

$f(a + (2-1)h_3)$

$f(a + (1-1)h_3)$

Next: $n = 10\ldots$



Left 3rd Riemann Sum from a to b of f

$L_3 S_a^b f$ = total shaded area

$h_3 = \dfrac{b-a}{3}$

§7.1

DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.

Let f be a function. Assume that f is continuous on $[a, b]$.

$\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

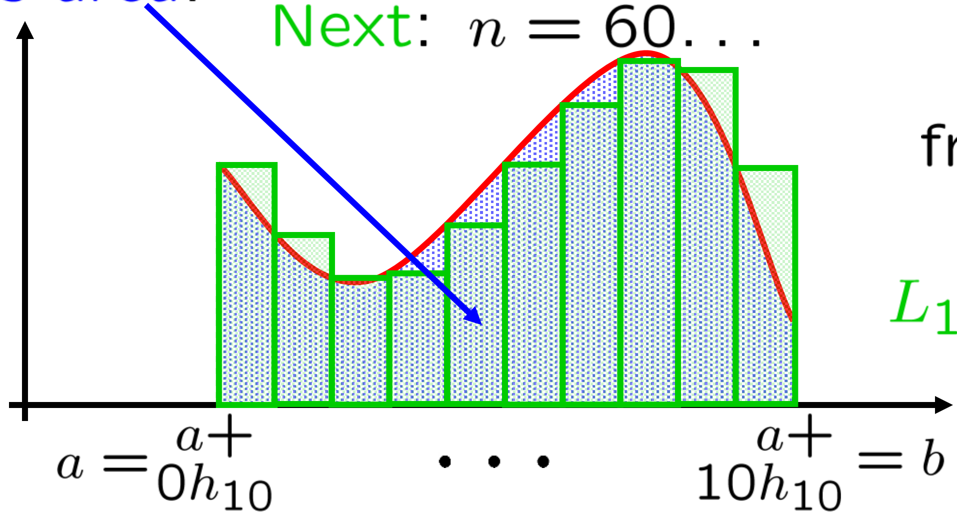
$$\text{let } \boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)],$$

$$\text{let } \boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \tfrac{1}{2})h_n)]$$

$$\text{\& let } \boxed{L_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - 1)h_n)].$$

Goal: Find this area.

Next: $n = 60\ldots$



Left 10th Riemann Sum from a to b of f

$L_{10}S_a^b f =$ total shaded area

$$h_{10} = \frac{b-a}{10}$$

§7.1

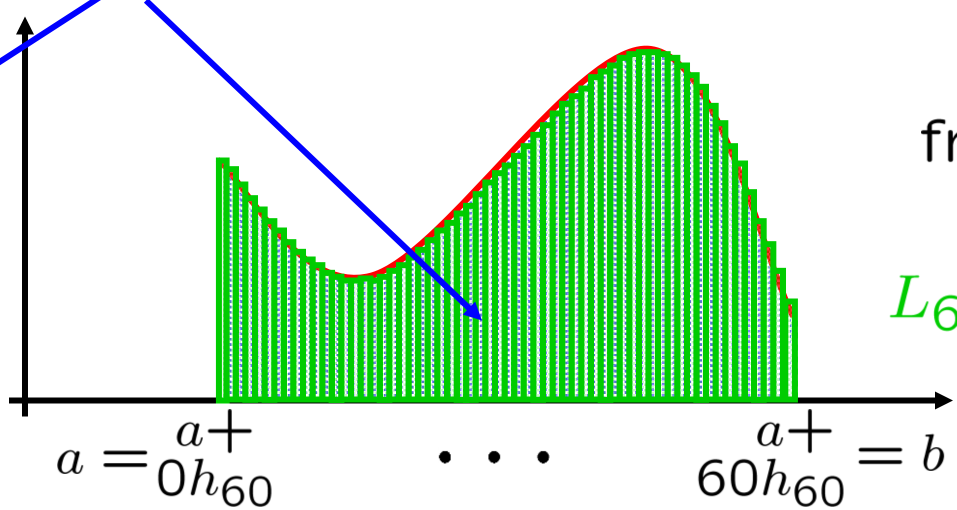
DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.
Let f be a function. Assume that f is continuous on $[a, b]$.
$\forall$integers $n \geq 1$, let $h_n := (b - a)/n$,

let $\boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)]$,

let $\boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \frac{1}{2})h_n)]$

& let $\boxed{L_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - 1)h_n)]$.

Goal: Find this area.
Take $\lim_{n \to \infty} \ldots$
Which one?



Left 60th Riemann Sum from a to b of f

$L_{60} S_a^b f$ = total shaded area

$h_{60} = \dfrac{b - a}{60}$

DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.
Let f be a function. Assume that f is continuous on $[a, b]$.
$\forall$integers $n \geq 1$, let $h_n := (b-a)/n$,

let $\boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)]$,

let $\boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \frac{1}{2})h_n)]$

& let $\boxed{L_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j-1)h_n)]$.

Theorem: If $a < b$, then

$$\lim_{n \to \infty} L_n S_a^b f = \lim_{n \to \infty} M_n S_a^b f = \lim_{n \to \infty} R_n S_a^b f$$

Note: True even if f has a finite number of jump discontinuities.

DEFINITION: Let $a, b \in \mathbb{R}$ satisfy $a < b$.
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let $$M_n S_a^b f := \sum_{j=1}^{n} [h_n][f(a + (j - \tfrac{1}{2})h_n)]$$

& let $$L_n S_a^b f := \sum_{j=1}^{n} [h_n][f(a + (j-1)h_n)].$$

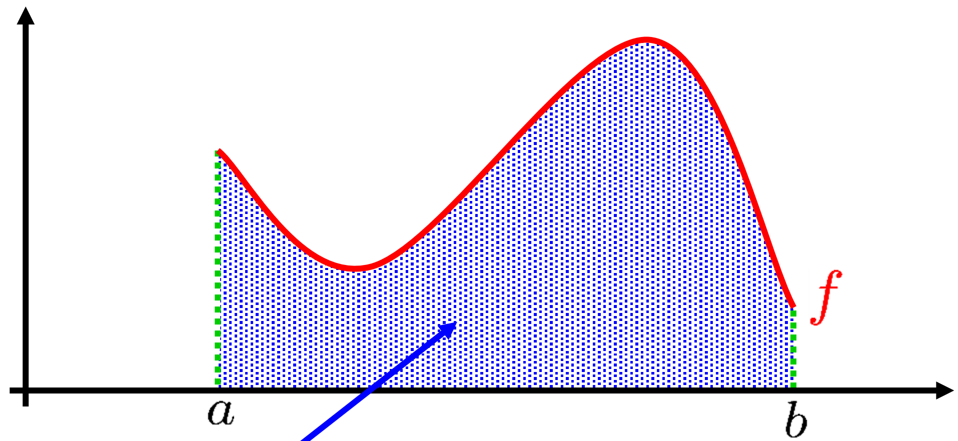
DEFINITION OF A DEFINITE INTEGRAL: If $a < b$, then

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} L_n S_a^b f = \lim_{n\to\infty} M_n S_a^b f = \lim_{n\to\infty} R_n S_a^b f$$

Note: True even if f has a finite number of jump discontinuities.

$$\int_a^b f(x)\,dx = \int_a^b f(v)\,dv = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds = \int_a^b f$$

Next: $\int_c^c$ and $\int_b^a$ . . .



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Note: True even if f has a finite number of jump discontinuities.

$$\int_a^b f(x)\,dx = \int_a^b f(v)\,dv = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds = \int_a^b f$$

Def'n: $\boxed{\int_c^c f(x)\,dx} := 0$

Next: $\int_c^c$ and $\int_b^a \ldots$

$$\boxed{\int_b^a f(x)\,dx} := -\int_a^b f(x)\,dx, \quad \text{if } a < b$$

Integrals of the form $\int f(x)\,dx$ are called **indefinite integrals**.
They are sets of expressions.

Integrals of the form $\int_a^b f(x)\,dx$ are called **definite integrals**.
They are numbers.

DEFINITION OF A DEFINITE INTEGRAL: If $a < b$, then

$$\boxed{\int_a^b f(x)\,dx} := \lim_{n\to\infty} L_n S_a^b f = \lim_{n\to\infty} M_n S_a^b f = \lim_{n\to\infty} R_n S_a^b f$$

Note: True even if f has a finite number of jump discontinuities.

ALGEBRA YIELDS SOME ALTERNATE VERSIONS:

$$L_n S_a^b f := \sum_{j=1}^{n} [h_n][f(a+(j-1)h_n)]$$

$$= h_n \sum_{j=1}^{n} f(a+(j-1)h_n) = h_n \sum_{j=0}^{n-1} f(a+jh_n)$$

$j :\to j+1$

$j-1 :\to (j+1)-1 = j$

§7.1

ALGEBRA YIELDS SOME ALTERNATE VERSIONS:

$$\boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)]$$

$$\boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \tfrac{1}{2})h_n)]$$

ALGEBRA YIELDS SOME ALTERNATE VERSIONS:

$$\boxed{L_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - 1)h_n)]$$

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$$= h_n \sum_{j=1}^{n} f(a + (j - \tfrac{1}{2})h_n) \overset{j :\to j+1}{=} h_n \sum_{j=0}^{n-1} f(a + (j + \tfrac{1}{2})h_n)$$

$$L_n S_a^b f := \sum_{j=1}^{n} [h_n][f(a + (j-1)h_n)]$$

$$= h_n \sum_{j=1}^{n} f(a + (j-1)h_n) = h_n \sum_{j=0}^{n-1} f(a + jh_n)$$

RIEMANN SUM FORMULAS:

$$\boxed{R_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + jh_n)]$$

$$= h_n \sum_{j=1}^{n} f(a + jh_n) = h_n \sum_{j=0}^{n-1} f(a + (j+1)h_n)$$

The "0 convention"

$$\boxed{M_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j - \tfrac{1}{2})h_n)]$$

$$= h_n \sum_{j=1}^{n} f(a + (j - \tfrac{1}{2})h_n) = h_n \sum_{j=0}^{n-1} f(a + (j + \tfrac{1}{2})h_n)$$

The "0 convention"

$$\boxed{L_n S_a^b f} := \sum_{j=1}^{n} [h_n][f(a + (j-1)h_n)]$$

Next: Back to $\int_0^1 x^2\,dx$

$$= h_n \sum_{j=1}^{n} f(a + (j-1)h_n) = h_n \sum_{j=0}^{n-1} f(a + jh_n)$$

The "0 convention"

EXAMPLE: $f(x) = x^2,\ a = 0,\ b = 1$

$R_n S_a^b f$

$$R_n S_a^b f = h_n \sum_{j=1}^{n} f(a + jh_n)$$

$$= h_n \sum_{j=1}^{n} f(a + jh_n)$$

Next: Back to $\int_0^1 x^2\, dx$

EXAMPLE: $f(x) = x^2$, $a = 0$, $b = 1$ $\qquad$ n varies

$$h_n = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$R_n S_0^1 f = R_n S_a^b f = h_n \sum_{j=1}^{n} f(a + j h_n)$$

$$\frac{1}{n}\sum_{j=1}^{n} f\left(\frac{j}{n}\right) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^2 = \frac{1}{n}\sum_{j=1}^{n}\frac{j^2}{n^2}$$

asymptotics, or . . .

$$\frac{2n^3+3n^2+n}{6n^3} = \frac{1}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] = \frac{1}{n^3}\sum_{j=1}^{n} j^2$$

$$\frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3}$$

$$\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Next: Back to $\int_0^1 x^2\,dx = \lim_{n\to\infty} R_n S_0^1 f$

EXAMPLE: $f(x) = x^2$, $a = 0$, $b = 1$ n varies

$$h_n = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$R_n S_0^1 f$

$$R_n S_0^1 f = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$$\int_0^1 x^2\,dx = \int_0^1 f(x)\,dx = \lim_{n\to\infty} R_n S_0^1 f$$

$$= \lim_{n\to\infty} \left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right]$$

$$=$$

Next: Back to $\int_0^1 x^2\,dx$

$\|$

$\lim_{n\to\infty} R_n S_0^1 f$

$$\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

EXAMPLE: $f(x) = x^2$, $a = 0$, $b = 1$ $\qquad$ n varies

$$h_n = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$R_n S_0^1 f = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$$\int_0^1 x^2\,dx = \int_0^1 f(x)\,dx = \lim_{n\to\infty} R_n S_0^1 f$$

$$= \lim_{n\to\infty} \left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right]$$

$$= \frac{1}{3} + 0 + 0 = \frac{1}{3} \quad \blacksquare$$

Kinda hard...

Next: Back to $\displaystyle\int_0^1 x^2\,dx = \lim_{n\to\infty} R_n S_0^1 f$

IOU: An easier approach, via the Fundamental Theorem of Calculus (Later topic.)