CALCULUS
The Fundamental Theorems of Calculus, statements and motivations

Notation: The *set* of *all* antiderivatives of $f(x)$ w.r.t. x is denoted $\boxed{\int f(x)\,dx}$.

$$\int_a^b f(x)\,dx$$

And now, for something completely different: Area

Or is it?

Connecting antidifferentiation to area:

The Fundamental Theorem of Calculus

The idea: The derivative of position is velocity,

So, position is an antiderivative of velocity.

antiderivative of velocity

We'll connect change in position

to the area under the graph of velocity. . .

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$
(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$v(6) = 36$, $v(8) = 64$, $v(10) = 100$

Connecting antidifferentiation to area:
The Fundamental Theorem of Calculus

The idea: The derivative of position is velocity,
So, position is an antiderivative of velocity.

antiderivative of velocity
We'll connect change in position
to the area under the graph of velocity...

Motion along a line: 3 subintervals: $[5, 7]$, $[7, 9]$, $[9, 11]$
(cf. §7.1, p. 139 EX 7.1)

midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5, 11]$.

Want: change in position $p(t)$ over $t \in [5, 11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$v(6) = 36$, $v(8) = 64$, $v(10) = 100$

$[p(11)] - [p(9)] \approx [2][100]$

$[p(9)] - [p(7)] \approx [2][64]$

$[p(7)] - [p(5)] \approx [2][36]$

For a better approximation, use a shorter subinterval.

Between time 5 and time 7, velocity ≈ 36

"Estimate velocity using the midpoint time."

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$
(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

ADD
$$[p(11)] - [p(9)] \approx [2][100]$$
$$[p(9)] - [p(7)] \approx [2]\ [64]$$
$$[p(7)] - [p(5)] \approx [2]\ [36]$$

$$[p(11)] - [p(9)] \approx [2][100]$$

$$[p(9)] - [p(7)] \approx [2][64]$$

$$[p(7)] - [p(5)] \approx [2][36]$$

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$
(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

ADD
$$
\begin{aligned}
&[p(11)] - \cancel{[p(9)]} &&\approx [2][100] \\
&\cancel{[p(9)]} - \cancel{[p(7)]} &&\approx [2]\ [64] \\
&\cancel{[p(7)]} - [p(5)] &&\approx [2]\ [36] \\
\hline
&[p(11)] - [p(5)] &&\approx [2][200]
\end{aligned}
$$

For a better approximation,
use more subintervals.

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$

(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

100	100
64	64
36	36

$$[p(11)] - [p(5)] \approx [2][200] - [p(5)] \approx [2][200]$$

Related Q: Compute $M_3 S_5^{11} v$.

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$

(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

100

64

36

$$[p(11)] - [p(5)] \approx [2][200]$$

Related Q: Compute $M_3 S_5^{11} v$. $h_3 = \frac{11-5}{3} = 2$

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$
(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

100
64
36

$[p(11)] - [p(5)] \approx [2][200]$

Related Q: Compute $M_3 S_5^{11} v$.

$$h_3 = \frac{11-5}{3} = 2$$

$a = 5$ $a + h_3 = 7$ $a + 2h_3 = 9$ $a + 3h_3 = 11$

Midpoints: 6 8 10

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$

(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

100

64

36

$[p(11)] - [p(5)] \approx [2][200]$

Related Q: Compute $M_3 S_5^{11} v$.

$$h_3 = \frac{11-5}{3} = 2$$

$a = 5$ $\quad a + h_3 = 7$ $\quad a + 2h_3 = 9$ $\quad a + 3h_3 = 11$

Midpoints: 6 8 10

$$\begin{aligned} M_3 S_5^{11} v &= [2][v(6)] + [2][v(8)] + [2][v(10)] \\ &= [2][36] + [2][64] + [2][100] \end{aligned}$$

7.1

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$
(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$$\begin{array}{r} 100 \\ 64 \\ 36 \\ \hline \end{array}$$

$$[p(11)] - [p(5)] \approx [2][200]$$

Related Q: Compute $M_3 S_5^{11} v$. $h_3 = \frac{11-5}{3} = 2$

$a = 5$ $a + h_3 = 7$ $a + 2h_3 = 9$ $a + 3h_3 = 11$

Midpoints: 6 8 10

$$\begin{aligned} M_3 S_5^{11} v &= [2][36] + [2][64] + [2][100] \\ &= [2][36] + [2][64] + [2][100] \end{aligned}$$

Motion along a line: 3 subintervals: $[5,7]$, $[7,9]$, $[9,11]$

(cf. §7.1, p. 139 EX 7.1) midpoints: 6 8 10

Know: velocity $v(t)$ at time t, $t \in [5,11]$.

Want: change in position $p(t)$ over $t \in [5,11]$. Split in 3.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$$\begin{array}{r} 100 \\ 64 \\ 36 \\ \hline \end{array}$$

$$[p(11)] - [p(5)] \approx [2][200] = M_3 S_5^{11} v$$

Related Q: Compute $M_3 S_5^{11} v$.

$$h_3 = \frac{11-5}{3} = 2$$

$a = 5$ $\quad a + h_3 = 7$ $\quad a + 2h_3 = 9$ $\quad a + 3h_3 = 11$

Midpoints: 6 8 10

$$M_3 S_5^{11} v = [2][36] + [2][64] + [2][100]$$

$$= [2][36 + 64 + 100]$$

7.1

Motion along a line:

(cf. §7.1, p. 139 EX 7.1)

Know: velocity $v(t)$ at time t, $t \in [5, 11]$.

Want: change in position $p(t)$ over $t \in [5, 11]$.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

100
64
36

$$[p(11)] - [p(5)] \approx [2][200] = M_3 S_5^{11} v$$

We'll connect change in position to the area under the graph of velocity...

For a better approximation, use more subintervals.

error $\to 0$, as $n \to \infty$

$$[p(11)] - [p(5)] \approx M_n S_5^{11} v$$

$$[p(11)] - [p(5)] = \lim_{n\to\infty} M_n S_5^{11} v$$

KINDA HARD TO CALCULATE, BUT...

$$= \int_5^{11} v(t)\, dt = \int_5^{11} t^2\, dt$$

7.1

Motion along a line:

(cf. §7.1, p. 139 EX 7.1)

Know: velocity $v(t)$ at time t, $t \in [5, 11]$.

Want: change in position $p(t)$ over $t \in [5, 11]$.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$p'(t) = v(t)$, *i.e.*, $p(t)$ is an antiderivative of t^2 w.r.t. t.

$$\{\text{antiderivatives of } t^2 \text{ w.r.t. } t\} = \int t^2\,dt = (t^3/3) + C$$

$$p(t) = (t^3/3) + C,$$

for some C

$$[(11^3/3) + C] - [(5^3/3) + C] = [11^3/3] - [5^3/3] \blacksquare$$

//

$$[p(11)] - [p(5)] = \lim_{n\to\infty} M_n S_5^{11} v$$

KINDA HARD TO CALCULATE, BUT...

$$= \int_5^{11} v(t)\,dt = \boxed{\int_5^{11} t^2\,dt}$$

☺

Motion along a line:

(cf. §7.1, p. 139 EX 7.1)

Know: velocity $v(t)$ at time t, $t \in [5, 11]$.

Want: change in position $p(t)$ over $t \in [5, 11]$.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$p'(t) = v(t)$, *i.e.*, $p(t)$ is an antiderivative of t^2 w.r.t. t.

$$\{\text{antiderivatives of } t^2 \text{ w.r.t. } t\} = \int t^2\,dt = (t^3/3) + C$$

$$\int_5^{11} t^2\,dt = [11^3/3] - [5^3/3] = [t^3/3]_{t:\to 5}^{t:\to 11}$$

$$= [11^3/3] - [5^3/3]$$

$$\int_5^{11} t^2\,dt$$

7.1

Motion along a line:

(cf. §7.1, p. 139 EX 7.1)

Know: velocity $v(t)$ at time t, $t \in [5, 11]$.

Want: change in position $p(t)$ over $t \in [5, 11]$.

e.g.: Assuming $v(t) = t^2$, find $[p(11)] - [p(5)]$.

$p'(t) = v(t)$, *i.e.*, $p(t)$ is an antiderivative of t^2 w.r.t. t.

$$\{\text{antiderivatives of } t^2 \text{ w.r.t. } t\} = \int t^2\,dt = (t^3/3) + C$$

$$\int_5^{11} t^2\,dt = [11^3/3] - [5^3/3] = [t^3/3]_{t:\to 5}^{t:\to 11}$$

$$= [(t^3/3) + C]_{t:\to 5}^{t:\to 11}$$

Key idea: To compute a definite integral,
find an antiderivative,
then evaluate at limits of integration,
then subtract.

7.1

 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let v be any function, contin. on $[a, b]$.

Let p be an antiderivative of v on $[a, b]$.

Then $$\int_a^b v(t)\,dt = [p(t)]_{t:\to a}^{t:\to b} = [p(b)] - [p(a)].$$

$$\{\text{antiderivatives of } t^2 \text{ w.r.t. } t\} = \int t^2\,dt = (t^3/3) + C$$

$$\int_5^{11} t^2\,dt = [11^3/3] - [5^3/3] = [t^3/3]_{t:\to 5}^{t:\to 11}$$

$$= [(t^3/3) + C]_{t:\to 5}^{t:\to 11}$$

Key idea: To compute a definite integral,
find an antiderivative,
then evaluate at limits of integration,
then subtract.

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let v be any function, contin. on $[a,b]$.

Let p be an antiderivative of v on $[a,b]$.

Then $$\int_a^b v(t)\,dt = [p(t)]_{t:\to a}^{t:\to b} = [p(b)] - [p(a)].$$

There's another version of this th'm, in which we integrate to a variable, then differentiate w.r.t. it.

$$\int_5^{11} t^2\,dt = [11^3/3] - [5^3/3]$$

Key idea: To compute a definite integral,
find an antiderivative,
then evaluate at limits of integration,
then subtract.

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let v be any function, contin. on $[a,b]$.

Let p be an antiderivative of v on $[a,b]$.

Then $$\int_a^b v(t)\,dt = [p(t)]_{t:\to a}^{t:\to b} = [p(b)] - [p(a)].$$

There's another version of this th'm, in which we integrate to a variable, then differentiate w.r.t. it.

$$\int_5^{11} t^2\,dt = [11^3/3] - [5^3/3]$$

e.g.: $$\int_5^x t^2\,dt = [x^3/3] - [5^3/3]$$

$$\frac{d}{dx}\int_5^x t^2\,dt = \frac{d}{dx}\Big([x^3/3] - [5^3/3]\Big) = x^2 = [t^2]_{t:\to x}$$

Key idea: To compute a definite integral, find an antiderivative, then evaluate at limits of integration, then subtract.

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let v be any function, contin. on $[a,b]$.

Let p be an antiderivative of v on $[a,b]$.

Then $\displaystyle\int_a^b v(t)\,dt = [p(t)]_{t:\to a}^{t:\to b} = [p(b)] - [p(a)]$.

---

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If v is continuous on $[a,b]$,

then $\displaystyle\frac{d}{dx}\int_a^x v(t)\,dt = [v(t)]_{t:\to x} = v(x)$

$$\frac{d}{dx}\int_5^x t^2\,dt = \frac{d}{dx}\Big([x^3/3] - [5^3/3]\Big) = x^2 = [t^2]_{t:\to x}$$

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let v be any function, contin. on $[a,b]$.

Let p be an antiderivative of v on $[a,b]$.

Then $$\int_a^b v(t)\,dt = [p(t)]_{t:\to a}^{t:\to b} = [p(b)] - [p(a)].$$

---

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If v is continuous on $[a,b]$,

then $$\frac{d}{dx}\int_a^x v(t)\,dt = [v(t)]_{t:\to x} = v(x),\ \text{for } x \in (a,b).$$

Domain: $a < x < b$ (for $\frac{d}{dx}\int_a^x v(t)\,dt$)

Domain: $a \le x \le b$ (for $\int_a^x v(t)\,dt$)

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let v be any function, contin. on $[a, b]$.

Let p be an antiderivative of v on $[a, b]$.

$v :\to f$, $p :\to F$, $t :\to x$

Then $$\int_a^b v(t)\,dt = [p(t)]_{t:\to a}^{t:\to b} = [p(b)] - [p(a)].$$

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If v is continuous on $[a, b]$,

$v :\to f$

then $$\frac{d}{dx}\int_a^x v(t)\,dt = [v(t)]_{t:\to x} = v(x), \text{ for } x \in (a, b).$$

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

IOU: Rigorous pf

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

---

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If f is continuous on $[a,b]$,

IOU: Rigorous pf

then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

Don't change t to x.

WARNING: $\displaystyle\int_a^x f$ is acceptable,

but $\displaystyle\int_a^x f(x)\,dx$ is not.

Don't use the same variable here and here.

Function notation is, as usual, more compact . . .

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.
Let F be an antiderivative of f on $[a,b]$.

Then $$\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)].$$

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If f is continuous on $[a,b]$,

then $$\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x), \text{ for } x \in (a,b).$$

Function notation is, as usual, more compact . . .

$$\int_a^b f = F|_a^b = [F(b)] - [F(a)]$$

$$\int_a^b F' = F|_a^b$$

sometimes sloppy
$$\int F' = F + C$$

$$\left(\int_a^\bullet f\right)' = f \text{ on } (a,b)$$

7.2 Integration and differentiation are inverses.

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a, b]$.

Let F be an antiderivative of f on $[a, b]$.

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

---

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If f is continuous on $[a, b]$,

then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a, b)$.

$\displaystyle\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ w.r.t. x on $a < x < b$.

continuous on $a \leq x \leq b$

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a, b]$.

Let F be an antiderivative of f on $[a, b]$.

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

---

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If f is continuous on $[a, b]$,

then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a, b)$.

$\displaystyle\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ w.r.t. x on $a \leq x \leq b$.

continuous on $a \leq x \leq b$

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

EXISTS!

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If f is continuous on $[a,b]$,

then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x)$, for $x \in (a,b)$.

$\displaystyle\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ w.r.t. x on $a \le x \le b$.

Can replace a by any $c \in [a,b]$.

Key point:

continuous on $[a,b]$ $\Rightarrow$ has an antiderivative on $[a,b]$

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a, b]$.

Let F be an antiderivative of f on $[a, b]$.

Then $$\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)].$$

---

cf. §7.2, p. 146 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.4

If f is continuous on $[a, b]$,

then $$\frac{d}{dx}\int_a^x f(t)\,dt = [f(t)]_{t:\to x} = f(x),\ \text{for } x \in (a, b).$$

$\forall c \in [a, b]$, $\displaystyle\int_c^x f(t)\,dt$ is an antiderivative of $f(x)$ w.r.t. x on $a \leq x \leq b$.

Next: Loose ends from previous topics

Key point:

continuous on $[a, b]$ $\Rightarrow$ has an antiderivative on $[a, b]$

 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.
Let F be an antiderivative of f on $[a,b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

$\int_0^x e^{t^2}\,dt$ is an antiderivative of e^{x^2} w.r.t. x on $-100 \le x \le 100$.

$\forall x \in (-100, 100)$,

$$\frac{d}{dx}\left[\int_0^x e^{t^2}\,dt\right] = e^{x^2}$$

IOU: Fund. Th'm of Calc gives an answer.

There is NO "elementary" antiderivative.

$\forall c \in [a,b]$, $\int_c^x f(t)\,dt$ is an antiderivative of $f(x)$ w.r.t. x on $a \le x \le b$.

Next: Loose ends from previous topics

Key point:
continuous on $[a,b]$ $\Rightarrow$ has an antiderivative on $[a,b]$

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

$\int_0^x e^{t^2}\,dt$ is an antiderivative of e^{x^2} w.r.t. x on $-1000000 \leq x \leq 1000000$.

$\forall x \in (-1000000, 1000000)$,

$$\frac{d}{dx}\left[\int_0^x e^{t^2}\,dt\right] = e^{x^2}$$

IOU: Fund. Th'm of Calc gives an answer.

There is NO "elementary" antiderivative.

$\forall c \in [a,b]$, $\int_c^x f(t)\,dt$ is an antiderivative of $f(x)$ w.r.t. x on $a \leq x \leq b$.

Next: Loose ends from previous topics

Key point:

continuous on $[a,b]$ $\Rightarrow$ has an antiderivative on $[a,b]$

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

Then $$\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)].$$

---

not "elementary"

A variant of H is quite important in probability theory.

$$H(x) = \int_0^x e^{t^2}\,dt$$ defines a NEW fn.

IOU: Fund. Th'm of Calc gives an answer.

$$\frac{d}{dx}\left[\int_0^x e^{t^2}\,dt\right] = e^{x^2}$$

There is NO "elementary" antiderivative.

$\forall c \in [a,b]$, $\int_c^x f(t)\,dt$ is an antiderivative of $f(x)$ w.r.t. x on $a \le x \le b$.

Next: Loose ends from previous topics

Key point:

continuous on $[a,b]$ $\Rightarrow$ has an antiderivative on $[a,b]$

 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a, b]$.

Let F be an antiderivative of f on $[a, b]$.

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

IOU from previous topic.

An easier way to show $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_{x:\to 0}^{x:\to 1} = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} \blacksquare$$

ANTIDIFF AND EVALUATE

SKILL
Definite integration

 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a, b]$.

Let F be an antiderivative of f on $[a, b]$.

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

IOU from previous topic.

An easier way to show $\displaystyle\int_2^7 3x^2 + 4x^3\,dx = 2720$

$$\int_2^7 3x^2 + 4x^3\,dx = \left[x^3 + x^4\right]_{x:\to 2}^{x:\to 7} = [7^3 + 7^4] - [2^3 + 2^4]$$
$$= 2720 \blacksquare$$

ANTIDIFF AND EVALUATE

SKILL
Definite integration

Next: An example comparing a Riemann sum to a definite integral

EXAMPLE: Let $f(x) = x^3 - 6x$.

(a) Evaluate $R_6 S_0^3 f$. (b) Evaluate $\int_0^3 (x^3 - 6x)\, dx$.

$a = 0,\ b = 3,\ n = 6$ $\quad h_6 = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$

IOU from previous topic.

An easier way to show $\int_2^7 3x^2 + 4x^3\, dx = 2720$

$$\int_2^7 3x^2 + 4x^3\, dx = \left[x^3 + x^4\right]_{x:\to 2}^{x:\to 7} = [7^3 + 7^4] - [2^3 + 2^4]$$

$$= 2720$$

ANTIDIFF AND EVALUATE

SKILL
Definite integration

Next: An example comparing a Riemann sum to a definite integral

EXAMPLE: Let $f(x) = x^3 - 6x$.

(a) Evaluate $R_6 S_0^3 f$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\, dx$.

$a = 0, \; b = 3, \; n = 6$

$$h_6 = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

$$R_6 S_0^3 f = h_6 \sum_{j=1}^{6} f(a + j h_6)$$

$$= \frac{1}{2} \sum_{j=1}^{6} f\left(0 + \overbrace{j}^{1,\ldots,6}\left(\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left[f\left(\cancel{0} + 1\left(\frac{1}{2}\right)\right) + \cdots + f\left(\cancel{0} + 6\left(\frac{1}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\left(f\left(\frac{1}{2}\right)\right) + \cdots + \left(f\left(\frac{6}{2}\right)\right)\right]$$

VISUALIZATION . . . EXAMPLE: Let $f(x) = x^3 - 6x$.

(a) Evaluate $R_6 S_0^3 f$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\, dx$.

$a = 0,\ b = 3$

$$h_6 = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

. . . .

$$R_6 S_0^3 f = \frac{1}{2}\left[\left(f\left(\frac{1}{2}\right)\right) + \cdots + \left(f\left(\frac{6}{2}\right)\right)\right]$$

$$= \frac{1}{2}[-2.875 - 5 - 5.625 - 4 + 0.625 + 9]$$

$$= -3.9375$$

SKILL

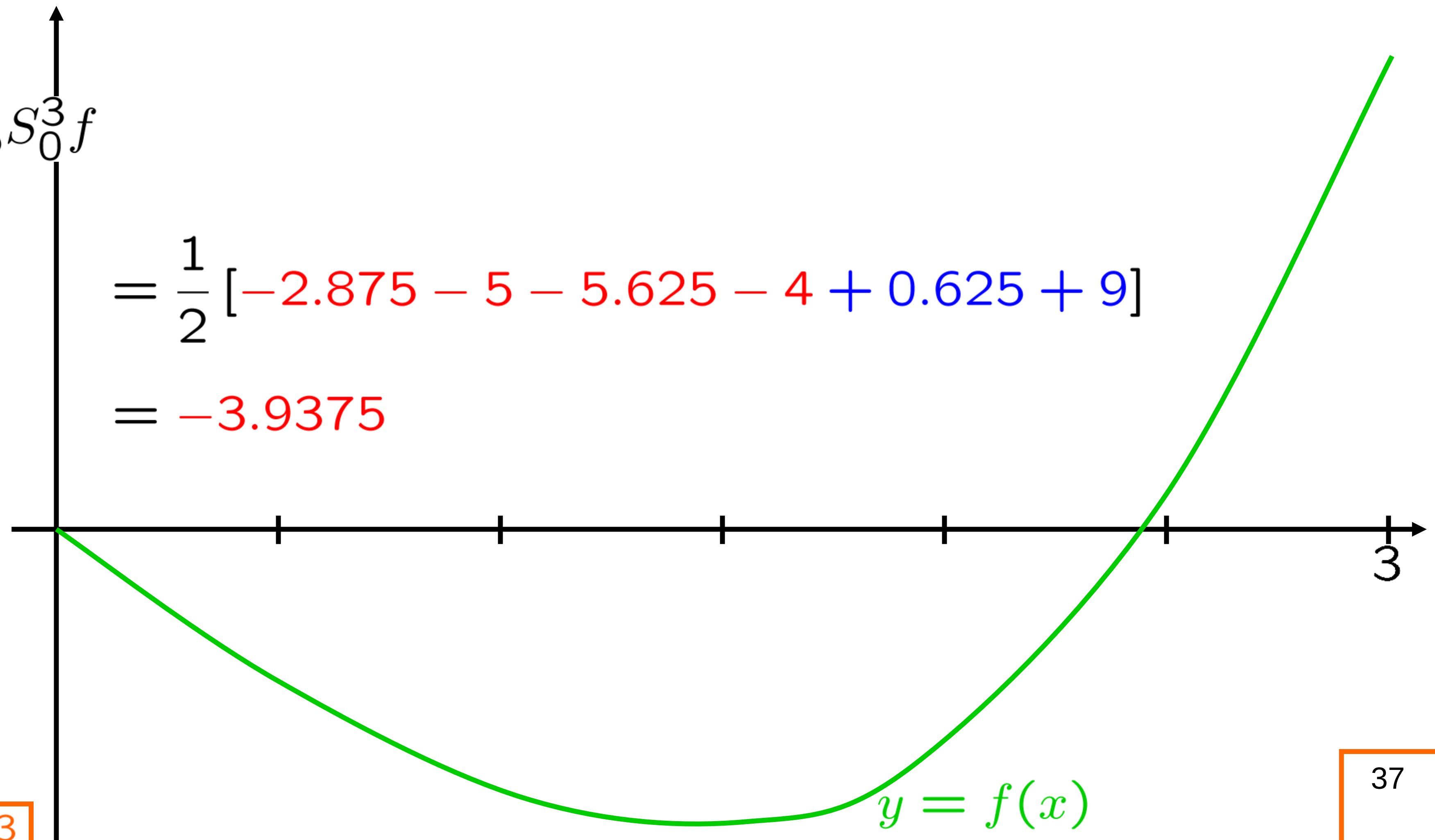
Riemann sums

$$= \frac{1}{2}\left[\left(f\left(\frac{1}{2}\right)\right) + \cdots + \left(f\left(\frac{6}{2}\right)\right)\right]$$

VISUALIZATION . . . EXAMPLE: Let $f(x) = x^3 - 6x$.

(a) Evaluate $R_6 S_0^3 f$. (b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

$$R_6 S_0^3 f = \frac{1}{2}[-2.875 - 5 - 5.625 - 4 + 0.625 + 9] = -3.9375$$

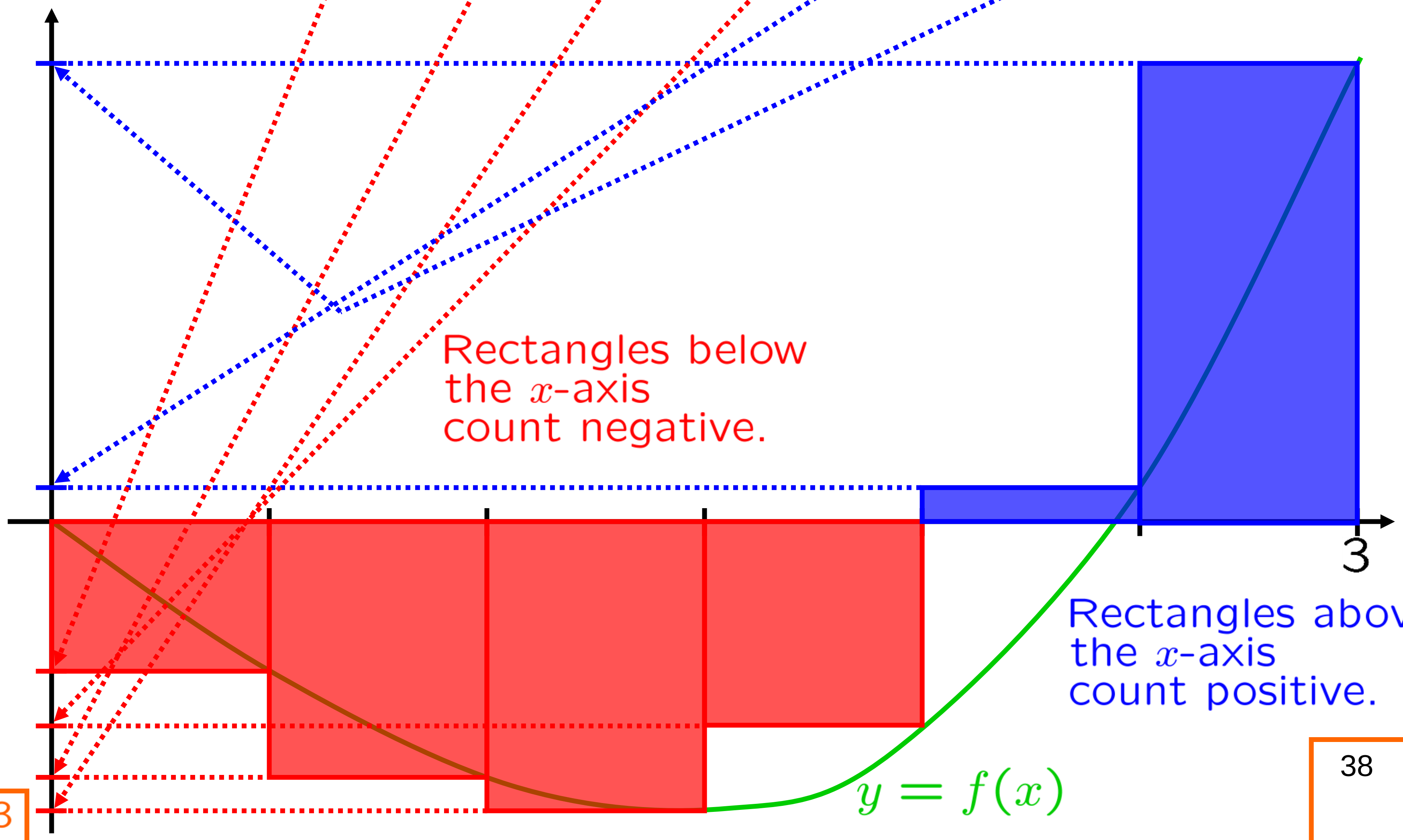


 EXAMPLE: Let $f(x) = x^3 - 6x$.

(a) Evaluate $R_6 S_0^3 f$. (b) Evaluate $\int_0^3 (x^3 - 6x)\, dx$.

VISUALIZATION OF (b) . . .

$$R_6 S_0^3 f = \frac{1}{2}[-2.875 - 5 - 5.625 - 4 + 0.625 + 9] = -3.9375$$



EXAMPLE: Let $f(x) = x^3 - 6x$.

(a) Evaluate $R_6 S_0^3 f$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\, dx$.

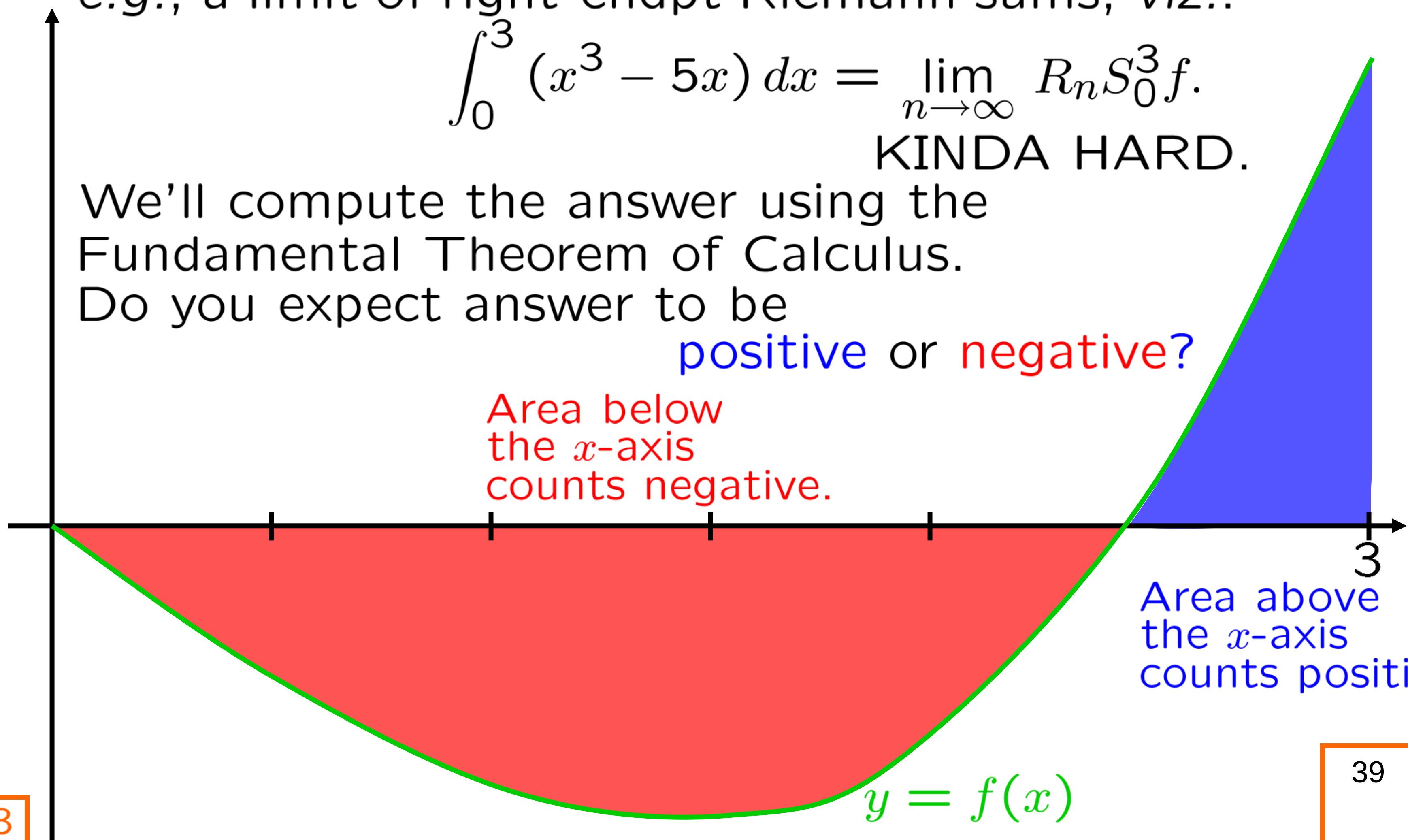
VISUALIZATION OF (b) . . .

Can solve (b) as a limit of Riemann sums, *e.g.*, a limit of right-endpt Riemann sums, *viz.*:

$$\int_0^3 (x^3 - 5x)\, dx = \lim_{n\to\infty} R_n S_0^3 f.$$

KINDA HARD.

We'll compute the answer using the Fundamental Theorem of Calculus.
Do you expect answer to be positive or negative?



Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a, b]$.

Let F be an antiderivative of f on $[a, b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

ANTIDIFF & EVALUATE

$$\left[\left(\frac{x^4}{4}\right) - 6\left(\frac{x^2}{2}\right)\right]_{x:\to 0}^{x:\to 3}$$

|| LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$\left(\left[\frac{x^4}{4}\right]_{x:\to 0}^{x:\to 3}\right) - 6\left(\left[\frac{x^2}{2}\right]_{x:\to 0}^{x:\to 3}\right)$$

|| LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$\left(\frac{[x^4]_{x:\to 0}^{x:\to 3}}{4}\right) - 6\left(\frac{[x^2]_{x:\to 0}^{x:\to 3}}{2}\right) = \left(\frac{3^4 - 0^4}{4}\right) - 6\left(\frac{3^2 - 0^2}{2}\right)$$

SKILL
Definite integration

-6.75
//

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

$$\| \quad \text{ANTIDIFF \& EVALUATE}$$

$$\left[\left(\frac{x^4}{4}\right) - 6\left(\frac{x^2}{2}\right)\right]_{x:\to 0}^{x:\to 3}$$

$$\|\text{LINEARITY OF } [\bullet]_{x:\to a}^{x:\to b}$$

$$\left(\left[\frac{x^4}{4}\right]_{x:\to 0}^{x:\to 3}\right) - 6\left(\left[\frac{x^2}{2}\right]_{x:\to 0}^{x:\to 3}\right)$$

$$\|\text{LINEARITY OF } [\bullet]_{x:\to a}^{x:\to b}$$

$$\left(\frac{[x^4]_{x:\to 0}^{x:\to 3}}{4}\right) - 6\left(\frac{[x^2]_{x:\to 0}^{x:\to 3}}{2}\right) = \left(\frac{3^4 - 0^4}{4}\right) - 6\left(\frac{3^2 - 0^2}{2}\right)$$

Back up a few steps...

SKILL

Definite integration

■

-6.75

//

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

$$\|$$ ANTIDIFF & EVALUATE

$$\left[\left(\frac{x^4}{4}\right) - 6\left(\frac{x^2}{2}\right)\right]_{x:\to 0}^{x:\to 3}$$

$\|$ LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$\left(\left[\frac{x^4}{4}\right]_{x:\to 0}^{x:\to 3}\right) - 6\left(\left[\frac{x^2}{2}\right]_{x:\to 0}^{x:\to 3}\right)$$

$$\|$$

$$\left(\int_0^3 x^3\,dx\right) - 6\left(\int_0^3 x\,dx\right)$$

---

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a, b]$.

Let F be an antiderivative of f on $[a, b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

$$||$$

$$\left(\int_0^3 x^3\,dx\right) - 6\left(\int_0^3 x\,dx\right)$$

DEFINITE INTEGRATION IS LINEAR.

$$\left(\int_0^3 x^3\,dx\right) - 6\left(\int_0^3 x\,dx\right)$$

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a,b]$.

Let F be an antiderivative of f on $[a,b]$.

Then $\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.

Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\,dx$.

$$\|$$

$$\left(\int_0^3 x^3\,dx\right) - 6\left(\int_0^3 x\,dx\right)$$

DEFINITE INTEGRATION
IS LINEAR.

MORE ON THIS IN
A LATER TOPIC . . .

cf. §7.2, p. 145 THE FUNDAMENTAL THEOREM OF CALCULUS, THEOREM 7.3

Let f be any function, contin. on $[a, b]$.

Let F be an antiderivative of f on $[a, b]$.

Then $\displaystyle\int_a^b f(x)\,dx = [F(x)]_{x:\to a}^{x:\to b} = [F(b)] - [F(a)]$.