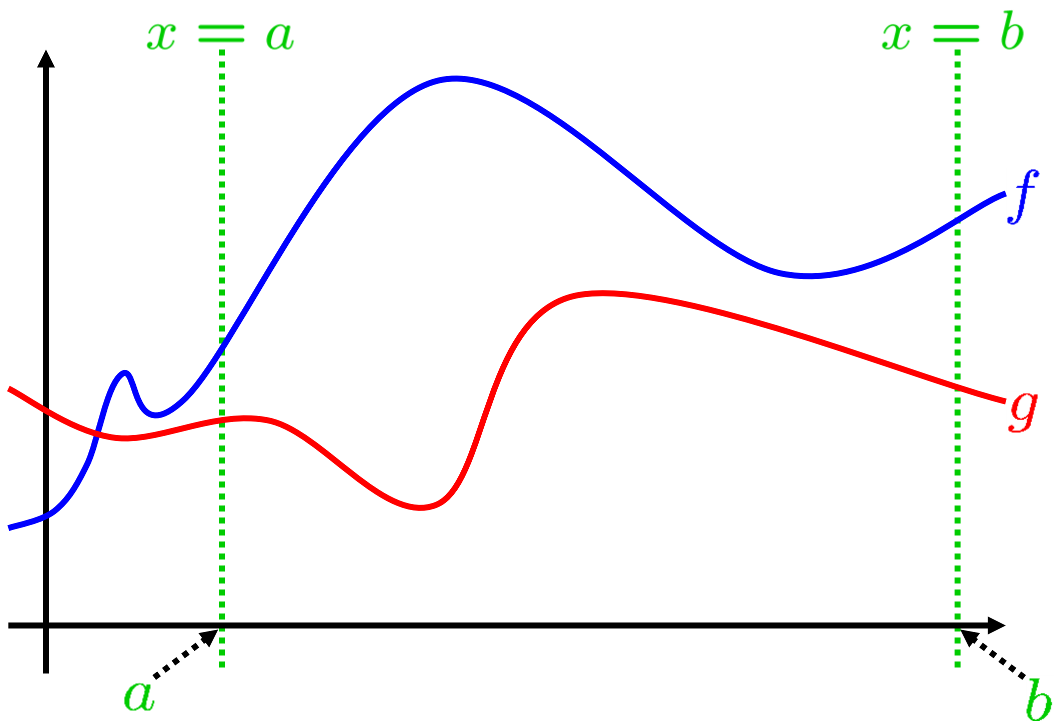
CALCULUS
Area between curves

REMARK:

Suppose $f(x) \geq g(x)$, for all $x \in [a,b]$.

Then the area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and by the lines $x = a$ and $x = b$



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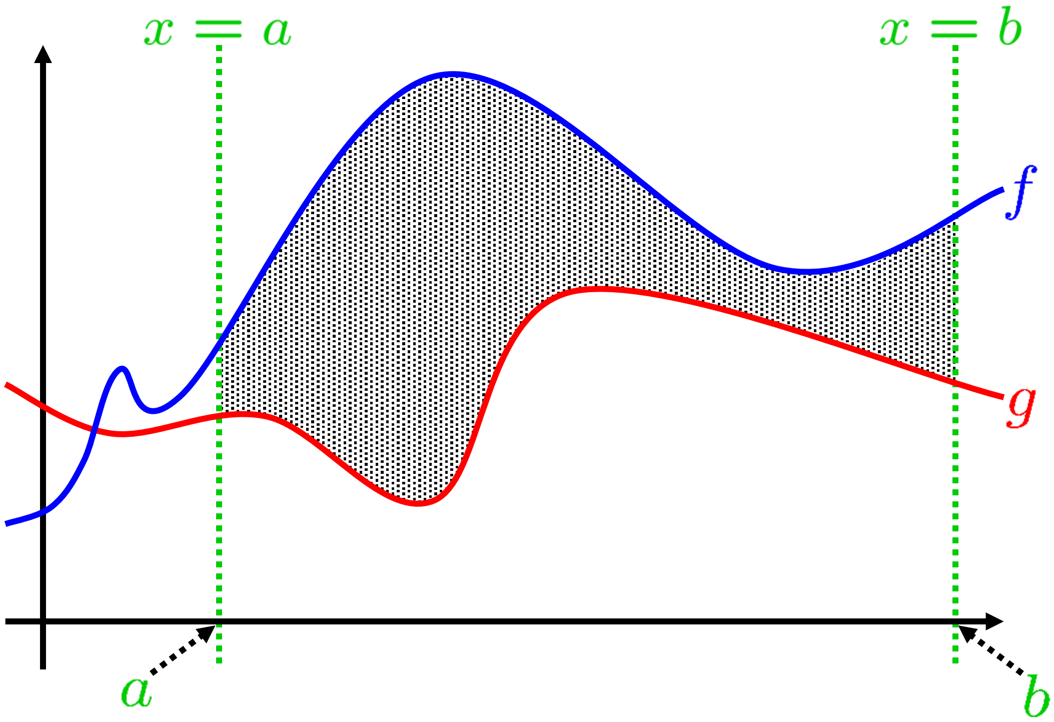

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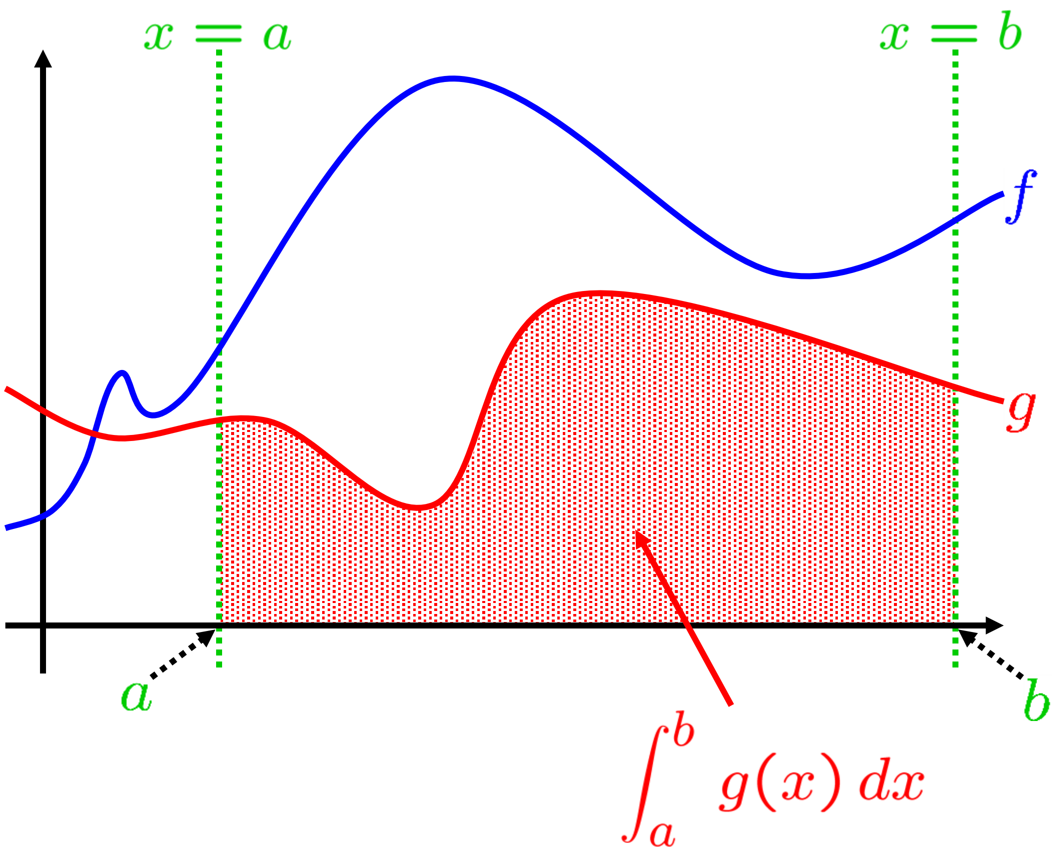

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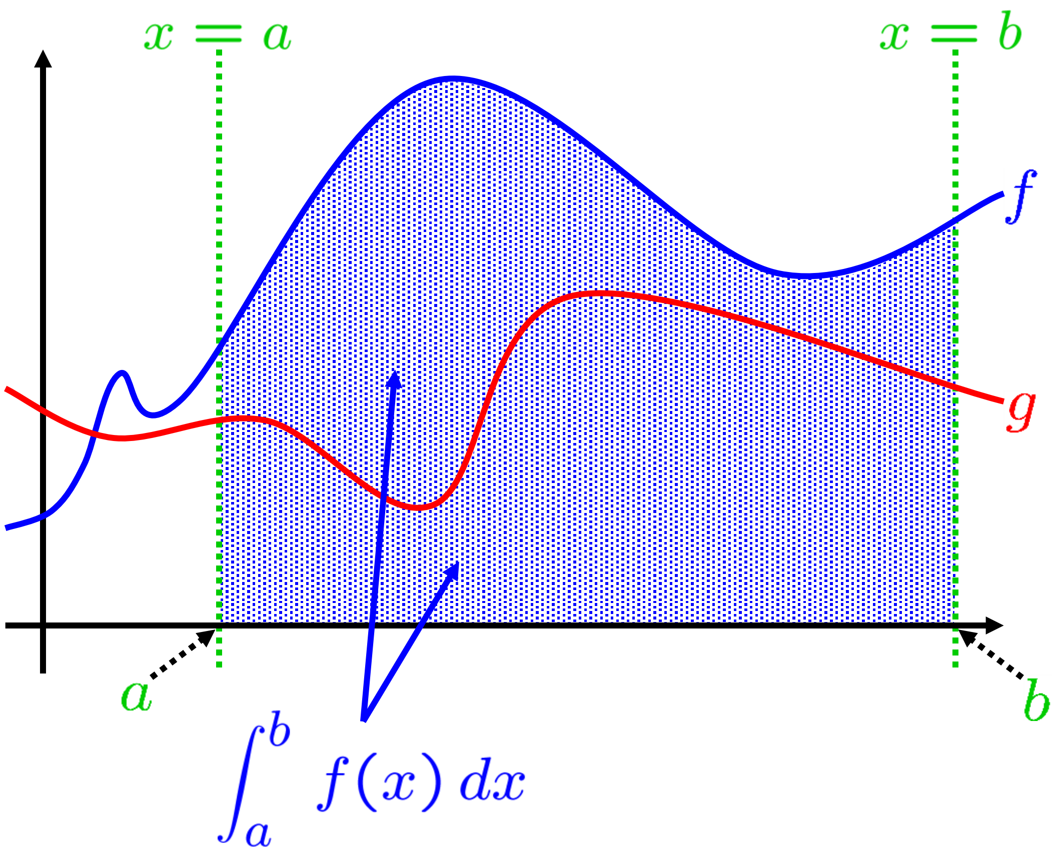

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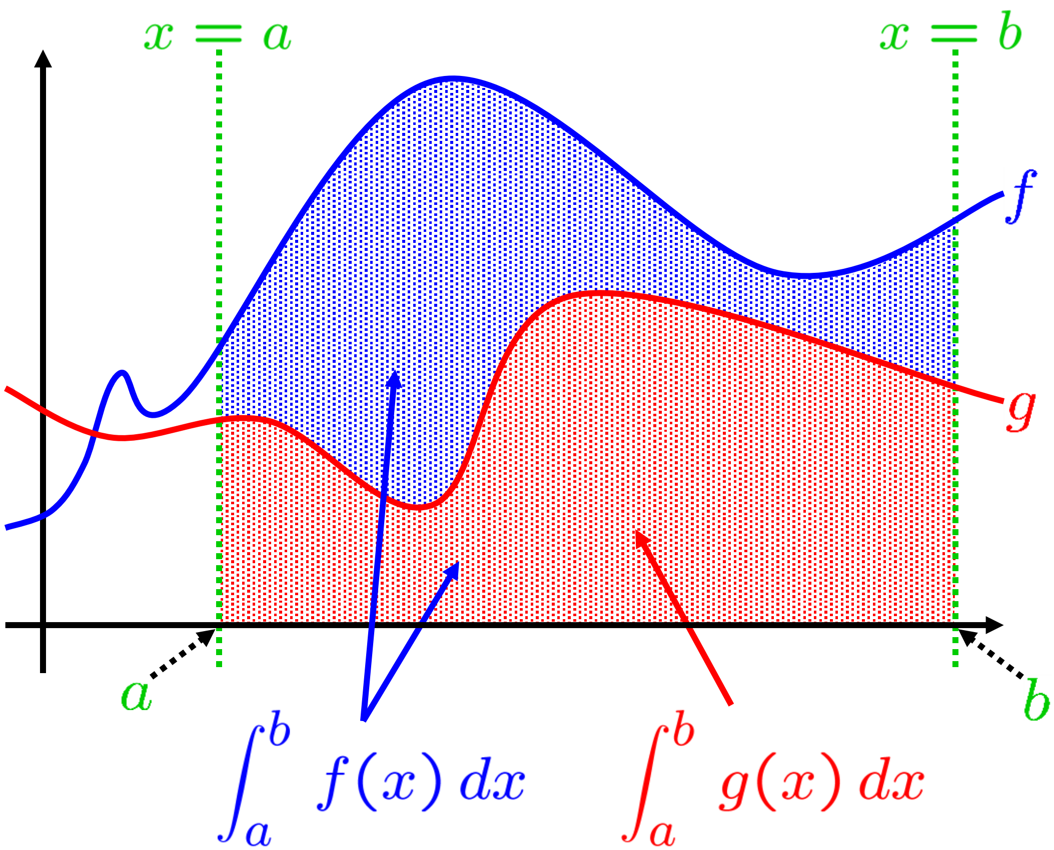

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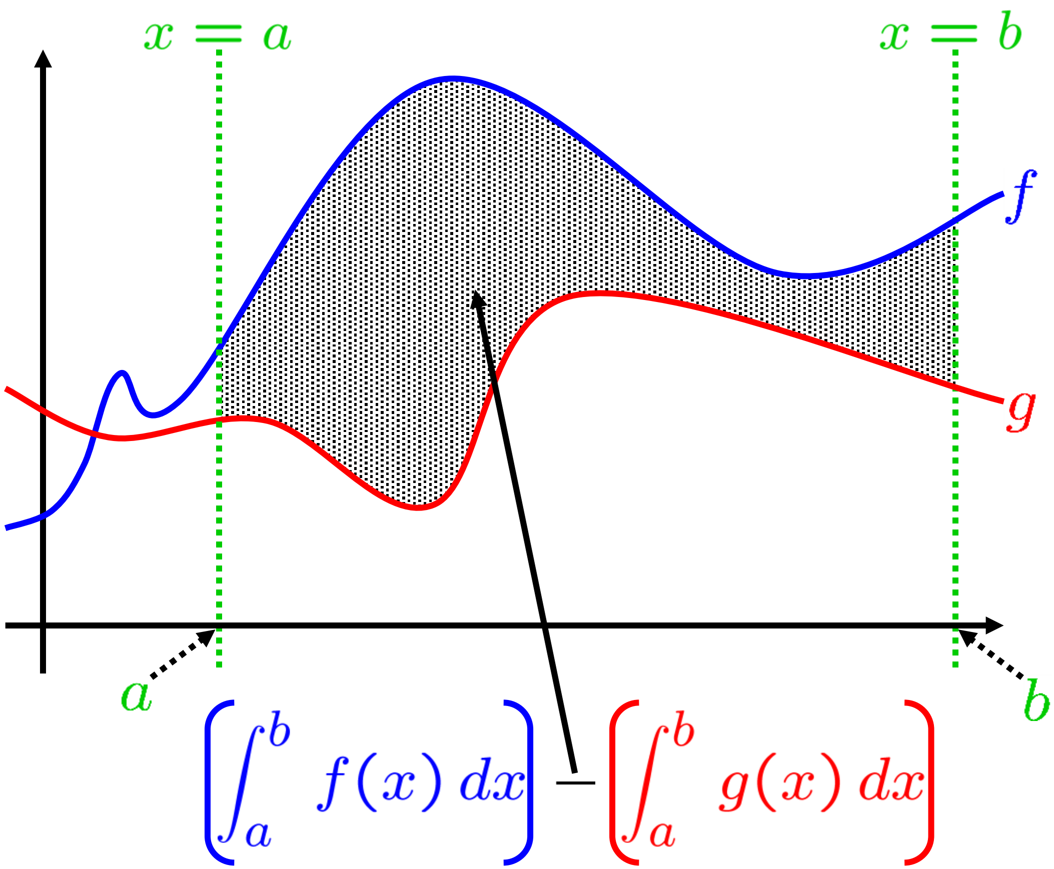
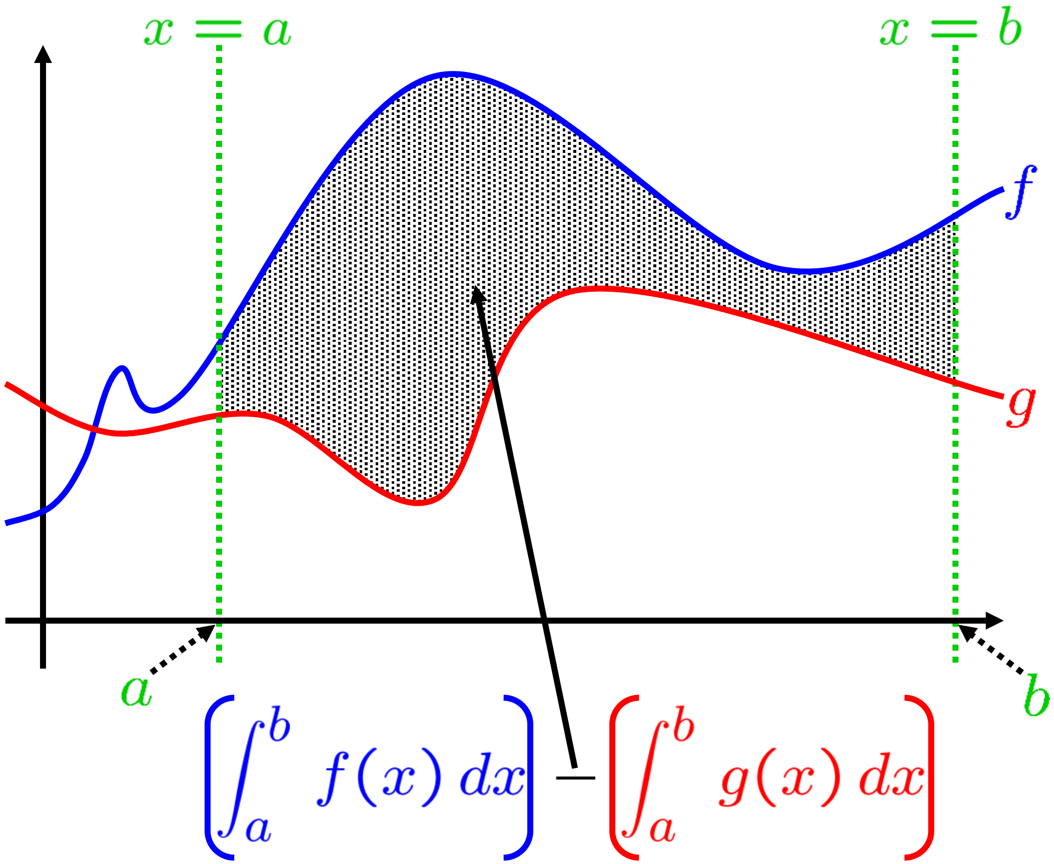


REMARK:

Suppose $f(x) \geq g(x)$, for all $x \in [a,b]$. What happens if we drop this hypothesis?

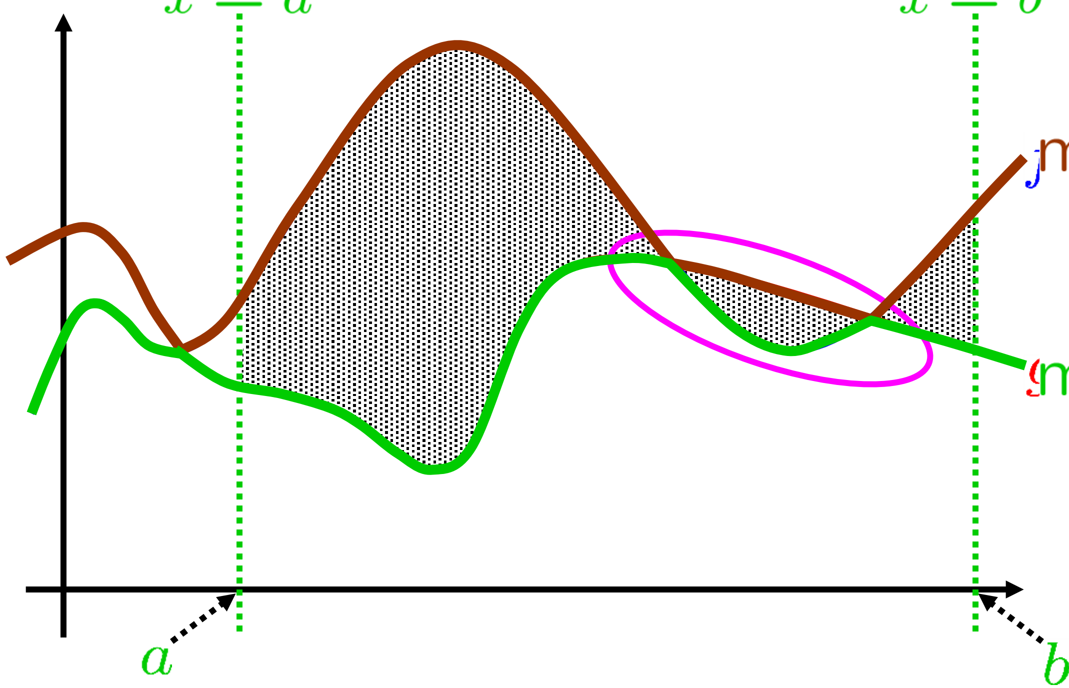
Then the area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and by the lines $x = a$ and $x = b$ is equal to $\int_a^b [((f(x)) - (g(x))] \, dx$.



REMARK:

The area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and by the lines $x = a$ and $x = b$

is equal to $\int_a^b \underbrace{[\max\{f(x), g(x)\}] - [\min\{f(x), g(x)\}]}_{|\,[f(x)] - [g(x)]\,|} \, dx.$

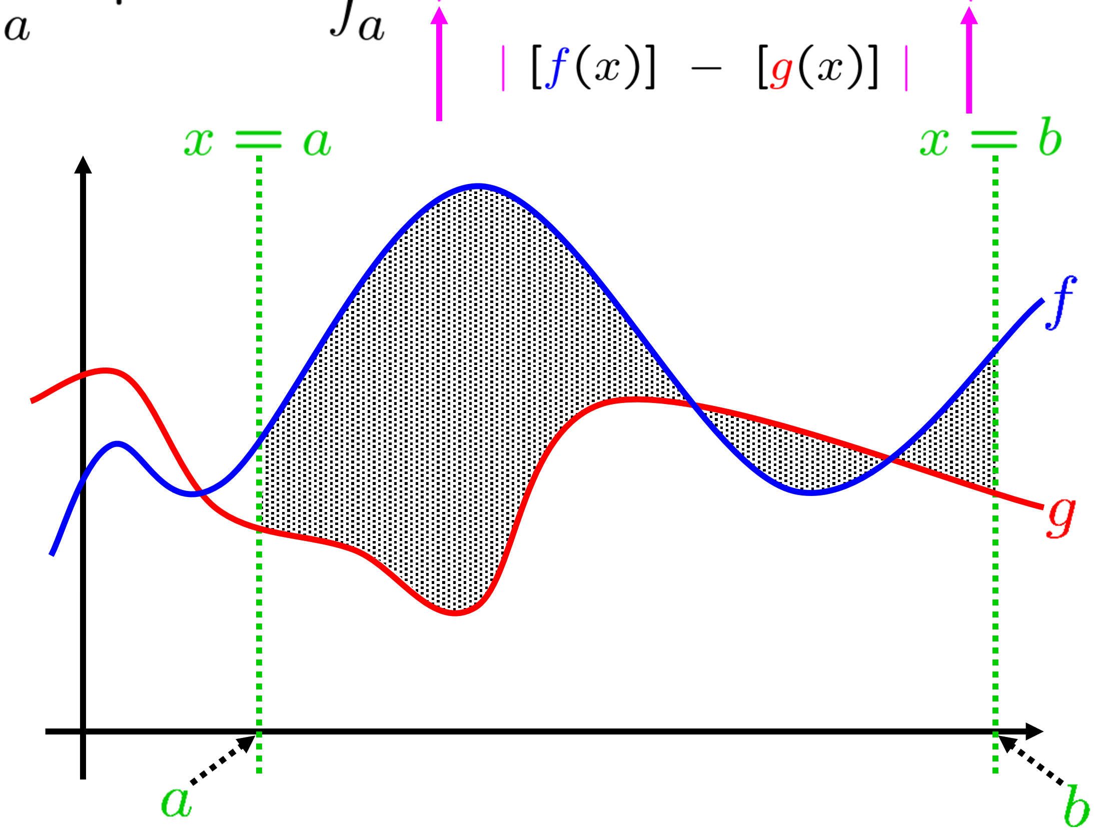


$|\, 2 - 5 \,| \neq [\max\{2, 5\}] - [\min\{2, 5\}]$

$|\, S - T \,| = [\max\{S, T\}] - [\min\{S, T\}]$

REMARK:

The area of the region bounded
by the curves $y = f(x)$ and $y = g(x)$
and by the lines $x = a$ and $x = b$
is equal to $\int_a^b$ equal to $\int_a^b \left| [f(x)] - [g(x)] \right| dx.$ $dx.$



EXAMPLE: Find the area enclosed by the parabolas $y = x^2$ and $y = 4x - x^2$.

Solution:

$$x^2 - (4x - x^2) = 2x^2 - 4x = 2x(x-2)$$

$2x(x-2)$: pos 0 neg 0 pos

x: 0, 2

$$\int_0^2 \left| 2x - 4x^2 \right| dx = \int_0^2 -(2x^2 - 4x)\, dx$$

$$= -\int_0^2 2x^2 - 4x\, dx$$

$$= -\left[2\frac{x^3}{3} - 4\frac{x^2}{2} \right]_{x:\to 0}^{x:\to 2}$$

$$= -\left(\left[\frac{16}{3} - \frac{16}{2} \right] - [0] \right) = \frac{8}{3} \blacksquare$$

EXAMPLE: Find the area enclosed by the curves $y = x^5 + x^4$ and $y = 2x^4 + 2x^3$.

Solution:

$$(x^5 + x^4) - (2x^4 + 2x^3) = x^5 - x^4 - 2x^3 = x^3(x^2 - x - 2)$$
$$= x^3(x - 2)(x + 1)$$

$x^3(x-2)(x+1)$

$$\int_{-1}^{2} \left| x^5 - x^4 - 2x^3 \right| dx$$

COCYCLE IDENTITY

$$= \left[\int_{-1}^{0} \left| x^5 - x^4 - 2x^3 \right| dx\right] + \left[\int_{0}^{2} \left| x^5 - x^4 - 2x^3 \right| dx\right]$$

$$= \left[\int_{-1}^{0} +(x^5 - x^4 - 2x^3)\, dx\right] + \left[\int_{0}^{2} -(x^5 - x^4 - 2x^3)\, dx\right]$$

$$= \left[\int_{-1}^{0} x^5 - x^4 - 2x^3\, dx\right] - \left[\int_{0}^{2} x^5 - x^4 - 2x^3\, dx\right]$$

EXAMPLE: Find the area enclosed by the curves $y = x^5 + x^4$ and $y = 2x^4 + 2x^3$.

Solution:

$$\int_{-1}^{2} \left| x^5 - x^4 - 2x^3 \right| dx$$

$$= \left[\int_{-1}^{0} x^5 - x^4 - 2x^3 \, dx\right] - \left[\int_{0}^{2} x^5 - x^4 - 2x^3 \, dx\right]$$

$$\int_{-1}^{2} \left| x^5 - x^4 - 2x^3 \right| dx$$

$$= \left[\frac{x^6}{6} - \frac{x^5}{5} - 2\frac{x^4}{4}\right]_{x:\to -1}^{x:\to 0} - \left[\frac{x^6}{6} - \frac{x^5}{5} - 2\frac{x^4}{4}\right]_{x:\to 0}^{x:\to 2}$$

$$= \left[\int_{-1}^{0} x^5 - x^4 - 2x^3 \, dx\right] - \left[\int_{0}^{2} x^5 - x^4 - 2x^3 \, dx\right]$$

EXAMPLE: Find the area enclosed by the curves $y = x^5 + x^4$ and $y = 2x^4 + 2x^3$.

Solution:

$$\int_{-1}^{2} \left| x^5 - x^4 - 2x^3 \right| dx$$

$$= \left[\int_{-1}^{0} x^5 - x^4 - 2x^3\,dx\right] - \left[\int_{0}^{2} x^5 - x^4 - 2x^3\,dx\right]$$

$$= \left[\frac{x^6}{6} - \frac{x^5}{5} - 2\frac{x^4}{4}\right]_{x:\to -1}^{x:\to 0} - \left[\frac{x^6}{6} - \frac{x^5}{5} - 2\frac{x^4}{4}\right]_{x:\to 0}^{x:\to 2}$$

$$= \left[0 - \left(\frac{(-1)^6}{6} - \frac{(-1)^5}{5} - 2\frac{(-1)^4}{4}\right)\right] - \left[\left(\frac{2^6}{6} - \frac{2^5}{5} - 2\frac{2^4}{4}\right) - 0\right]$$

$$= \left[-\frac{1}{6} - \frac{1}{5} + \frac{1}{2}\right] - \left[\frac{64}{6} - \frac{32}{5} - \frac{32}{4}\right]$$

$$= \left[\frac{8}{60}\right] - \left[-\frac{224}{60}\right] = \frac{58}{15} \blacksquare$$

EXAMPLE: Find the area enclosed by the line $y = \frac{1}{2}x - 1$ and the parabola $\boxed{y^2 = x + 6}$.

not hard to solve for x: $x = y^2 - 6$

hard to solve for y

EXAMPLE: Find the area enclosed by the line $y = \frac{1}{2}x - 1$ and the parabola $y^2 = x + 6$.

Solution: $x = 2y + 2$ $\qquad$ $x = y^2 - 6$

$$(2y + 2) - (y^2 - 6) = -y^2 + 2y + 8 = -(y + 2)(y - 4)$$

$-(y + 2)(y - 4)$: neg, 0, pos, 0, neg

y: -2, 4

We'll work this problem with expressions of y instead of expressions of x.

Slightly uncommon, but doable . . .

EXAMPLE: Find the area enclosed by the line $y = \frac{1}{2}x - 1$ and the parabola $y^2 = x + 6$.

Solution: $x = 2y + 2$ $\qquad$ $x = y^2 - 6$

$$(2y+2) - (y^2-6) = -y^2 + 2y + 8 = -(y+2)(y-4)$$

$-(y+2)(y-4)$ neg 0 pos 0 neg

y $\qquad -2 \qquad 4$

$$\int_{-2}^{4} \left| -y^2 + 2y + 8 \right| dy = + \int_{-2}^{4} -y^2 + 2y + 8 \, dy$$

$$= \left[-\frac{1}{3}y^3 + y^2 + 8y \right]_{y:\to -2}^{y:\to 4}$$

$$= \left[-\frac{1}{3}4^3 + 4^2 + 8 \cdot 4 \right] - \left[-\frac{1}{3}(-2)^3 + (-2)^2 + 8(-2) \right]$$

$$= \left[-\frac{64}{3} + 16 + 32 \right] - \left[\frac{8}{3} + 4 - 16 \right] = 36 \ \blacksquare$$

SKILL

area between curves

You might choose to interchange x and y...

EXAMPLE: Find the area enclosed by the line $x = \frac{1}{2}y - 1$ and the parabola $x^2 = y + 6$.

Solution: $y = 2x + 2$ $y = x^2 - 6$

$$(2x+2) - (x^2 - 6) = -x^2 + 2x + 8 = -(x+2)(x-4)$$

$-(x+2)(x-4)$ neg 0 pos 0 neg

x -2 4

$$\int_{-2}^{4} \left| -x^2 + 2x + 8 \right| \, dx = + \int_{-2}^{4} -x^2 + 2x + 8 \, dx$$

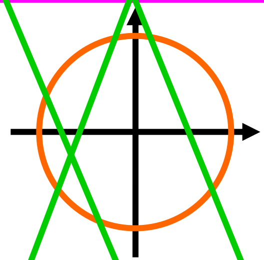
$$= \left[-\tfrac{1}{3}x^3 + x^2 + 8x \right]_{x:\to -2}^{x:\to 4}$$

$$= \left[-\tfrac{1}{3}4^3 + 4^2 + 8 \cdot 4 \right] - \left[+\tfrac{1}{3}(+2)^3 + (+2)^2 + 8(-2) \right]$$

$$= \left[-\tfrac{64}{3} + 16 + 32 \right] - \left[\tfrac{8}{3} + 4 - 16 \right] = 36 \blacksquare$$

EXAMPLE Find the area enclosed in a circle of radius 7.

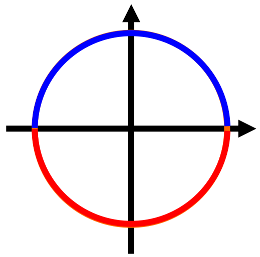
$y^2 = 7^2 - x^2$ $\quad$ $y = \pm\sqrt{7^2 - x^2}$



$x^2 + y^2 = 7^2$

EXAMPLE Find the area enclosed in a circle of radius 7.

$y^2 = 7^2 - x^2 \qquad y = \pm\sqrt{7^2 - x^2}$



$x^2 + y^2 = 7^2$

$y = \sqrt{7^2 - x^2}$

$y = -\sqrt{7^2 - x^2}$

$$\int_{-7}^{7} \left| \left(\sqrt{7^2 - x^2}\right) + \left(+\sqrt{7^2 - x^2}\right) \right| dx$$

UNNEEDED

UNNEEDED

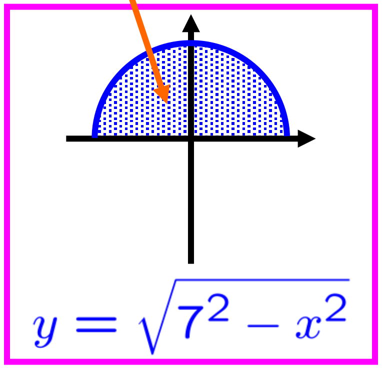
$$\| \qquad \int_{-7}^{7} 2\sqrt{7^2 - x^2}\, dx$$

LINEARITY OF DEFINITE INTEGRATION

$$\| \qquad 2\int_{-7}^{7} \sqrt{7^2 - x^2}\, dx$$

EXAMPLE Find the area enclosed in a circle of radius 7.

$$2\int_{-7}^{7} \sqrt{7^2 - x^2}\, dx$$

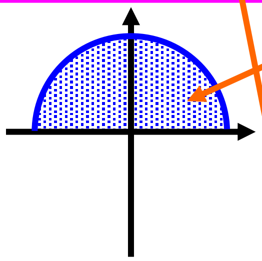


$y = \sqrt{7^2 - x^2}$

$$2\int_{-7}^{7} \sqrt{7^2 - x^2}\, dx$$

EXAMPLE Find the area enclosed in a circle of radius 7.

$$2\int_{-7}^{7} \sqrt{7^2 - x^2}\, dx = 4\int_{0}^{7} \sqrt{7^2 - x^2}\, dx$$



$y = \sqrt{7^2 - x^2}$

INTEGRATING SYMMETRIC FUNCTIONS

Suppose f is continuous on $[-a, a]$.

(a) If f is even, *i.e.*, $f(-x) = f(x)$,

then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$

(b) If f is odd, *i.e.*, $f(-x) = -(f(x))$,

then $\int_{-a}^{a} f(x)\, dx = 0.$

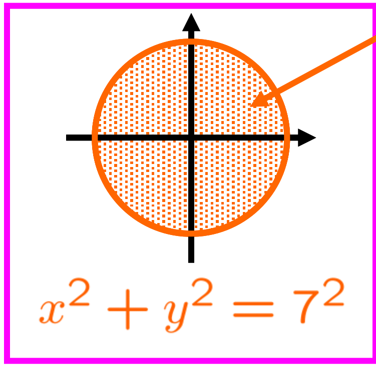
EXAMPLE Find the area enclosed in a circle of radius 7.

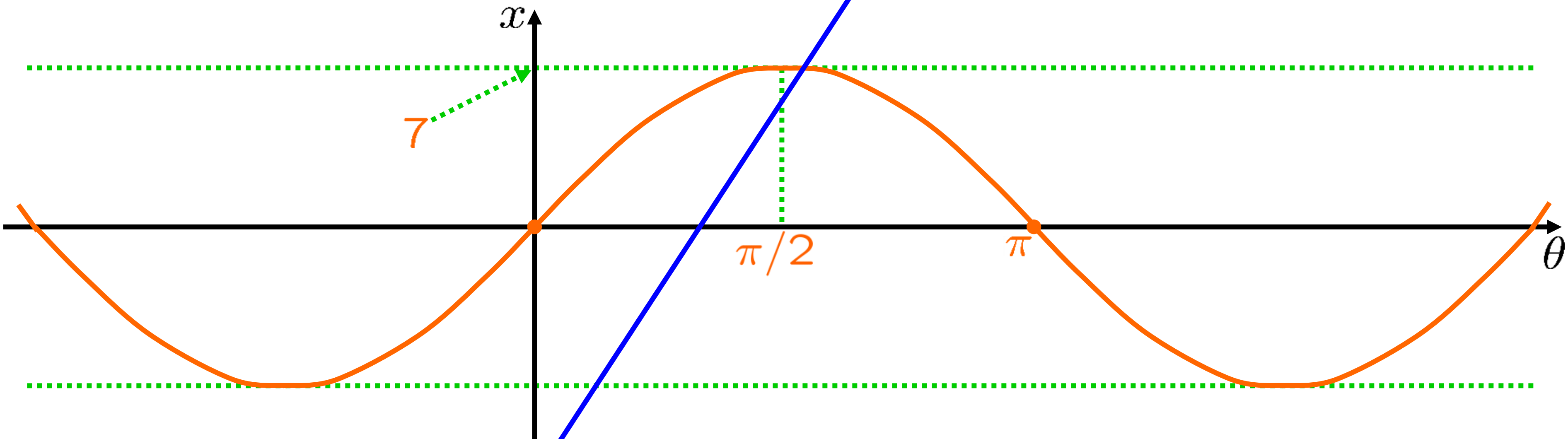
$$2\int_{-7}^{7}\sqrt{7^2-x^2}\,dx = 4\int_{0}^{7}\sqrt{7^2-x^2}\,dx$$

$0 \le \theta \le \pi/2$

$x = 7\sin\theta$

$0 \le x \le 7$





$x = 7\sin\theta$

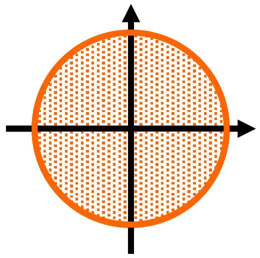
EXAMPLE Find the area enclosed in a circle of radius 7.

$$2\int_{-7}^{7}\sqrt{7^2-x^2}\,dx = 4\int_{0}^{7}\sqrt{7^2-x^2}\,dx$$

$0 \le \theta \le \pi/2$

$x = 7\sin\theta$

$dx = 7\cos\theta\,d\theta$



$$= 4\int_{0}^{\pi/2}\sqrt{7^2-(7\sin\theta)^2}\,[7\cos\theta]\,d\theta$$

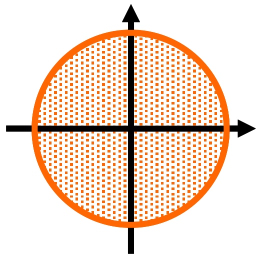
EXAMPLE Find the area enclosed in a circle of radius 7.

$$2\int_{-7}^{7}\sqrt{7^2-x^2}\,dx = 4\int_0^7 \sqrt{7^2-x^2}\,dx$$

$$= 4\int_0^{\pi/2}\sqrt{7^2-(7\sin\theta)^2}\,[7\cos\theta]\,d\theta$$

$$= 4\int_0^{\pi/2}[7\cos\theta]\,[7\cos\theta]d\theta$$

$$= 4(7^2)\int_0^{\pi/2}\cos^2\theta\,d\theta$$



$x^2+y^2=7^2$

$(7\cos\theta)^2 = 7^2\cos^2\theta = 7^2(1-\sin^2\theta) = 7^2 - 7^2\sin^2\theta = 7^2-(7\sin\theta)^2$

$\cos(2\theta) =$

$(\cos^2\theta) - (\sin^2\theta) =$

$1-(\cos^2\theta)$

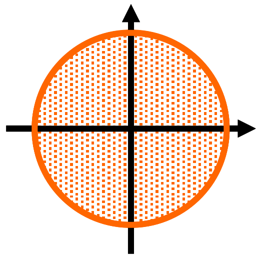
$$0\le\theta\le\pi/2 \quad\Rightarrow\quad \cos\theta\ge 0$$

$$\Rightarrow\quad 7\cos\theta\ge 0$$

$$\Rightarrow\quad \sqrt{(7\cos\theta)^2} = 7\cos\theta$$

EXAMPLE Find the area enclosed in a circle of radius 7.

$$2\int_{-7}^{7}\sqrt{7^2-x^2}\,dx = 4\int_{0}^{7}\sqrt{7^2-x^2}\,dx$$



$$= 4\int_{0}^{\pi/2}\sqrt{7^2-(7\sin\theta)^2}\,[7\cos\theta]\,d\theta$$

$$= 4\int_{0}^{\pi/2}[7\cos\theta]\,[7\cos\theta]d\theta$$

$$= 4(7^2)\int_{0}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 4(7^2)\int_{0}^{\pi/2}\frac{1+(\cos(2\theta))}{2}\,d\theta$$

$1-(\cos^2\theta)$

$\cos(2\theta) = (\cos^2\theta)-(\sin^2\theta) = 2(\cos^2\theta)-1$

$$\cos^2\theta = \frac{1+(\cos(2\theta))}{2}$$

EXAMPLE Find the area enclosed in a circle of radius 7.

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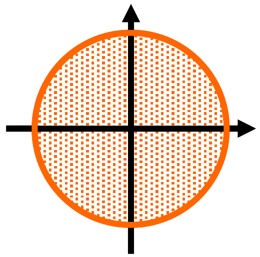


$x^2+y^2=7^2$

$$= 4\int_0^{\pi/2}\sqrt{7^2-(7\sin\theta)^2}\,[7\cos\theta]\,d\theta$$

$$= 4\int_0^{\pi/2}[7\cos\theta]\,[7\cos\theta]d\theta$$

$$= 4(7^2)\int_0^{\pi/2}\cos^2\theta\,d\theta$$

ANTIDIFFERENTIATE $\rightarrow \theta + \frac{\sin(2\theta)}{2}$

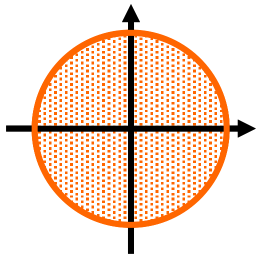
$$= \cancel{4}\,2\,(7^2)\int_0^{\pi/2}\frac{1+(\cos(2\theta))}{\cancel{2}}\,d\theta$$

$$= 2(7^2)\left[\theta+\frac{\sin(2\theta)}{2}\right]_{\theta:\to 0}^{\theta:\to\pi/2}$$

$$= 2(7^2)\left[\left[\frac{\pi}{2}+\frac{\sin\pi}{2}\right]-\left[0+\frac{\sin 0}{2}\right]\right]$$

EXAMPLE Find the area enclosed in a circle of radius 7.

$$2\int_{-7}^{7}\sqrt{7^2-x^2}\,dx = 2(7^2)\left[\left[\frac{\pi}{2}+\frac{\sin\pi}{2}\right]-\left[0+\frac{\sin 0}{2}\right]\right]$$



$$= 2(7^2)\left[\left[\frac{\pi}{2}+\frac{0}{2}\right]-\left[0+\frac{0}{2}\right]\right]$$

$$= 2(7^2)\left[\left[\frac{\pi}{2}+\frac{\sin\pi}{2}\right]-\left[0+\frac{\sin 0}{2}\right]\right]$$

EXAMPLE Find the area enclosed in a circle of radius 7.

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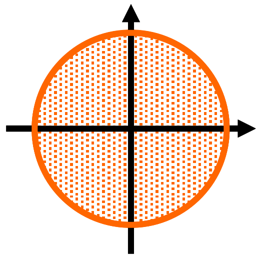

$$= 2(7^2)\left[\left[\frac{\pi}{2}+\frac{0}{2}\right]-\left[0+\frac{0}{2}\right]\right]$$

$$= 2(7^2)\left[\frac{\pi}{2}\right] = (\pi)(7^2)$$ ■

STOP

SKILL
Area by integration