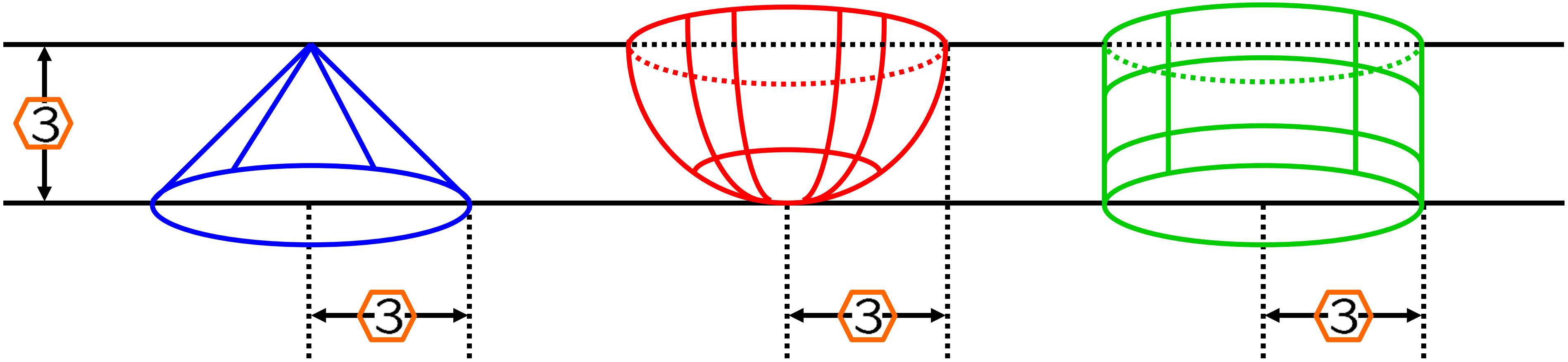
CALCULUS

The disk and washer methods

EXAMPLE: Find the volume enclosed in a sphere of radius 7.



Vol(cone) + Vol(hemisphere) = Vol(cylinder)

$3 :\to 7$

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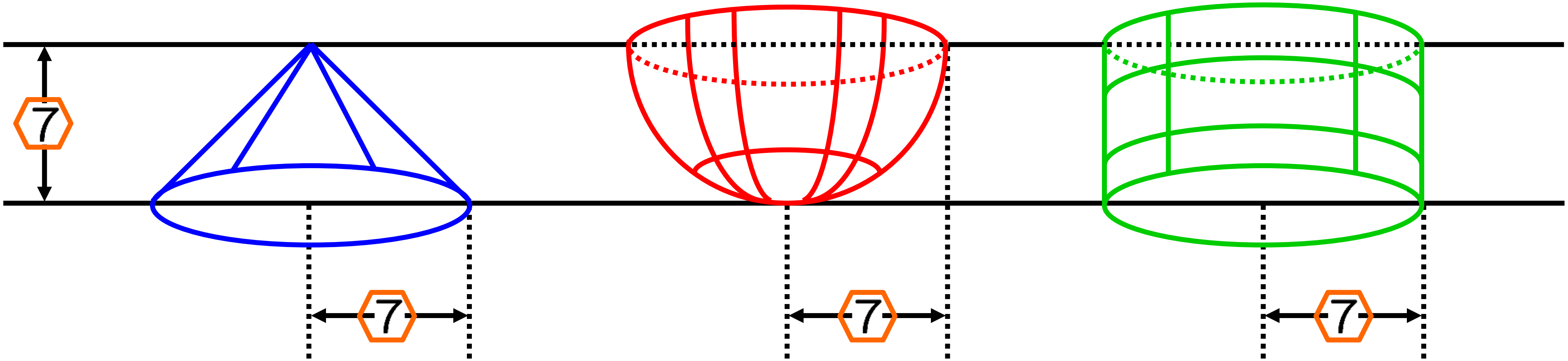

$$\underbrace{\text{Vol(cone)}}_{\frac{\pi 7^3}{3}} + \text{Vol(hemisphere)} = \underbrace{\text{Vol(cylinder)}}_{(\pi 7^2)7 = \pi 7^3}$$

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In 3-D, $\quad \text{Vol(generalized cone)} = \dfrac{\text{Vol(generalized cylinder)}}{3}$

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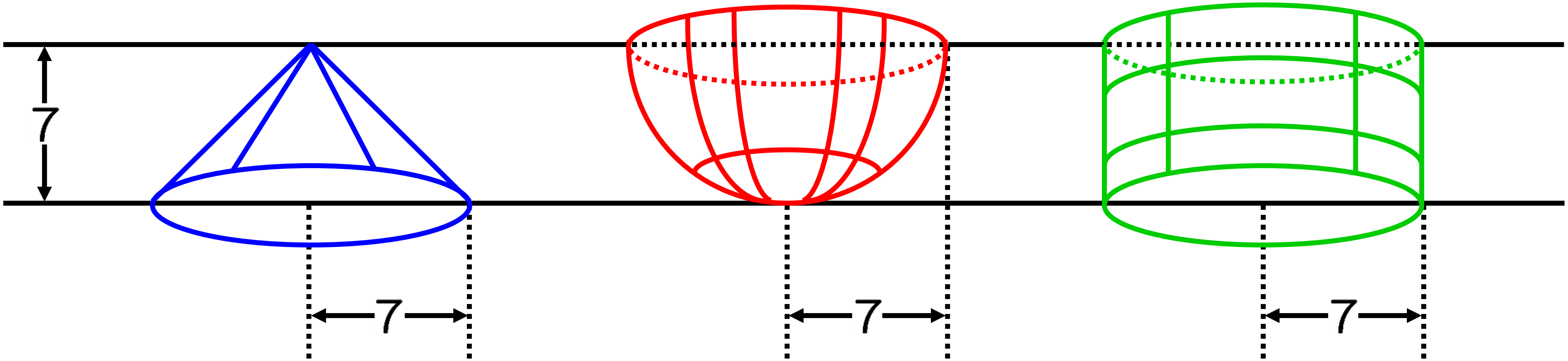


$$\underbrace{\text{Vol(cone)}}_{\frac{\pi 7^3}{3}} + \text{Vol(hemisphere)} = \underbrace{\text{Vol(cylinder)}}_{(\pi 7^2)7 \, = \, \pi 7^3}$$

$$\text{Vol(hemisphere)} = \pi 7^3 - \frac{\pi 7^3}{3}$$

$$= \left(1 - \frac{1}{3}\right)\pi 7^3 = \frac{2}{3}\pi 7^3$$

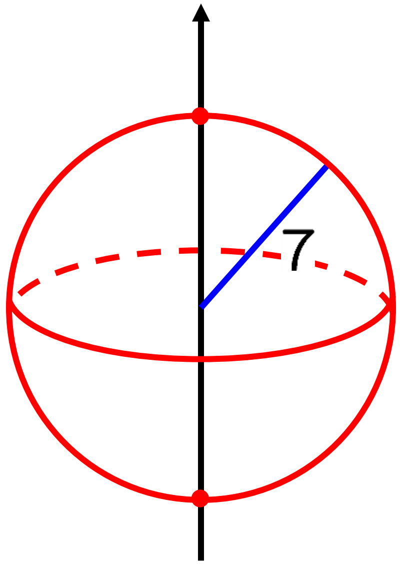
$$\text{Vol(sphere)} = \frac{4}{3}\pi 7^3 \blacksquare$$

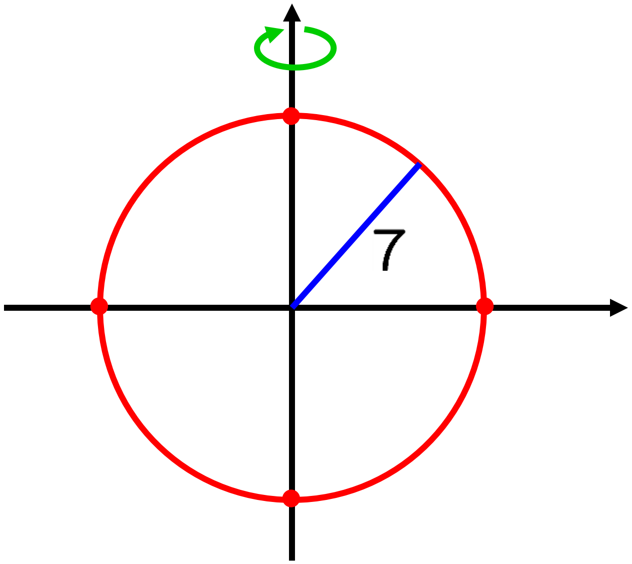
A MORE DIRECT APPROACH . . .

EXAMPLE: Find the volume enclosed in a sphere of radius 7.

Solution:

3-D





2-D is easier to draw,
but you have to imagine
the 3-D visualization.

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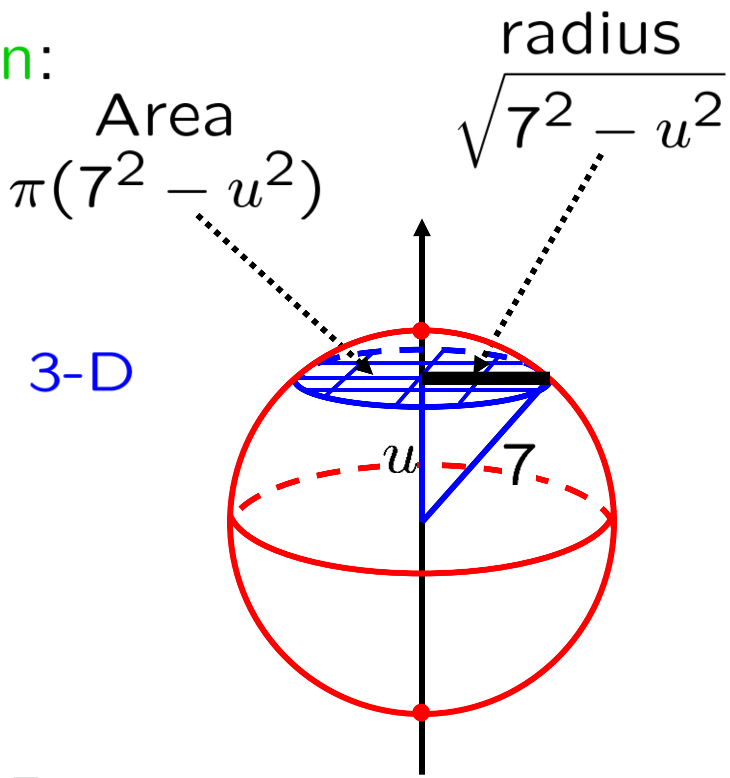

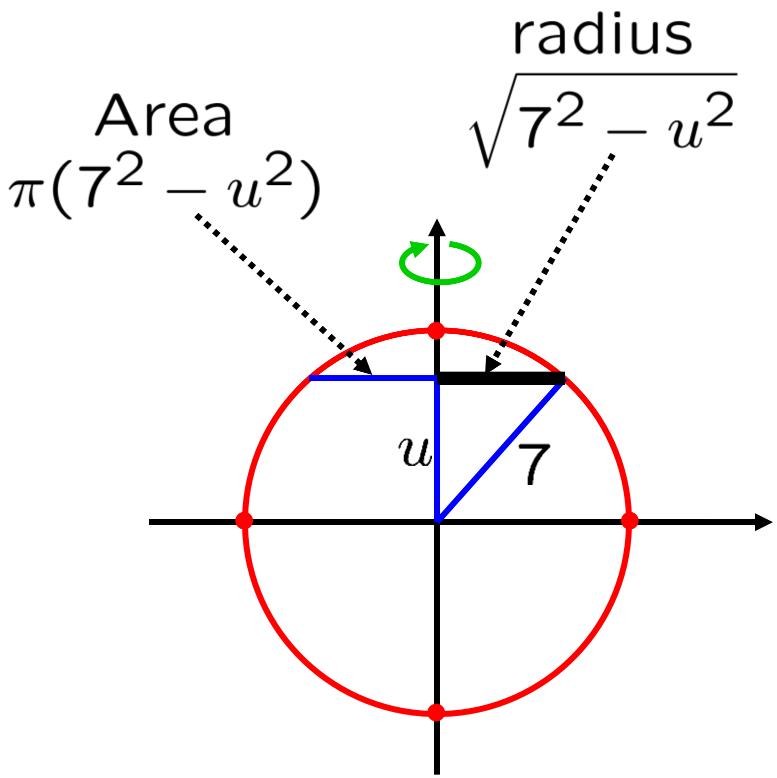

Goal: $\int_{-7}^{7} \pi(7^2 - u^2)\, du$

2-D is easier to draw,
but you have to imagine
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DEFINITION:
A solid obtained by revolving, about a line, a subset of a plane, is called a **solid of revolution**.

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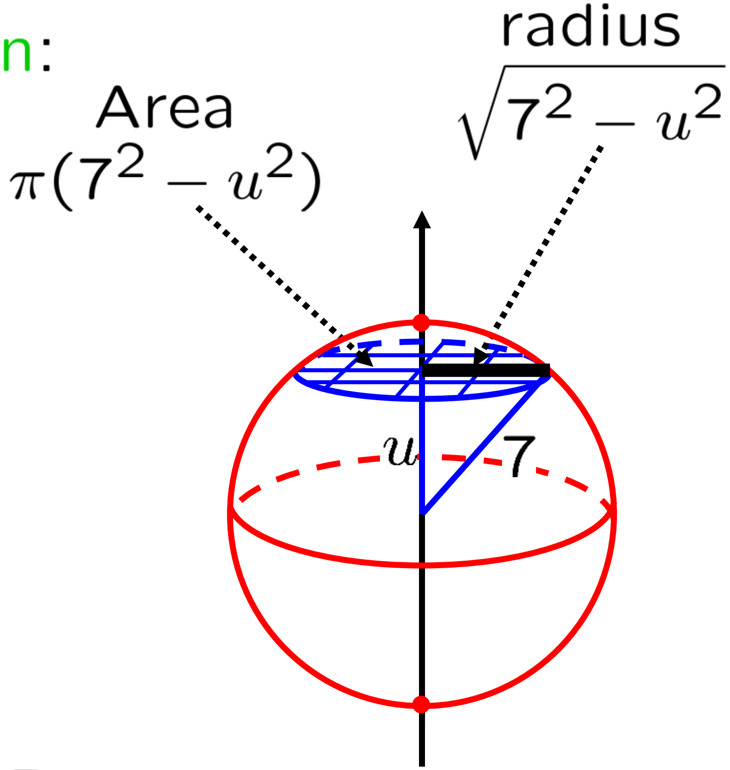

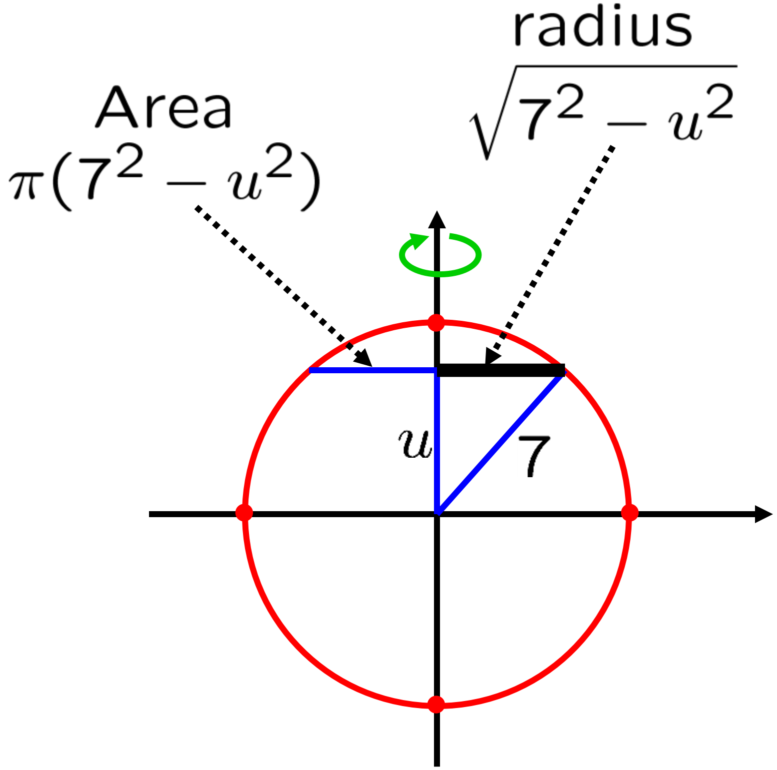

Goal: $\int_{-7}^{7} \pi(7^2 - u^2)\, du = 2\int_{0}^{7} \pi(7^2 - u^2)\, du$

$$= 2\left[\pi\left(7^2 u - \frac{u^3}{3}\right)\right]_{u:\to 0}^{u:\to 7}$$

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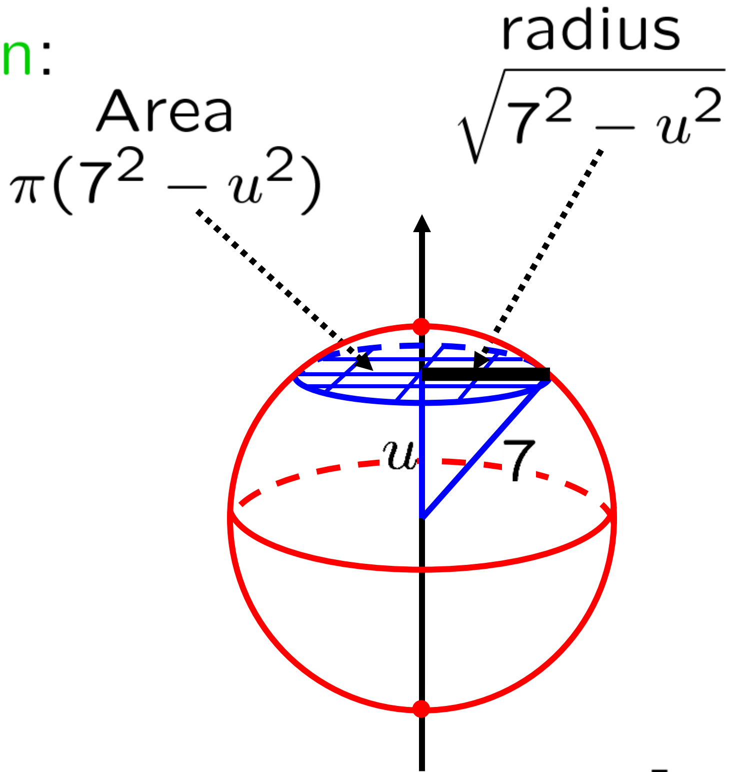

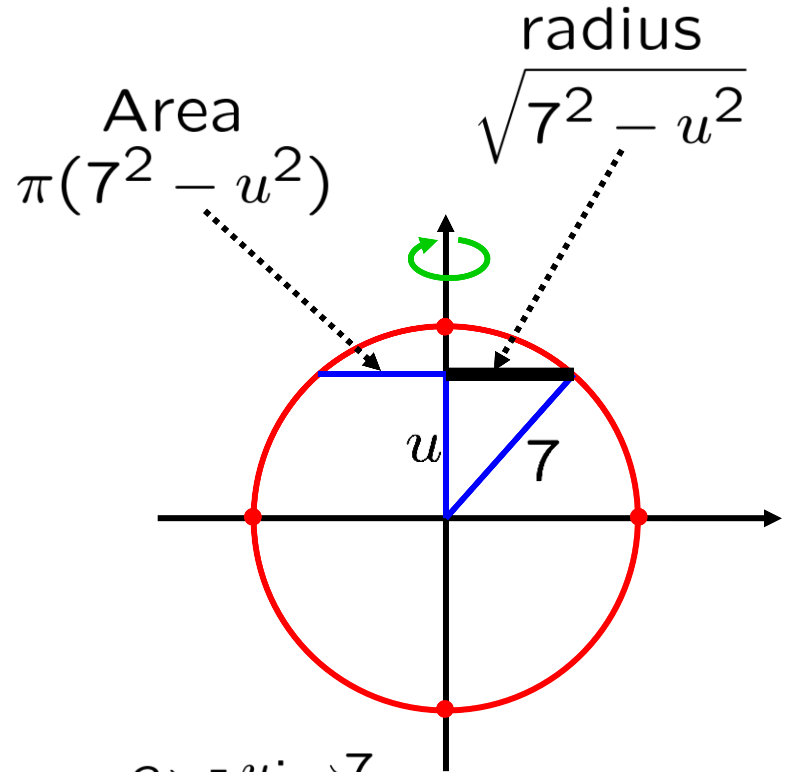

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$= 2\left[\pi\left(7^2\cdot 7 - \frac{7^3}{3}\right)\right]_{:\to 0}$

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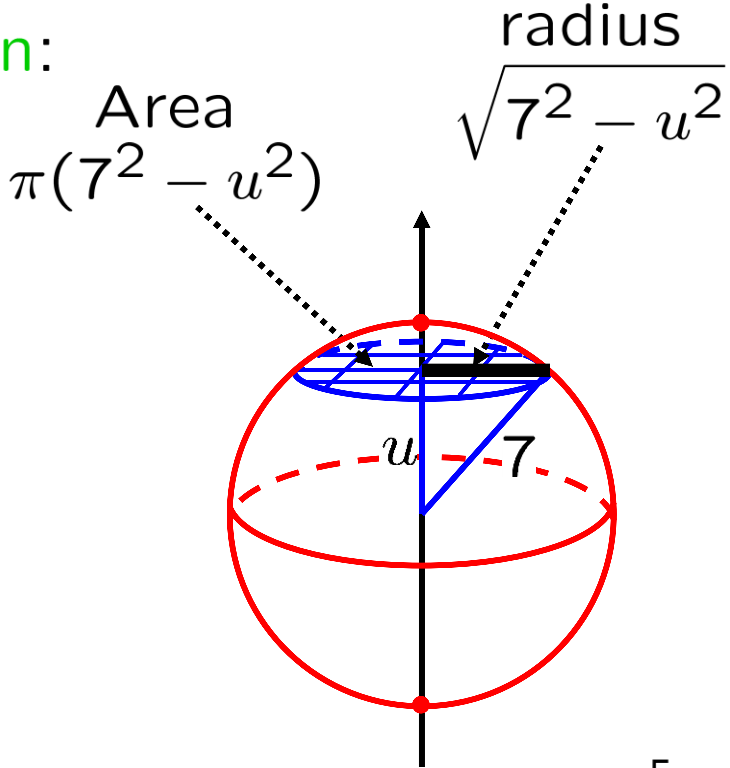

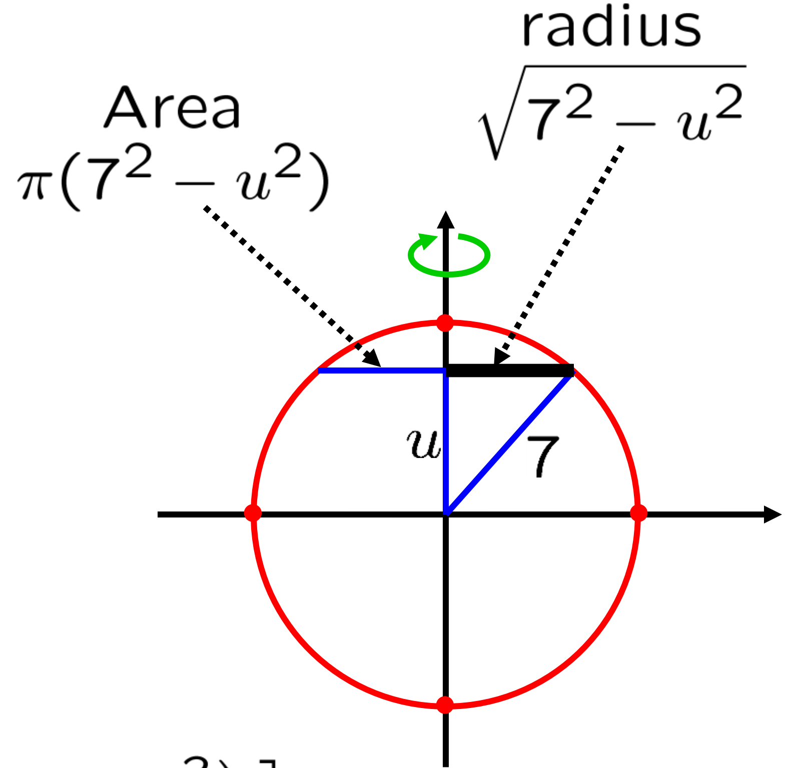

Goal: $\displaystyle\int_{-7}^{7} \pi(7^2 - u^2)\,du = 2\left[\pi\left(7^2 \cdot 7 - \frac{7^3}{3}\right)\right]$

$$= 2\left[\pi\left(7^3 - \frac{7^3}{3}\right)\right]$$

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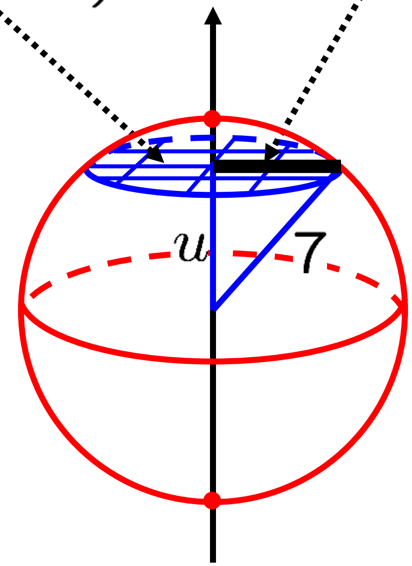

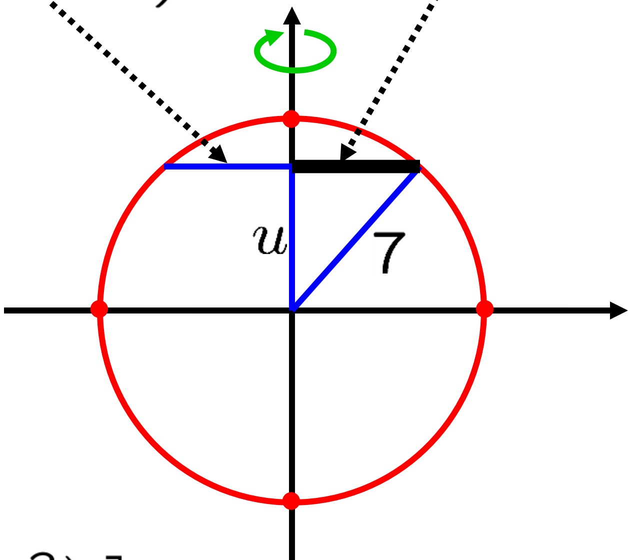

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$$= 2\left[\pi\left(7^3 - \frac{7^3}{3}\right)\right] = 2\left[\pi(7^3)\left(1 - \frac{1}{3}\right)\right]$$

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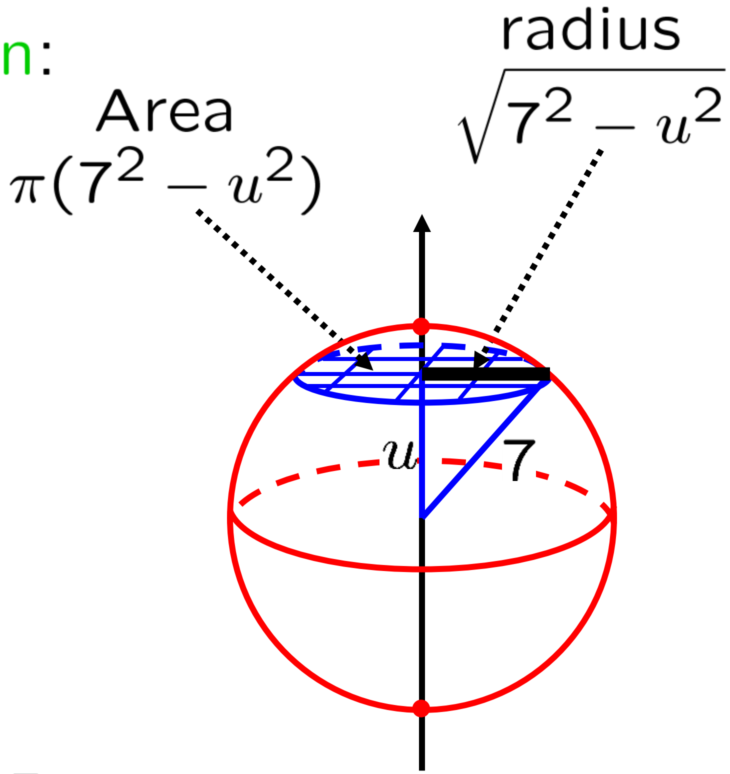

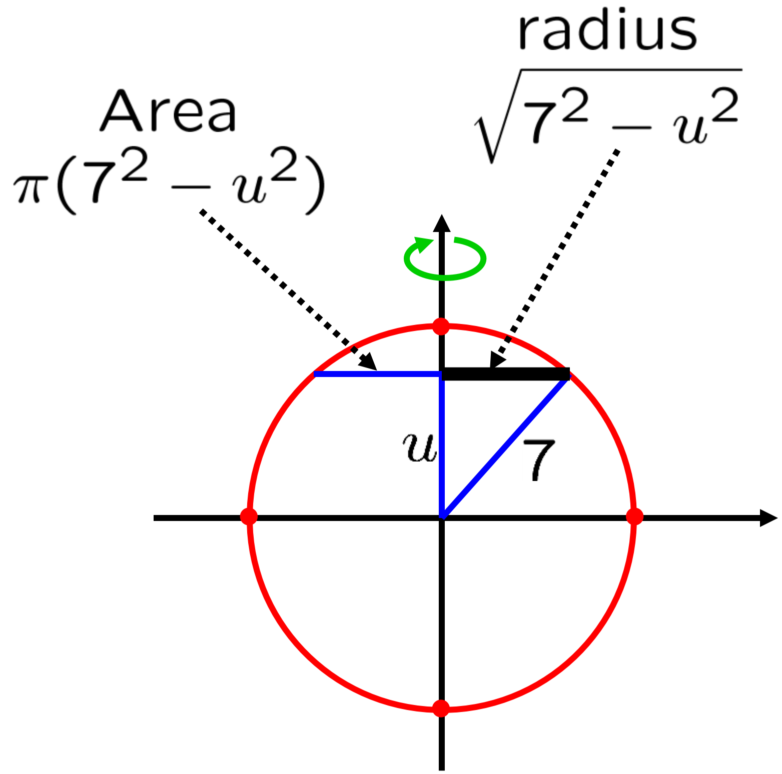

Goal: $\int_{-7}^{7} \pi(7^2-u^2)\,du = 2\left[\pi(7^3)\left(1-\frac{1}{3}\right)\right]$

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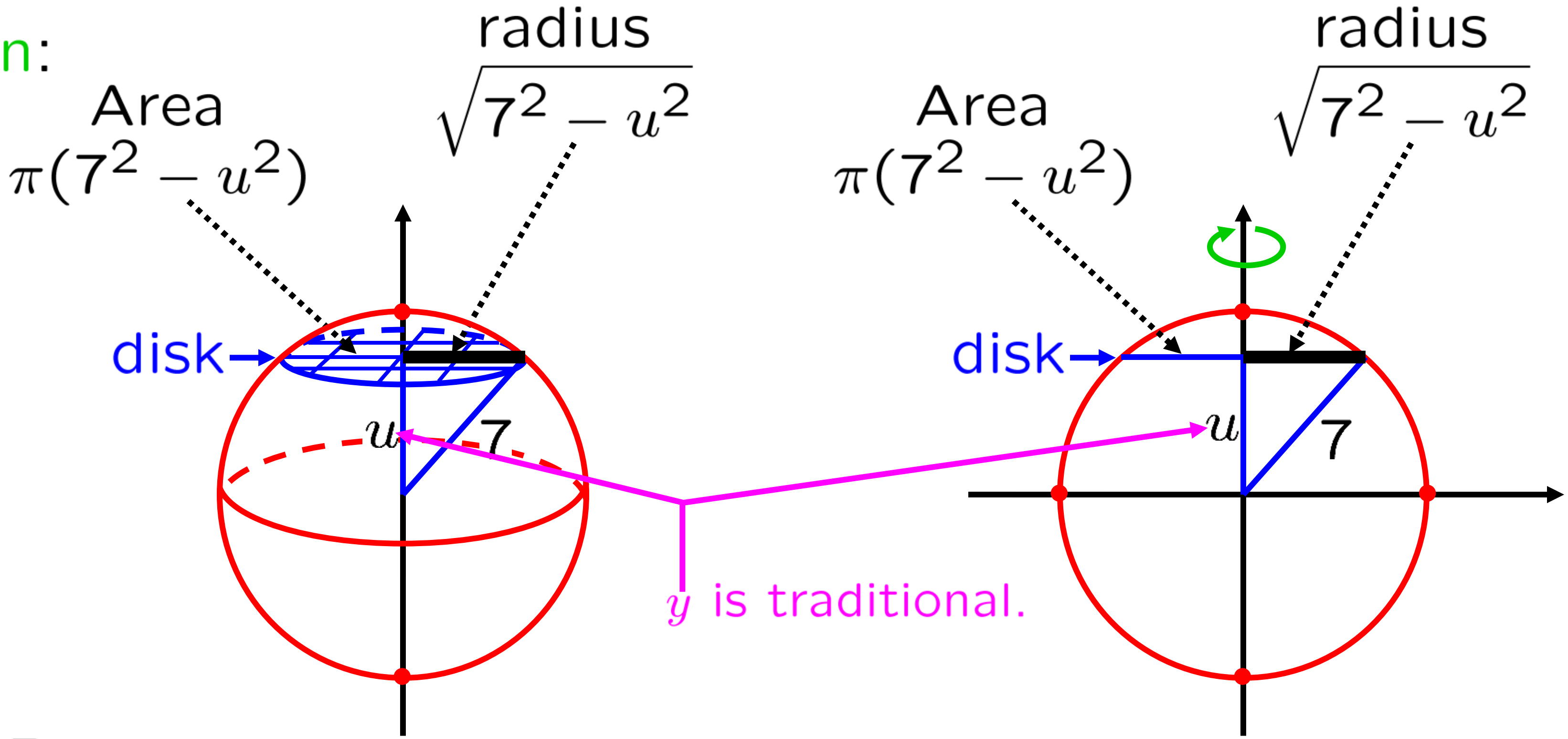

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y

$= 2\left[\pi(7^3)\left(\frac{2}{3}\right)\right] = \frac{4}{3}\pi 7^3$ ■

Could we use vertical disks?

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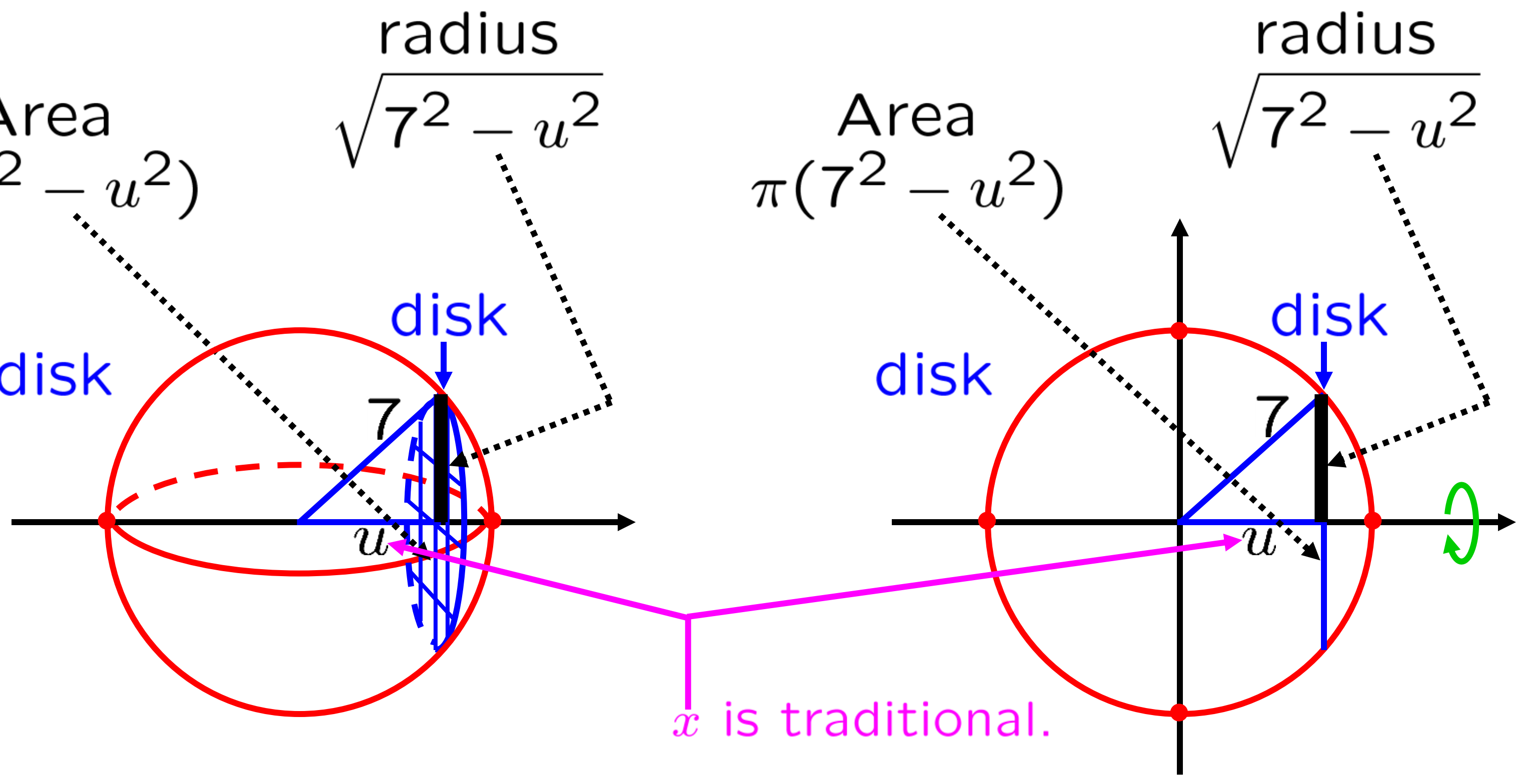

Goal: $\int_{-7}^{7} \pi(7^2 - u^2)\, du = 2\left[\pi(7^3)\left(1 - \frac{1}{3}\right)\right]$

x

SKILL
vol solid

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Could we use vertical disks? Yes. Next: Generalization...

DEFINITION:
A solid obtained by revolving, about a line, a subset of a plane, is called a **solid of revolution**.

THE DISK METHOD:

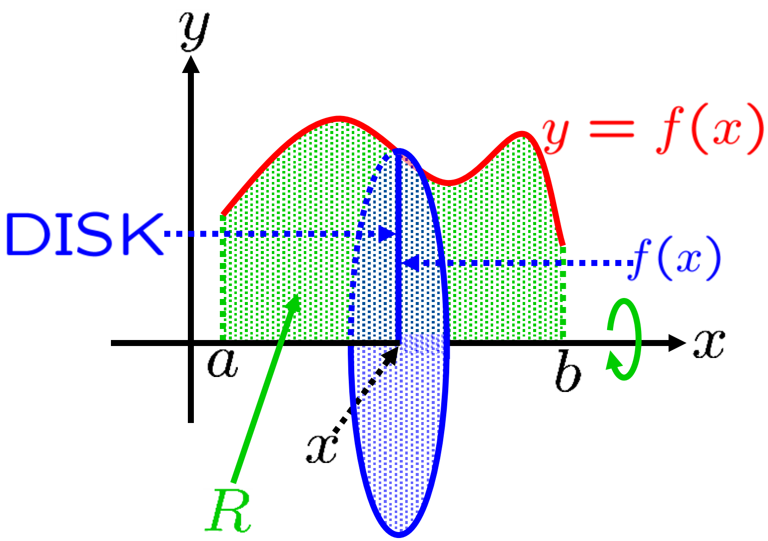
Assume $f \geq 0$ on $[a, b]$.

Let R be the region between the x-axis and the graph of $y = f(x)$ from $x = a$ to $x = b$.

If S is the solid obtained by revolving R about the horizontal axis,

then the volume of S is $\int_a^b \pi[f(x)]^2\, dx$.

area of disk $= \pi[f(x)]^2$



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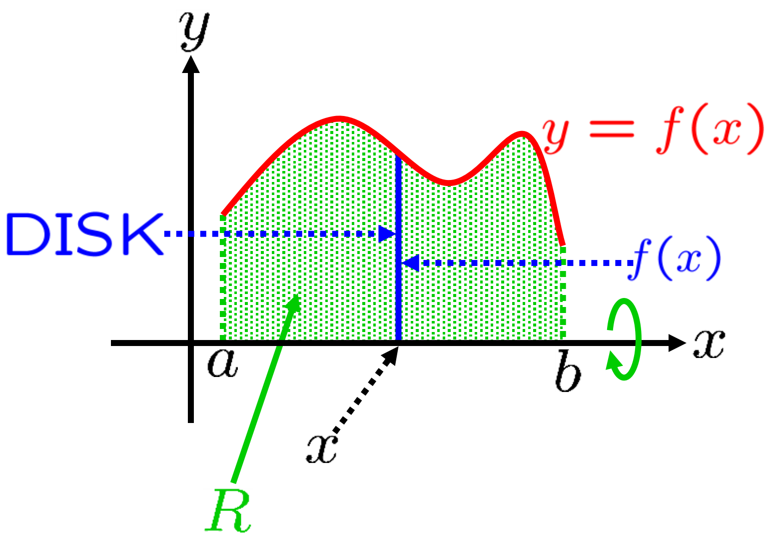
if desired



Next: General setup with horizontal disks . . .

$x \leftrightarrow y$

THE DISK METHOD:

Assume $f \geq 0$ on $[a, b]$.

Let R be the region between the y-axis and the graph of $x = f(y)$ from $y = a$ to $y = b$.

If S is the solid obtained by revolving R about the vertical axis,

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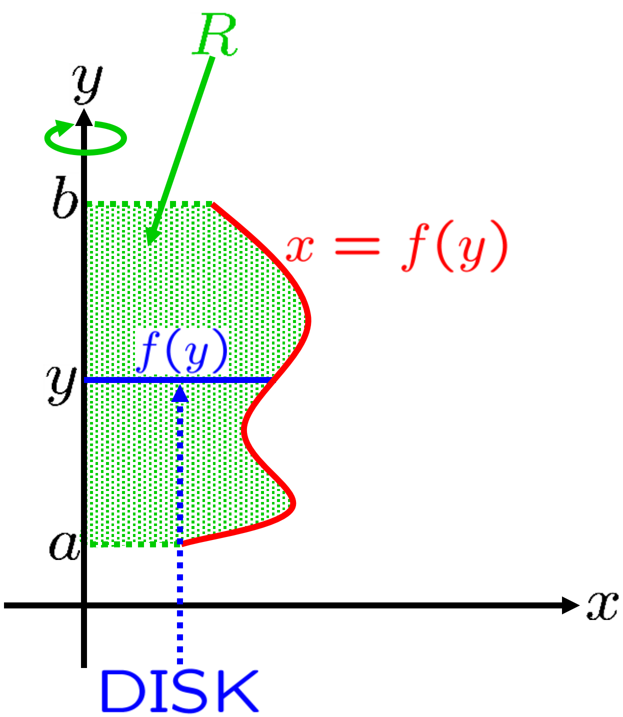
if desired



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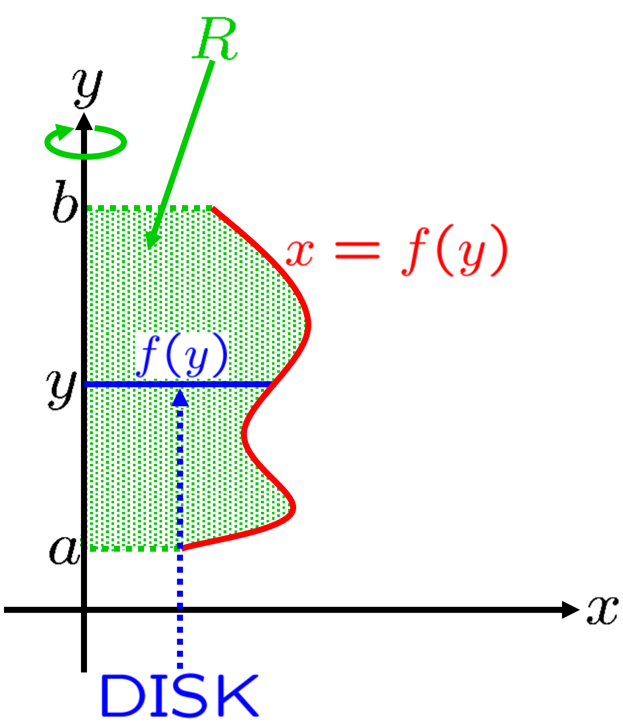


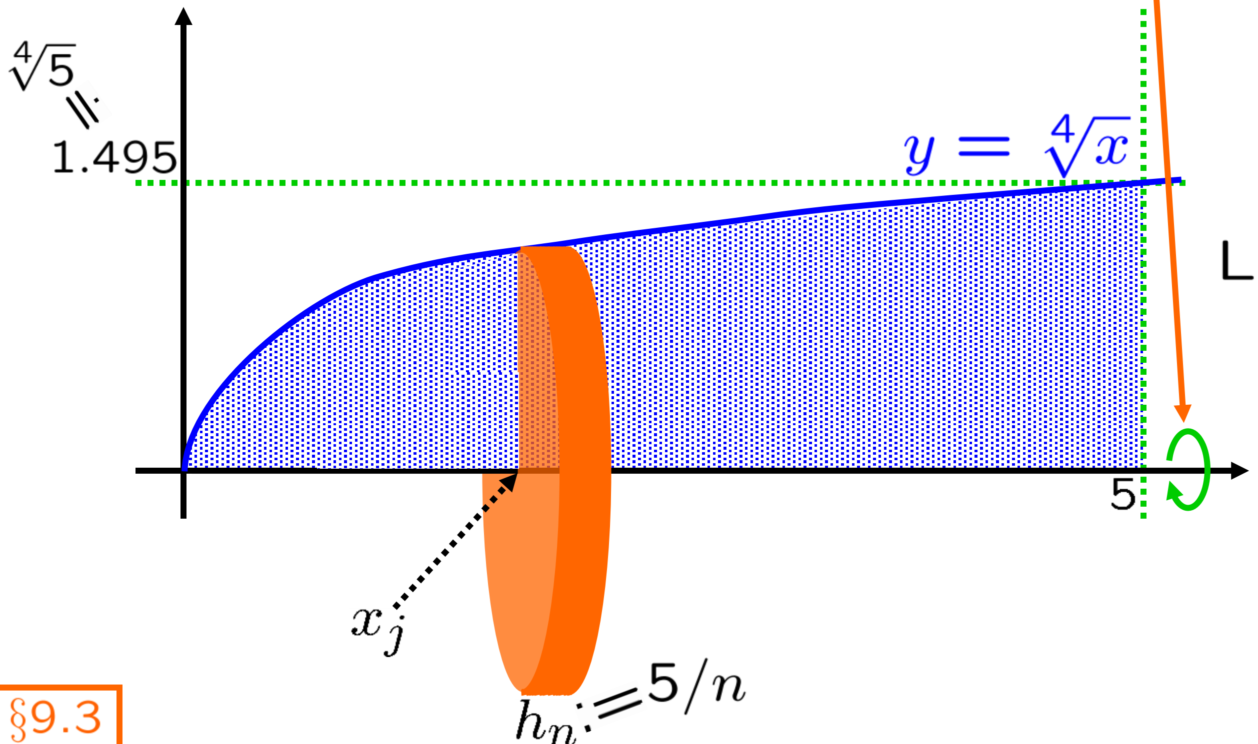
The axis of rotation may be neither horizontal nor vertical.

It might point left or down.

The variable may be neither “x” nor “y”.

We only require that all cross-sections be disks.

EXAMPLE: Find the volume of the solid obtained by revolving, about the x-axis, the region under the curve $y = \sqrt[4]{x}$ from 0 to 5. Illustrate the definition of volume by sketching a typical approximating cylinder.



Left endpoints:

$x_j = 0 + (j-1)(5/n),$

$j = 1, \ldots, n$

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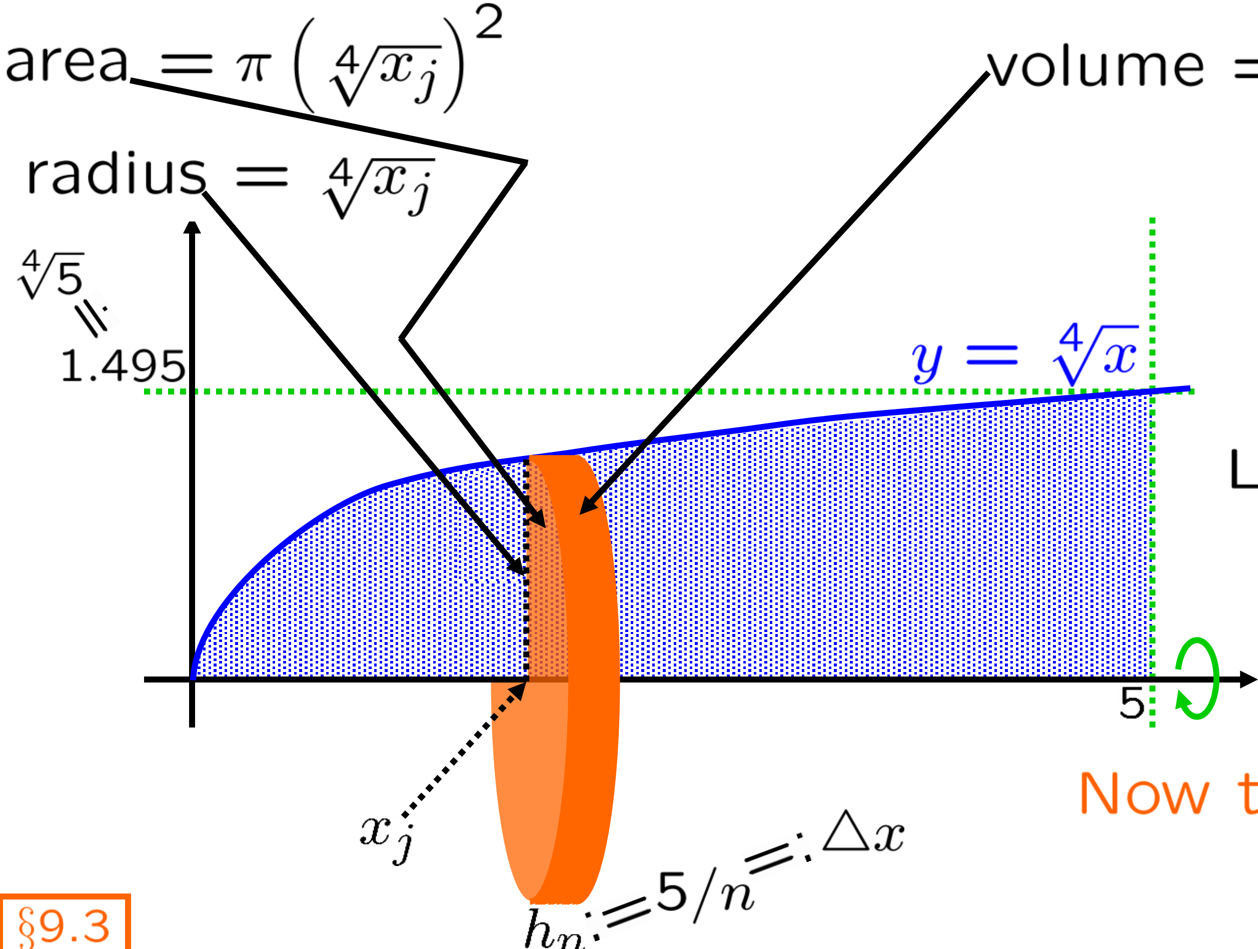
total volume of solid of revolution $\approx \sum_{j=1}^{n} \pi\left(\sqrt[4]{x_j}\right)^2 \triangle x$



Left endpoints:
$x_j = 0 + (j-1)(5/n)$,
$j = 1, \ldots, n$

Now take limit as $n \to \infty$.

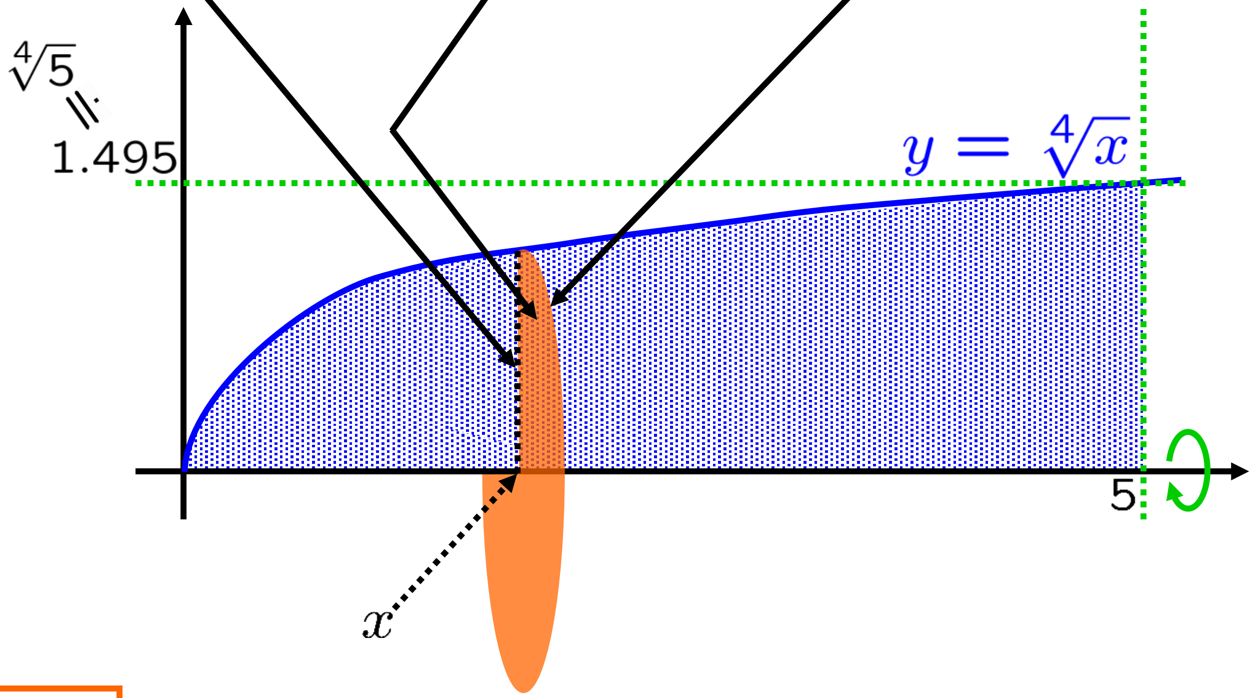
EXAMPLE: Find the volume of the solid obtained by revolving, about the x-axis, the region under the curve $y = \sqrt[4]{x}$ from 0 to 5.

total volume of solid of revolution $= \int_0^5 \pi \left(\sqrt[4]{x} \right)^2 \, dx$

area $= \pi \left(\sqrt[4]{x} \right)^2$

volume $= \pi \left(\sqrt[4]{x} \right)^2 \, dx$

radius $= \sqrt[4]{x}$



 Intuitive: dx

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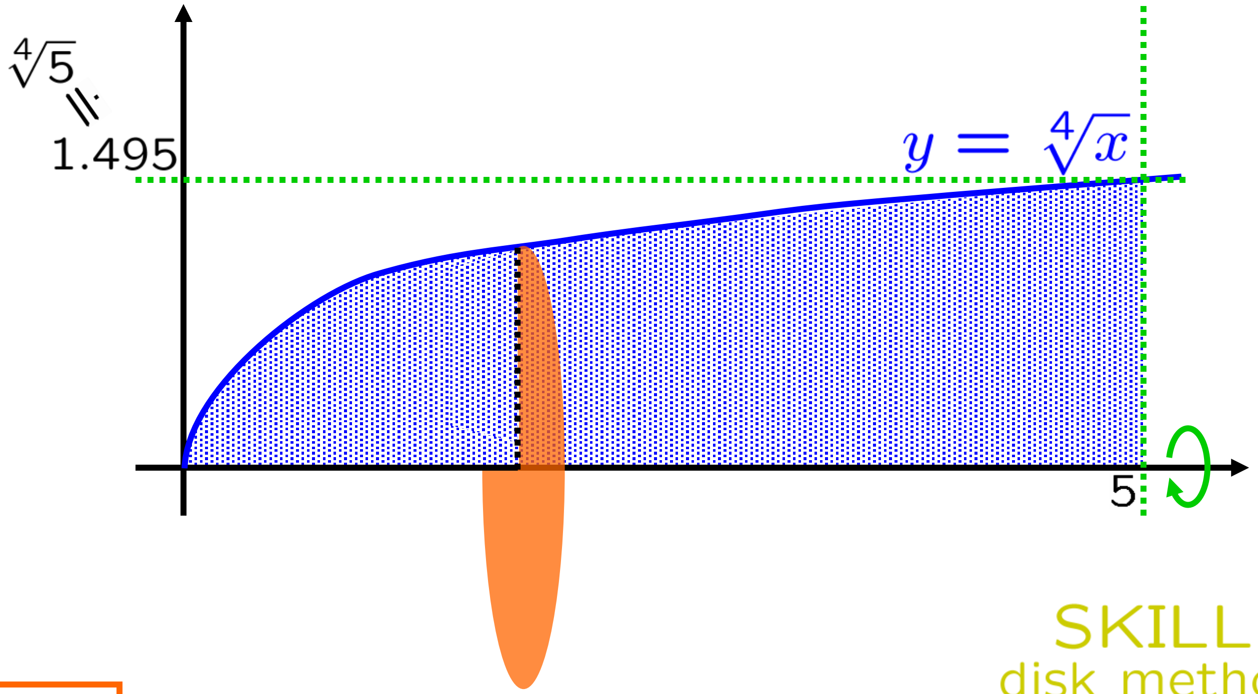
$$\| \quad \pi \int_0^5 \left(x^{1/4}\right)^2 \, dx$$

$$\| \quad \pi \int_0^5 x^{1/2} \, dx$$

$$\| \quad \pi \left[\frac{x^{3/2}}{3/2}\right]_{x:\to 0}^{x:\to 5}$$

$$\| \quad \pi \left[\frac{5^{3/2}}{3/2}\right]$$



SKILL
disk method

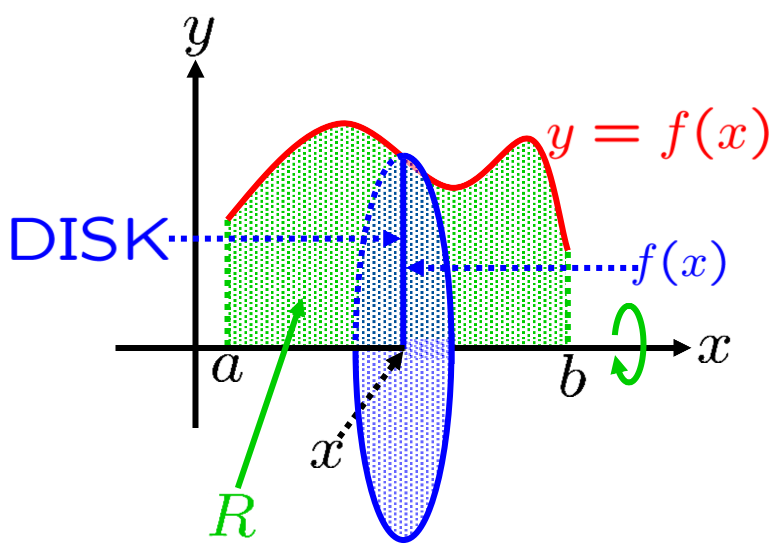
THE DISK METHOD:

Assume $f \geq 0$ on $[a, b]$. Let's try rotating a region between two functions. . .

Let R be the region between the x-axis and the graph of $y = f(x)$ from $x = a$ to $x = b$.

If S is the solid obtained by revolving R about the horizontal axis,

then the volume of S is $\int_a^b \pi [f(x)]^2 \, dx$. (if desired)



THE WASHER METHOD:

Assume $g \geq f \geq 0$ on $[a, b]$.

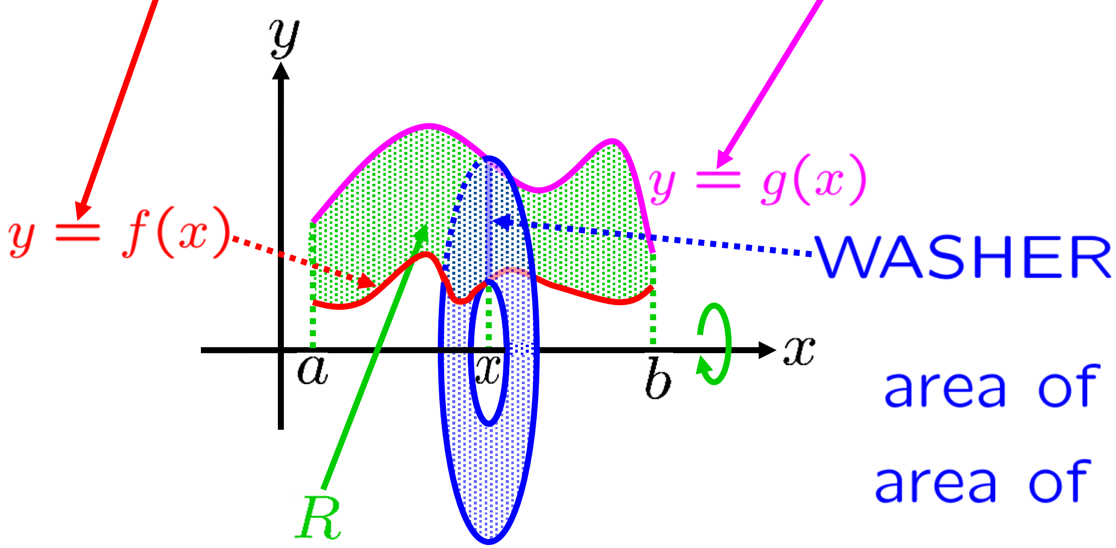
Let R be the region between the graphs of $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$.

If S is the solid obtained by revolving R about the horizontal axis,

if desired

then the volume of S is $\displaystyle\int_a^b \pi([g(x)]^2 - [f(x)]^2)\,dx$.

area of washer $= \pi([g(x)]^2 - [f(x)]^2)$



Horizontal washers . . .
$x \leftrightarrow y$

area of inner disk $= \pi[f(x)]^2$

area of outer disk $= \pi[g(x)]^2$

THE WASHER METHOD:

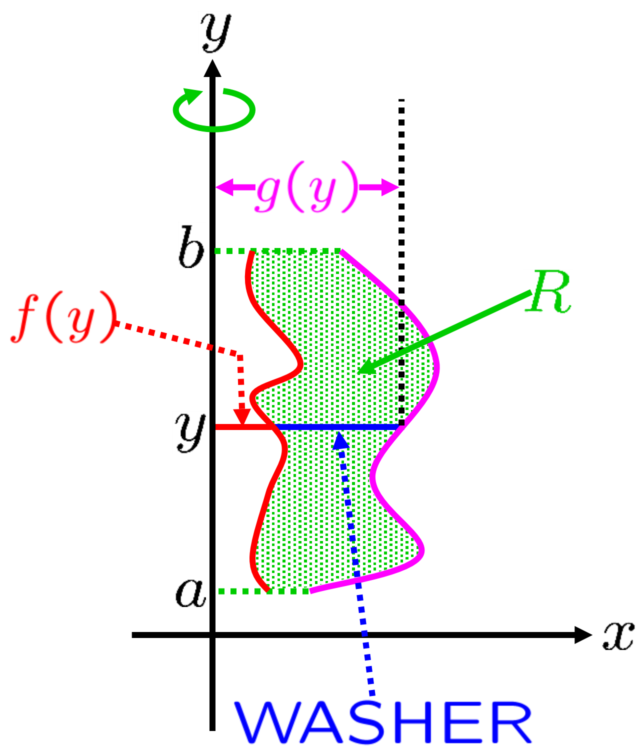
Assume $g \geq f \geq 0$ on $[a, b]$.

Let R be the region between the graphs of $x = f(y)$ and $x = g(y)$ from $y = a$ to $y = b$.

If S is the solid obtained by revolving R about the vertical axis,

then the volume of S is $\int_a^b \pi([g(y)]^2 - [f(y)]^2)\, dy$.

if desired



The axis of rotation may be neither horizontal nor vertical.

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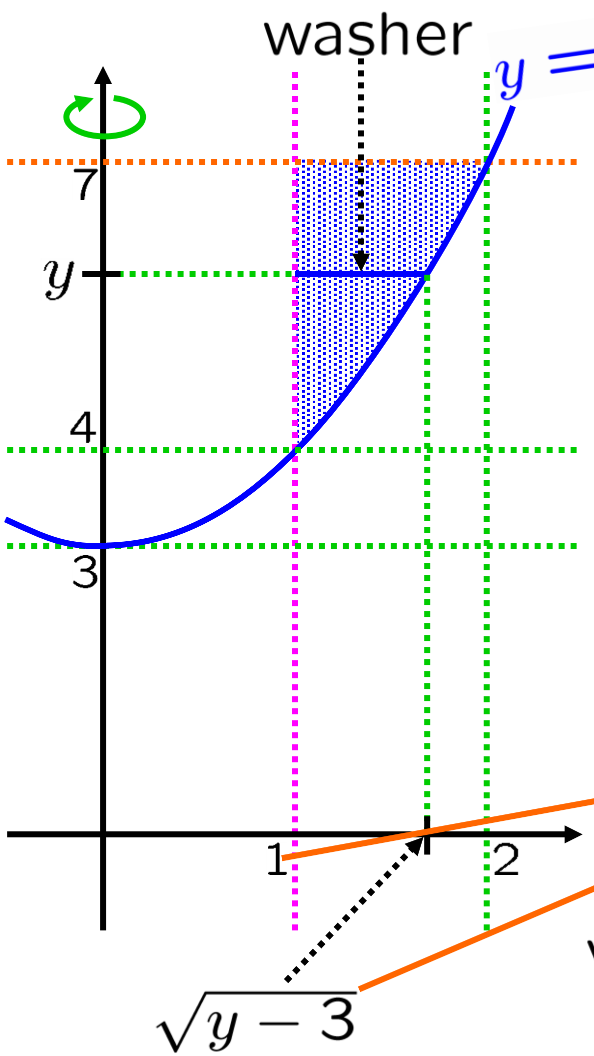
We only require that all cross-sections be washers.

EXAMPLE: Find the volume of the solid obtained by revolving, about the y-axis, the region bounded by $y = x^2 + 3$, $y = 7$ and $x = 1$.



outer radius(washer) = $\sqrt{y-3}$

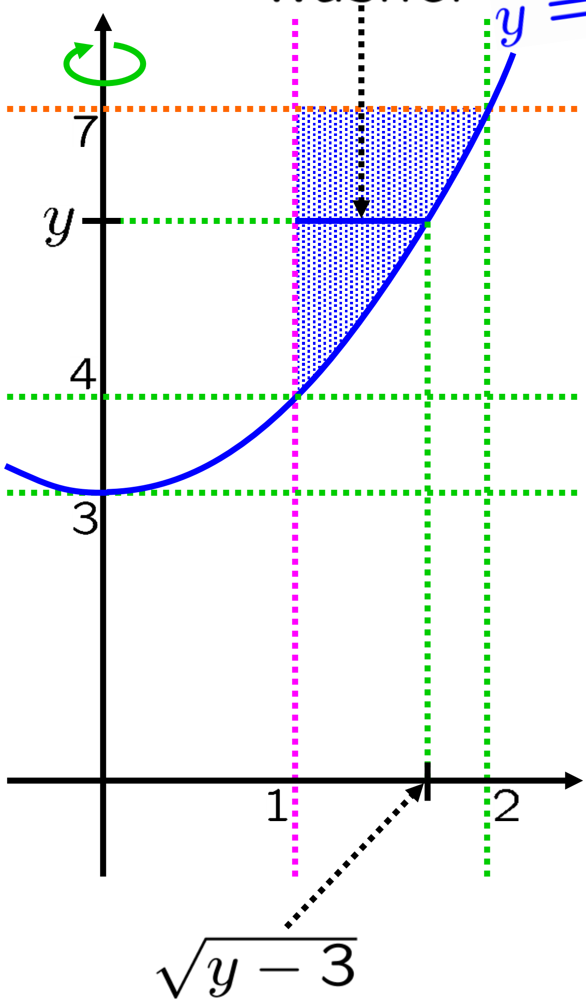
inner radius(washer) = 1

$$\text{area(washer)} = \pi\left(\sqrt{y-3}\right)^2 - \pi(1)^2$$

$$\text{vol(washer)} = \left[\pi\left(\sqrt{y-3}\right)^2 - \pi(1)^2\right] dy$$

$$\text{vol(solid)} = \int_4^7 \left[\pi\left(\sqrt{y-3}\right)^2 - \pi(1)^2\right] dy$$

EXAMPLE: Find the volume of the solid obtained by revolving, about the y-axis, the region bounded by $y = x^2 + 3$, $y = 7$ and $x = 1$.



$$\left[\frac{\pi y^2}{2} - 4\pi y\right]_{y:\to 4}^{y:\to 7}$$

$$||$$

$$\int_4^7 [\pi y - 4\pi]\, dy$$

$$||$$

$$\int_4^7 [\pi y - 3\pi - \pi]\, dy$$

$$||$$

$$\int_4^7 [\pi (y-3) - \pi]\, dy$$

$$||$$

$$\text{vol(solid)} = \int_4^7 \left[\pi\left(\sqrt{y-3}\right)^2 - \pi (1)^2\right] dy$$

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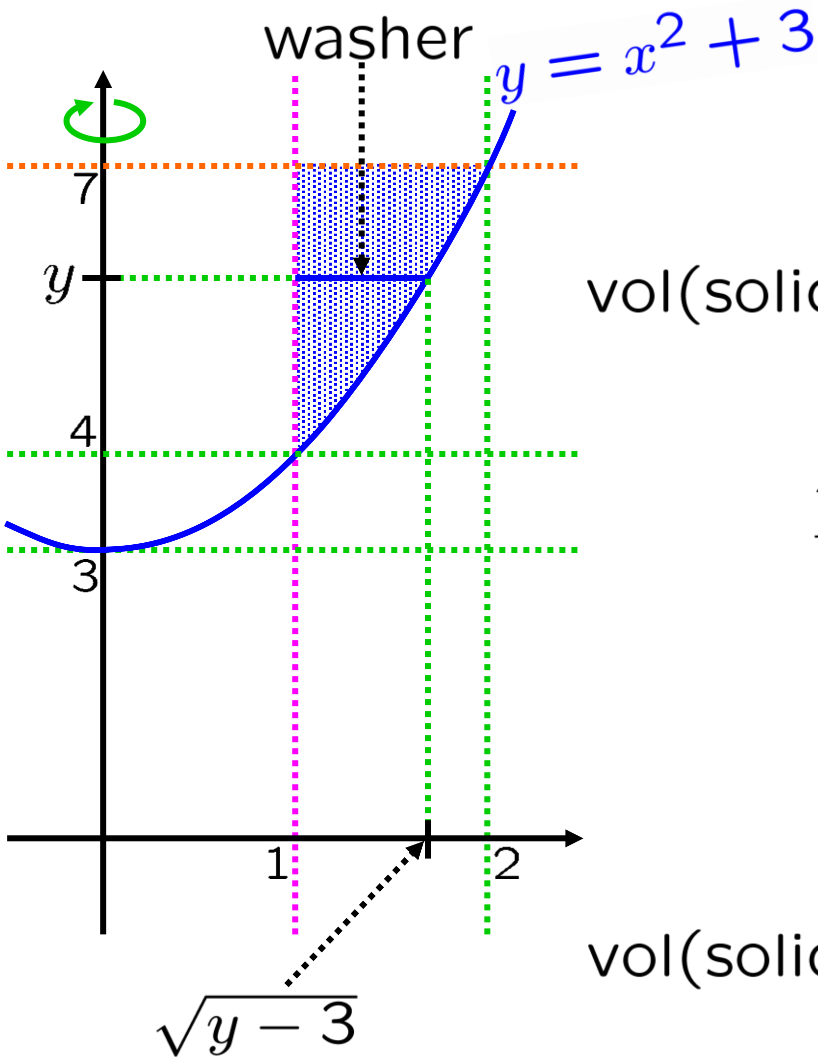

$$\left[\frac{\pi y^2}{2} - 4\pi y\right]_{y:\to 4}^{y:\to 7}$$

$$\text{vol(solid)} = \left[\frac{\pi y^2}{2} - 4\pi y\right]_{y:\to 4}^{y:\to 7}$$

$$\| $$

$$\frac{\pi(7^2 - 4^2)}{2} - 4\pi(7-4)$$

$$\text{vol(solid)} =$$

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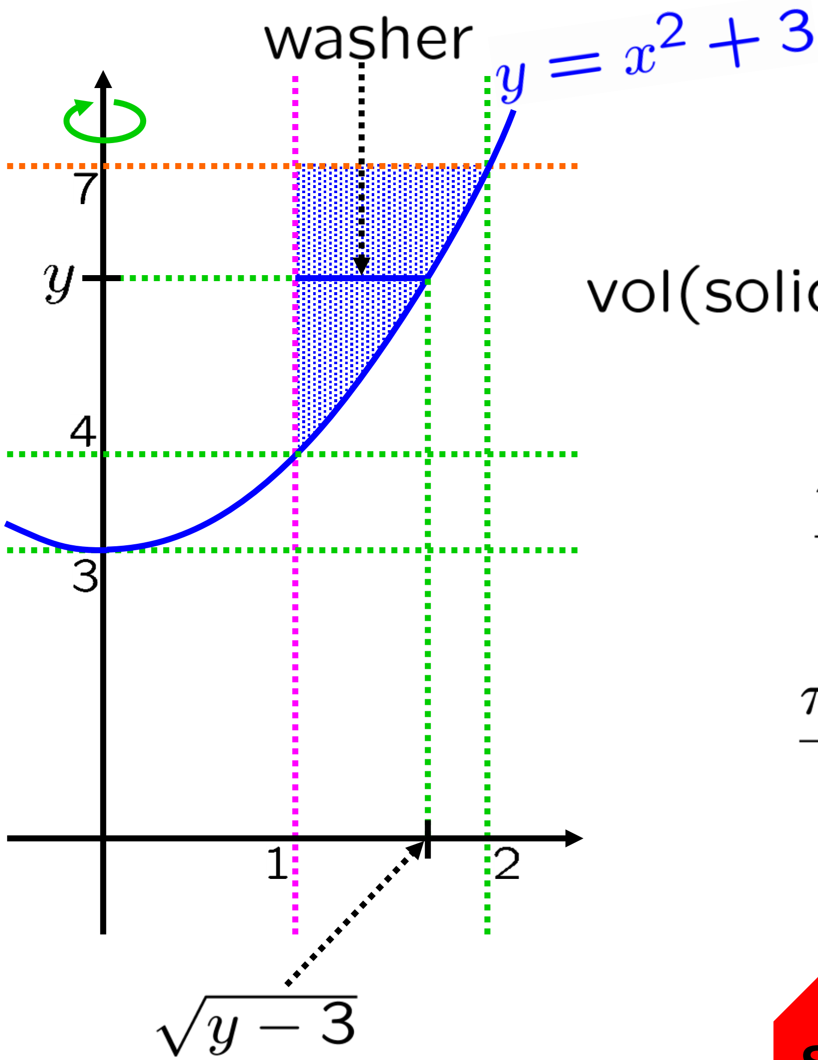

$$\text{vol(solid)} = \left[\frac{\pi y^2}{2} - 4\pi y\right]_{y:\to 4}^{y:\to 7}$$

$$\| $$

$$\frac{\pi(7^2 - 4^2)}{2} - 4\pi(7-4)$$

$$\|$$

$$\frac{\pi(49-16)}{2} - 12\pi$$

$$\|$$

$$\frac{33\pi}{2} - \frac{24\pi}{2} = \frac{9\pi}{2} \blacksquare$$

STOP

SKILL
washer method