Central Limit Theorem and Finance
Duluth 10 November 2008

Scot Adams

Kyle wants right, but not obligation, to buy 5000 shares of ABC for \$5000, 30 days from now.

Call option

Assume: Spot price = \$1/share.

REAL WORLD
uptick prob.:50.001%
dntick prob.:49.999%

7% ann exp. incr.
drift-vol assumption
20% ann volatility

Each second, price changes
either by a factor of 1.000035616
or by a factor of 0.999964386.

The one-second risk-free factor is 1.000000001.

3.2038527% ann int rate

Goal: Find the "right" price, *i.e.*, the price that can be used to set up a "perfect hedge".

Difficulty: $30 \times 24 \times 60 \times 60$ adjustments

Salvation: The Central Limit Theorem!

Kyle wants right, but not obligation, to buy 5000 shares of ABC for \$5000, 30 days from now.

Gail, seller

Call option

Assume: Spot price = \$1/share.

Each second, price changes either by a factor of 1.000035616 $=: u$ or by a factor of 0.999964386 $=: d$.

$1 + \iota = \rho$

The one-second risk-free factor is 1.000000001 $=: \rho$.

$\iota := 0.00000001 = \rho - 1$

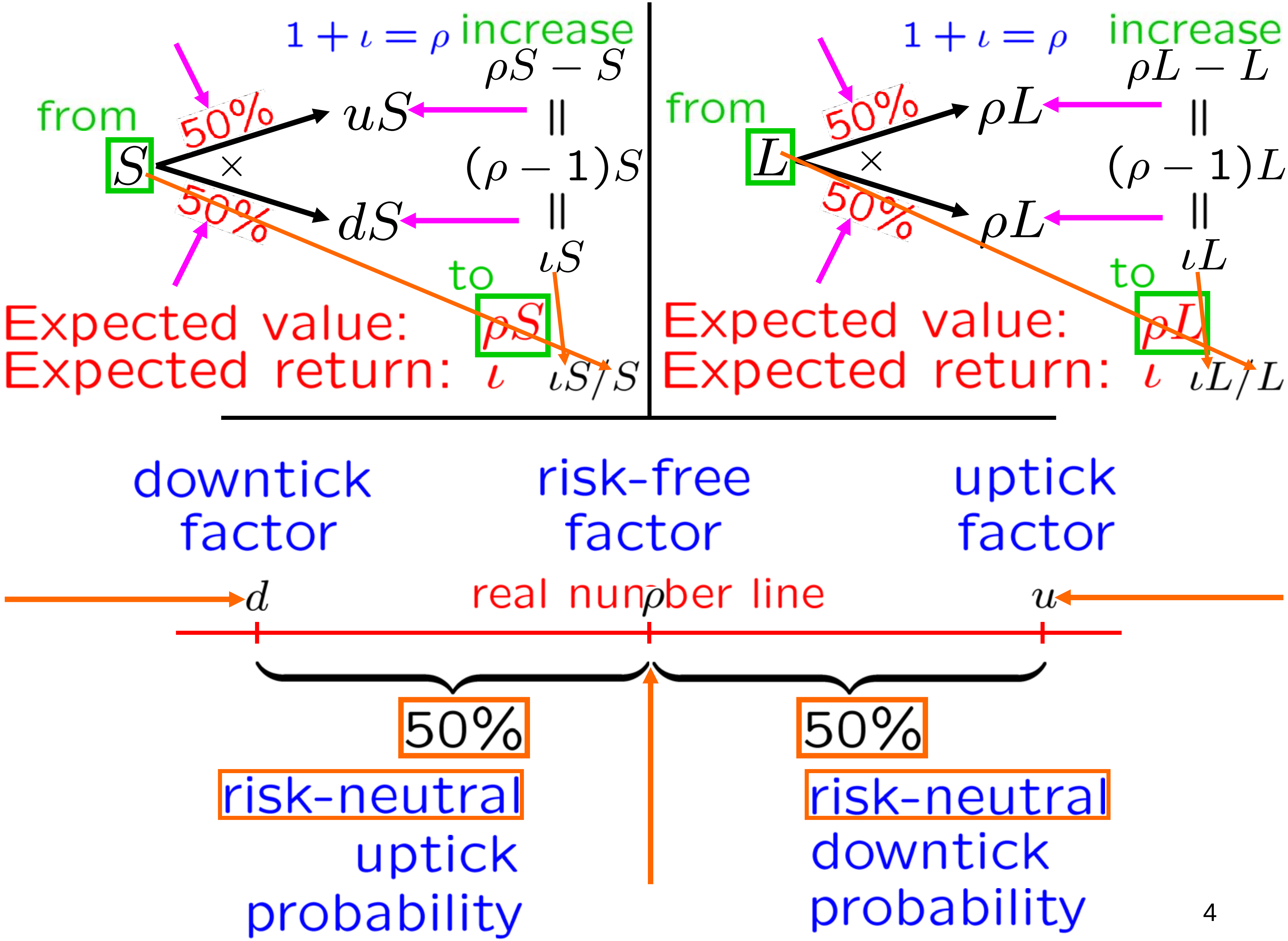
Goal: Find the "right" price, *i.e.*, the price that can be used to set up a "perfect hedge".

Payoff function:

$f(S) = (5000S - 5000)_+$

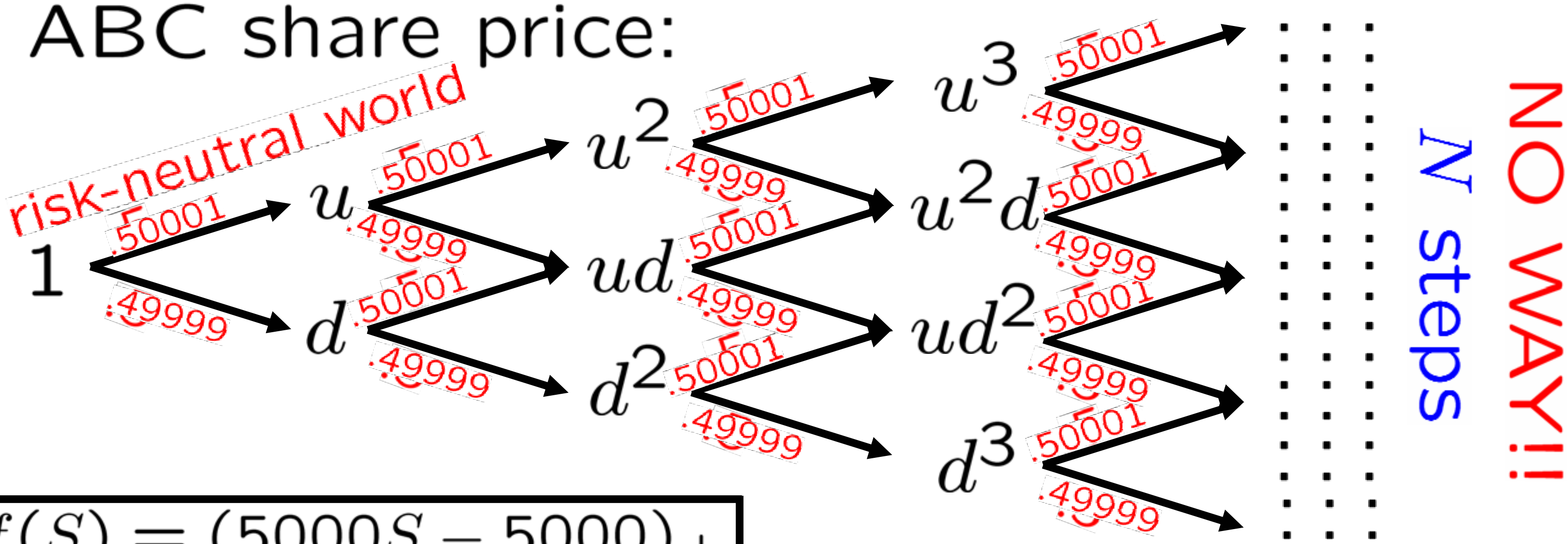
Exercise: Graph f.

$N := 30 \times 24 \times 60 \times 60 = 2,592,000$



1 + ι = ρ increase
ρS − S
from
S
50%
50%
uS
dS
(ρ − 1)S
ιS
to
ρS
Expected value: ρS
Expected return: ι ιS/S
1 + ι = ρ increase
ρL − L
from
L
50%
50%
ρL
ρL
(ρ − 1)L
ιL
to
ρL
Expected value: ρL
Expected return: ι ιL/L
downtick factor
risk-free factor
uptick factor
d
real number line
ρ
u
50%
risk-neutral uptick probability
50%
risk-neutral downtick probability

ABC share price:



$f(S) = (5000S - 5000)_+$

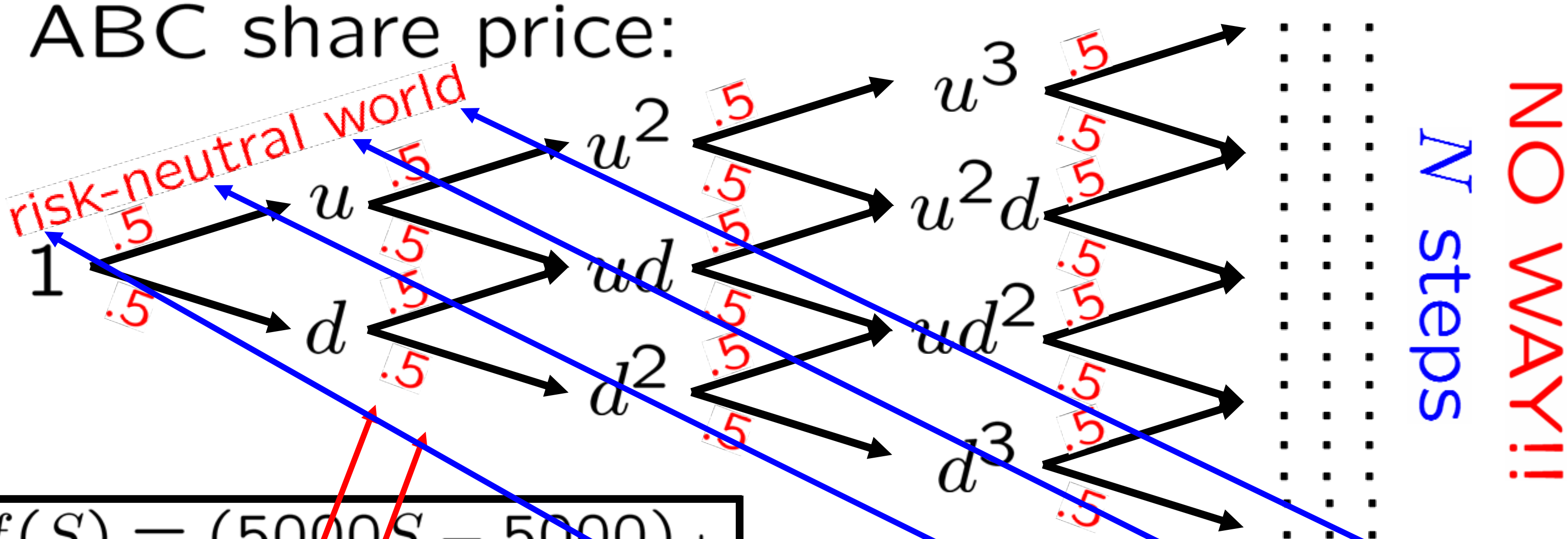
Ending ABC share price:

1.000035616 $= u$

0.999964386 $= d$

Ending ABC share price	Contingent claim:
u^N	$f(u^N)$
$u^{N-1}d$	$f(u^{N-1}d)$
$u^{N-2}d^2$	$f(u^{N-2}d^2)$
$\vdots$	$\vdots$
d^N	$f(d^N)$

ABC share price:



$f(S) = (5000S - 5000)_+$

$\boxed{P}$:= price of option

Goal = initial value of hedging portfolio

$1 + \iota = \rho$

$\rho^N \boxed{P} = (1+\iota)^N P =$

expected final value of hedging portfolio =

expected contingent claim

Contingent claim:

$f(u^N)$	prob. N, 0
$f(u^{N-1}d)$	prob. $N-1$, 1
$f(u^{N-2}d^2)$	prob. $N-2$, 2
$\vdots$	$\vdots$
$f(d^N)$	prob. 0, N

Coin-flipping game: Flip a fair coin N times.

If H heads and T tails,

pay $f(u^H d^T)$,

30 days from now.

ρ^{-N}

$1 + \iota = \rho$

$\rho^N P =$ expected payout $=: E$

$P = \rho^{-N} E$ equal: $(1 + \iota)^{-N} E$

$f(S) = (5000S - 5000)_+$

$P :=$ price of option

Goal $=$ initial value of hedging portfolio

$\rho^N P = (1+\iota)^N P =$

expected final value of hedging portfolio $=$

expected contingent claim

Contingent claim:

$f(u^N)$	N, 0
$f(u^{N-1}d)$	$N-1$, 1
$f(u^{N-2}d^2)$	$N-2$, 2
⋮	⋮
$f(d^N)$	0, N

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $f(u^H d^T)$,
30 days from now.

$1+\iota=\rho$

$\rho^N P =$ expected payout $=: E =$???

Hard problem
expected value problem

$P = \rho^{-N} E = (1+\iota)^{-N} E$
$=$ discounted expected payout

$P :=$ price of option
$=$ initial value of hedging portfolio

$\rho^N P = (1+\iota)^N P =$
expected final value of hedging portfolio $=$
expected contingent claim

Contingent claim:

$f(u^N)$	N, 0
$f(u^{N-1}d)$	$N-1$, 1
$f(u^{N-2}d^2)$	$N-2$, 2
$\vdots$	$\vdots$
$f(d^N)$	0, N

probability problems, *then* expected value problems

Easier problem:
Compute the probability that

$$-\sqrt{N} < H - T < \sqrt{N}.$$

DIVIDE BY $\sqrt{N}$

$$X := (H - T)/\sqrt{N}$$

Easier problem after restatement:
Compute the probability that

$$-1 < X < 1.$$

$H_1 :=$ number of heads after first flip
$H_2 :=$ number of heads after second flip
$\vdots$
$H_N :=$ number of heads after Nth flip $= H$

Easier problem:

Compute the probability that

$$-\sqrt{N} < H - T < \sqrt{N}.$$

$$X := (H - T)/\sqrt{N}$$

Easier problem after restatement:

Compute the probability that

$$-1 < X < 1.$$

X is hard . . .

For all integers $j \in [1, N]$,

$H_j :=$ number of heads after jth flip

$T_j :=$ number of tails after jth flip

$D_j := H_j - T_j$

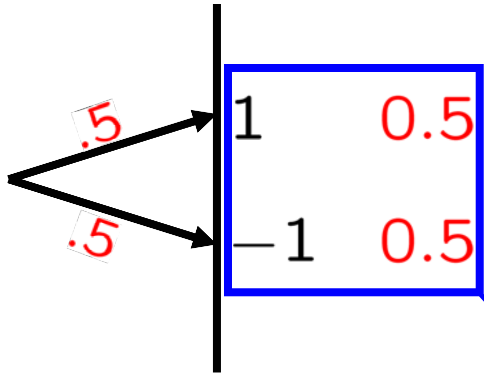
Easier: D_1, $D_1/7$, D_2, D_N

$$H = H_N, \quad T = T_N, \quad X = (H_N - T_N)/\sqrt{N} = D_N/\sqrt{N}$$

$D_1 = \overset{0}{H_1} - \overset{1}{T_1}$:

random variable

a variable whose value is determined by random events



(probability) measure of D_1

(probability) distribution of D_1

distribution of $T_1 - H_1$ is exactly the same

keep the distribution

forget its origin

divide by 7

$D_1 = H_1 - T_1$:

What about $D_1/7$?

Cannot recover the ra
Only its distribution.

0.5 — 1 $\qquad z^1 = z$

0.5 — -1 $\qquad z^{-1}$

Fourier transform of the distribution of D_1 is $\cos t$

$i = \sqrt{-1}$

$T_1 - H_1$ has the same distribution.

Replace z by e^{-it}

(expression of z)

Generating function:

$(0.5)z + (0.5)z^{-1}$

Fourier transform:

$(0.5)e^{-it} + (0.5)e^{it}$

$\|$

$\cos t$

$\xi\ t$ not time

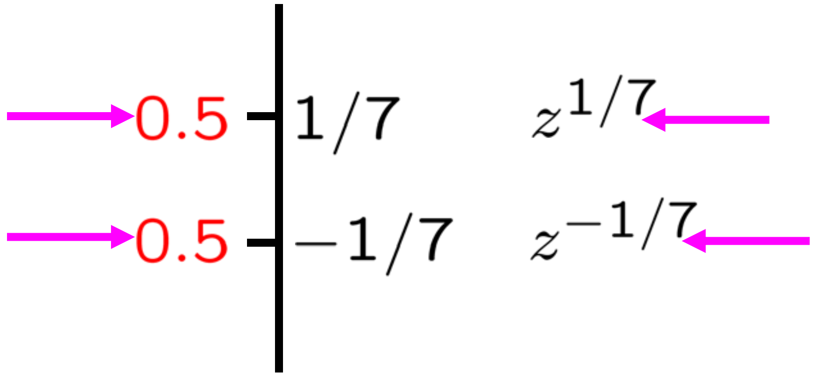
keep the distribution
forget its origin

Repl. t by $t/7$

0.5 × [$e^{it} = \cos t + i \sin t$] +
0.5 × [$e^{-it} = \cos t - i \sin t$]

Inverse Fourier transform

$D_1/7$:



What about $D_1/7$?

Replace t by $t/7$.

$i = \sqrt{-1}$

Replace z by e^{-it}

Generating function: $(0.5)z^{1/7} + (0.5)z^{-1/7}$

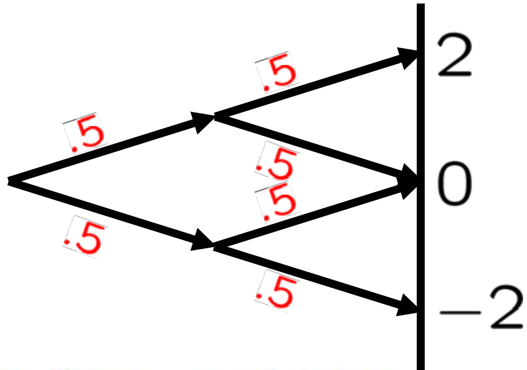
Fourier transform: $(0.5)e^{-it/7} + (0.5)e^{it/7}$

$$\| $$

$\cos(t/7)$

$$e^{it/7} = \cos(t/7) + i\,\sin(t/7)$$
$$e^{-it/7} = \cos(t/7) - i\,\sin(t/7)$$

$D_2 = \overset{0}{H_2} - \overset{2}{T_2}$:



2	0.25
0	$0.25 + 0.25 = 0.5$
-2	0.25

forget its origin keep the distribution

$D_2 = H_2 - T_2$:

0.25	2	z^2
0.5	0	$z^0 = 1$
0.25	-2	z^{-2}

forget its origin keep the distribution

Generating function:

$$(0.25)z^2 + 0.5 + (0.25)z^{-2}$$
$$= ((0.5)z + (0.5)z^{-1})^2$$

the generating function of the distribution of D_1

$$i = \sqrt{-1}$$

Replace z by e^{-it}

Fourier transform: $(\cos t)^2 = \cos^2 t$

$D_N = \cancel{H_N - T_N}$:

divide by $\sqrt{N}$

NO WAY!!

Goal: $X \approx$

What about $D_N/\sqrt{N}$?

Replace t by $t/\sqrt{N}$.

Generating function:

NO WAY!!

$$= ((0.5)z + (0.5)z^{-1})^N$$

the generating function of the distribution of D_1

$i = \sqrt{-1}$

Replace z by e^{-it}

Fourier transform: $(\cos t)^N = \cos^N t$

$X = D_N/\sqrt{N}$:

Goal: $X \approx D_N/\sqrt{N}$

What about $D_N/\sqrt{N}$?

Replace t by $t/\sqrt{N}$.

NO WAY!!

Fourier transform: $\cos^N(t/\sqrt{N})$

$X = D_N/\sqrt{N}$:

Generating functions
Fourier transforms

NO WAY!!

Fourier transform: $\cos^N(t/\sqrt{N})$

Fourier transform: $\cos^N(t/\sqrt{N})$

$X = D_N/\sqrt{N}$:

Generating functions
Fourier transforms
Fourier analysis
Spectral theory

Useful?

NO WAY!!!

Easier problem after restatement:

Compute the probability that

$$-1 < X < 1.$$

Exercise: $\lim_{n\to\infty} \cos^n(3/\sqrt{n}) = e^{-3^2/2}$

Fourier transform: $\boxed{\cos^N(t/\sqrt{N})}$

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

Verify for $t = 3$.

$X = D_N/\sqrt{N}$:

Fourier transform: $\cos^N(t/\sqrt{N})$

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

Fourier transform: $\cos^N(t/\sqrt{N})$

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

$X = D_N/\sqrt{N}$:

Fourier transform: $\cos^N(t/\sqrt{N})$

$$\approx \lim_{n \to \infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

in d How to find Z?

Inverse Fourier Transform

Easier problem after restatement:

Compute the probability that

$$-1 < X < 1.$$

Compute the probability that

$$-1 < Z < 1.$$

Z:

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ — infinitesimal	x

Do this for *all* $x \in \mathbb{R}$

$\exists$ RV Z with this dist.

NOTES

Mistake:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

$D_2 \in \{2, 0, -2\}$

distribution supported on three points

$D_N \in \{-N, -N+2, \ldots, N-2, N\}$

distribution supported on $N+1$ points

By contrast, the distribution of Z does not have finite support.

Z:

$$\frac{1}{\sqrt{2\pi}} \, e^{-x^2/2} \, dx \quad \vdash x$$

Do this for all $x \in \mathbb{R}$

Problem: Compute the probability that

$$Z = 7$$

Solution:

$$\int_7^7 \frac{1}{\sqrt{2\pi}} \, e^{-x^2/2} \, dx = 0$$

Z:

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \quad \Big|\, x$$

Do this for *all* $x \in \mathbb{R}$

Problem: Compute the probability that

$$2 < Z < 3$$

Solution: $$\int_2^3 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = [\Phi(x)]_{x=2}^{x=3}$$

$$= \Phi(3) - \Phi(2) = 0.0214$$

$$= 2.14\%$$

Z:

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \quad x \qquad z^x$$

Do this for *all* $x \in \mathbb{R}$

Generating function:

$$\int_{-\infty}^{\infty} z^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \text{Exercise}$$

Fourier transform:

Verify for $t = 3$.

$$\int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

Z:
$\wr\wr$
X

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ | x $\qquad z^x$

Do this for all $x \in \mathbb{R}$

Exercise: $$\int_{-\infty}^{\infty} e^{-3ix} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = e^{-3^2/2}$$

Fourier transform:

Verify for $t = 3$.

$$\int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

Z:

$\approx$

X

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ — x $\quad z^x$

probability problems,
then expected value problems

Do this for *all* $x \in \mathbb{R}$

Easier problem after restatement:

Compute the probability that
$-1 < X < 1.$

Approximately equal to the probability that
$-1 < Z < 1.$

Approximate solution:

Berry-Esseen Theorem

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = [\Phi(x)]_{x=-1}^{x=1}$$
$$= 68.27\%$$

Goal:

Compute the expected value of $f(u^H d^T)$.

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $f(u^H d^T)$,
30 days from now.

$\rho^N P =$ expected payout $=: E = ???$

$\boxed{P} = \rho^{-N} E = (1+\iota)^{-N} E$

$=$ discounted expected payout

$\iota := 0.00000001$

$$f(S) = (5000S - 5000)_+$$

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $f(D_2)$.

$D_2 = H_2 - T_2$:

0.25	2
0.5	0
0.25	-2

$f(2)$

$f(0)$

$f(-2)$

$f :\to g$

works for any function

$$[0.25][f(2)] + [0.5][f(0)] + [0.25][f(-2)] = 1,250$$

Define: $g(S) = 5e^S + S^2$

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $g(D_2)$.

$D_2 = H_2 - T_2$:

0.25	2	$g(2)$
0.5	0	$g(0)$
0.25	-2	$g(-2)$

$$[0.25][g(2)] \quad + \quad [0.5][g(0)] \quad + \quad [0.25][g(-2)] \quad = \quad \text{Exercise}$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $f(Z)$.

Z:	
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$	x

$f(x)$ ← Do this for all $x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} [f(x)] \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$f(x) = (5000x - 5000)_+$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)]\, e^{-x^2/2}\, dx = \text{exercise}$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $f(X)$.

Z:

$\approx$

X

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ | x $\quad f(x)$

Do this for all $x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} [f(x)] \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$f(x) = (5000x - 5000)_+$

Approx. Sol'n: $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)]\, e^{-x^2/2}\, dx$ = exercise

Recall: $f(S) = (5000S - 5000)_+$

write H, T as expr.s of X

Goal:

Compute the expected value of $\boxed{f(u^H d^T)}$.

$\| ?$

New easier problem:

Compute the expected value of $\boxed{g(X)}$.

Approx. Sol'n: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

Recall: $f(S) = (5000S - 5000)_+$ write H, T as expr.s of X

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $g(X)$.

$X = (H - T)/\sqrt{N}$ $\quad N = 2,592,000$

$\times\sqrt{N}$ $\quad \times\sqrt{N}$

$H + T = N$ $\quad H + T = N$

$H - T = X\sqrt{N}$ ADD NEGATE $\quad -H + T = -X\sqrt{N}$ ADD

$2H = N + X\sqrt{N}$ $\quad 2T = N - X\sqrt{N}$

Approx. Sol'n: $\dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

write H, T
as expr.s of X

Recall: $f(S) = (5000S - 5000)_+$

Goal:

Compute the expected value of $f(u^H d^T)$.

$\|?$

New easier problem:

Compute the expected value of $g(X)$.

$$H = N/2 + X\sqrt{N}/2 \qquad T = N/2 - X\sqrt{N}/2$$

$$u^H = u^{N/2} u^{X\sqrt{N}/2} \qquad d^T = d^{N/2} d^{-X\sqrt{N}/2}$$

$$u^H d^T =$$

$N := 30 \times 24 \times 60 \times 60 = 2,592,000$

$$2H = N + X\sqrt{N} \qquad 2T = N - X\sqrt{N}$$

Approx. Sol'n: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

Compute the expected value of $f(u^H d^T)$.

$\| ?$

New easier problem:

Compute the expected value of $g(X)$.

$$H = N/2 + X\sqrt{N}/2 \qquad T = N/2 - X\sqrt{N}/2$$

$$u^H = u^{N/2} u^{X\sqrt{N}/2} \qquad d^T = d^{N/2} d^{-X\sqrt{N}/2}$$

$$u^H d^T = u^{N/2}\, d^{N/2} \quad u^{X\sqrt{N}/2}\, d^{-X\sqrt{N}/2}$$

$$= (ud)^{N/2} \quad (u/d)^{X\sqrt{N}/2}$$

$$= C \quad e^{kX}$$

$$C := (ud)^{N/2}$$

$$k := \ln((u/d)^{\sqrt{N}/2})$$

Approx. Sol'n: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

$$e^k = (u/d)^{\sqrt{N}/2}$$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

Compute the expected value of $f(u^H d^T)$.

$\parallel$

New easier problem:

Compute the expected value of $g(X)$.

$$f(u^H d^T) = f(Ce^{kX}) = g(X)$$

$$g(x) := f(Ce^{kx})$$

$$\begin{aligned} u^H d^T &= u^{N/2}\, d^{N/2}\; u^{X\sqrt{N}/2}\, d^{-X\sqrt{N}/2} \\ &= (ud)^{N/2}\,(u/d)^{X\sqrt{N}/2} \\ &= C\; e^{kX} \end{aligned}$$

$$C := (ud)^{N/2}$$

$$k := \ln((u/d)^{\sqrt{N}/2})$$

Approx. Sol'n: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

Recall: $f(S) = (5000S - 5000)_+$

Goal:

Compute the expected value of $f(u^H d^T)$.

$\| $

Restatement of goal:

Compute the expected value of $g(X)$.

$$g(x) := f(Ce^{kx})$$

$$f(u^H d^T) = f(Ce^{kX}) = g(X)$$

$$\begin{aligned} u^H d^T &= u^{N/2}\, d^{N/2}\, u^{X\sqrt{N}/2}\, d^{-X\sqrt{N}/2} \\ &= (ud)^{N/2}\, (u/d)^{X\sqrt{N}/2} \\ &= C\, e^{kX} \end{aligned}$$

$$C := (ud)^{N/2}$$

$$k := \ln((u/d)^{\sqrt{N}/2})$$

Approx. Sol'n:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(S) = (5000S - 5000)_+$
$= 5000(S - 1)_+$

$g(x) := f(Ce^{kx}) = 5000(Ce^{kx} - 1)_+$ reasonable??

0.0573390439

$$\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1)_+ \, e^{-x^2/2}\, dx\right]$$

1.000948567

$N = 2,592,000$

1.00010005 $=: u$

0.99989997 $=: d$

$C := (ud)^{N/2}$

$k := \ln((u/d)^{\sqrt{N}/2})$

Approx. Sol'n: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

0.0573390439

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

1.000948567

$$= \frac{5000}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{a}^{\infty} (Ce^{kx} - 1) \, e^{-x^2/2} \, dx$$

$$Ce^{ka} - 1 = 0$$

$$Ce^{ka} = 1$$

$$e^{ka} = 1/C$$

$$ka = \ln(1/C) = -\ln C$$

$$a = -(\ln C)/k$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 5000(Ce^{kx} - 1)_+ \, e^{-x^2/2}\, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1)_+ \, e^{-x^2/2}\, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \int_{a}^{\infty} (Ce^{kx} - 1) \, e^{-x^2/2}\, dx$$

$$= \frac{5000}{\sqrt{2\pi}} \left[C \int_{a}^{\infty} e^{kx} \, e^{-x^2/2}\, dx - \int_{a}^{\infty} e^{-x^2/2}\, dx \right]$$

$$a = -(\ln C)/k$$

$$= \frac{5000}{\sqrt{2\pi}} \left[C \int_a^\infty e^{kx} e^{-x^2/2}\, dx - \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\sqrt{2\pi}\,\Phi(-a)} \right]$$

$$= \frac{5000}{\sqrt{2\pi}} \left[C \int_a^\infty e^{kx} e^{-x^2/2}\, dx - \int_a^\infty e^{-x^2/2}\, dx \right]$$

$$a = -(\ln C)/k$$

$$= \frac{5000}{\sqrt{2\pi}}\left[C\int_a^\infty e^{kx} e^{-x^2/2}\,dx \;-\; \int_a^\infty e^{-x^2/2}\,dx \right]$$

DON'T FORGET $\sqrt{2\pi}\,\Phi(-a)$ — NEGATE THE LOWER LIMIT

THE LOWER LIMIT

$$\int_{a-k}^\infty e^{k(x+k)} e^{-(x+k)^2/2}\,dx$$

$$e^{kx}\, e^{k^2}\, e^{-x^2/2}\, e^{-k^2/2}\, e^{-kx}$$

$$e^{k^2/2}\int_{a-k}^\infty e^{-x^2/2}\,dx$$

THE LOWER LIMIT

$$\sqrt{2\pi}\,\Phi(k-a)$$

DON'T FORGET

NEGATE THE LOWER LIMIT

$$a = -(\ln C)/k$$

$$= \frac{5000}{\sqrt{2\pi}}\left[C \underbrace{\int_a^{\infty} e^{kx}\, e^{-x^2/2}\, dx} - \overbrace{\int_a^{\infty} e^{-x^2/2}\, dx}^{\sqrt{2\pi}\,\Phi(-a)} \right]$$

$$\underbrace{\int_{a-k}^{\infty} \underbrace{e^{k(x+k)}\, e^{-(x+k)^2/2}}_{e^{k^2}\, e^{-x^2/2}\, e^{-k^2/2}}\, dx}$$

$$e^{k^2/2}\, \underbrace{\qquad\qquad}_{\sqrt{2\pi}\,\Phi(k-a)}$$

$$a = -(\ln C)/k$$

$$= \frac{5000}{\sqrt{2\pi}}\left[C \underbrace{\int_a^\infty e^{kx}\, e^{-x^2/2}\, dx}_{} - \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\sqrt{2\pi}\,\Phi(-a)} \right]$$

$$e^{k^2/2}\sqrt{2\pi}\,\Phi(k-a)$$

$$a = -(\ln C)/k$$

$$= \frac{5000}{\cancel{\sqrt{2\pi}}}\left[C \underbrace{\int_a^{\infty} e^{kx} e^{-x^2/2}\, dx}_{e^{k^2/2}\cancel{\sqrt{2\pi}}\,\Phi(k-a)} - \overbrace{\int_a^{\infty} e^{-x^2/2}\, dx}^{\cancel{\sqrt{2\pi}}\,\Phi(-a)} \right]$$

$$= 5000\left[\overbrace{C\, e^{k^2/2}}^{1.002595363} [\Phi(\overbrace{k-a}^{0.073874328})] - [\Phi(\overbrace{-a}^{0.01653528434})] \right]$$

$$= \boxed{121.0704439}$$

$a = -0.01653528434$

$k = 0.0573390439$

$C = 1.000948567$

$$\boxed{a = -(\ln C)/k}$$

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $f(u^H d^T)$,
30 days from now.

$\rho^N P =$ expected payout $=: E =$???

$P = \rho^{-N} E = (1+\iota)^{-N} E$

$=$ discounted expected payout

1.000000001

$=: \rho$

$P = \rho^{-N} E \approx 120.7570357$

120.76

$E \approx$ 121.0704439

$\rho^{-N} = 0.997411356$

$N := 30 \times 24 \times 60 \times 60 = 2{,}592{,}000$

$K = 5000$ "at the money" $\mu = \sout{0.003917149457}$

$K' := K/e^r = 4987.056782$ "implied volatility" $\sigma = 0.057338217$

$e^r = 1.002595362$

$S_0 = 5000$

$$d_{\pm} := \frac{\ln(S_0/K')}{\sigma} \pm \frac{\sigma}{2}$$

$$= \frac{0.002592000333}{0.057338217} \pm 0.028669108$$

$d_+ = 0.073874565$ $d_- = 0.016536349$

$\Phi(d_+) = 0.52944$ $\Phi(d_-) = 0.50660$

Black-Scholes Option Pricing Formula

Black-Scholes Price $= S_0[\Phi(d_+)] - K'[\Phi(d_-)]$

$= [5000][0.52944] - [4987.056782][0.5066]$

$= 120.7570357$

drift (μ) unused!!

QUESTIONS?

COMMENTS?