Professors and Grants

1. INTRODUCTION

This note is intended as a complement and complement to

B. Zhang's very enjoyable "Coconuts and Islanders",

which motivates the Boltzmann distribution in the case where every nonnegative integer is a possible energy-level.

Here, our initial focus is, instead, on Boltzmann distributions where 0 and 1 and 10 are the only possible energy-levels.

Taking our cue from "Coconuts and Islanders", we motivate by story.

From $\S3$ to \$13, we analyze three systems for

dispensing grant money to N professors. Congress allocates N dollars to award to the N professors. The grant rules stipulate: each professor receives \$0 or \$1 or \$10. Each professor is identified by a number, from 1 to N. By a **dispensation**, we mean a full complement of awards,

with a specific amount (0 or 1 or 10) to Professor#1,

a specific amount (0 or 1 or 10) to Professor#2,

etc., up to and including Professor#N, such that the total of the awards is the N allocated by Congress.

The first system (see $\S3$) for awarding grants is very simple:

There are many possible dispensations, and, among all of them, one is selected randomly,

giving equal probability to each possible dispensation.

The **main problem** is to figure out:

Using this first system, for a given professor,

what is the probability of being awarded \$0? \$1? \$10?

Later (see $\S5$), we describe

second and third probabilistic award systems.

Each of these systems depends on three parameters p, q, r

satisfying p, q, r > 0 and p + q + r = 1 = q + 10r. The **second system** uses

an iid system of random-variables, X_1, \ldots, X_N

such that, $\forall \ell$, $\Pr[X_{\ell} = 0] = p$,

 $\Pr[X_{\ell} = 1] = q,$

 $\Pr[X_{\ell} = 10] = r.$

For all ℓ , the second system awards X_{ℓ} dollars to Professor# ℓ . The total dollar payout $X_1 + \cdots + X_N$ is then random;

if $X_1 = \cdots = X_N = 0$, it could be as small as 0 dollars, and if $X_1 = \cdots = X_N = 10$, it could be as large as 10N dollars. The **third system** is obtained from the second

by conditioning on the event $X_1 + \cdots + X_N = N$, so that the total payout is exactly the N allocated by Congress.

KEY POINT:	With exactly the right choice of p, q, r ,
	the first and third systems are shown to be equivalent.
In $\S6$ and $\$7$,	we show that this parameter choice is Boltzmann,
meaning:	(p,q,r) is, for some real number β ,
	a scalar multiple of $(e^{-0\cdot\beta}, e^{-1\cdot\beta}, e^{-10\cdot\beta})$.
That is,	$\exists \beta, C \in \mathbb{R} \text{s.t.} (p,q,r) = (C, Ce^{-\beta}, Ce^{-10\beta}).$

The second and third systems are

accessible by basic tools of probability theory, while the above "main problem" involves the first system. However, once we know the first and third systems are equivalent,

we can bring these probabilistic tools to bear on the main problem. Thanks to J. Steif, for pointing out to me that

the Discrete Local Limit Theorem, which is described in §10, is the right tool for the main problem, which is solved in §13.

Boltzmann distributions are often motivated by entropy, but, from our perspective,

what's special about $(p,q,r) = (C, Ce^{-\beta}, Ce^{-10\beta})$ is: For any $i, j, k \ge 0$, we have $p^i q^j r^k = C^{i+j+k} \cdot e^{-\beta \cdot (j+10k)}$, so $p^i q^j r^k$ depends only on: i + j + k and j + 10k. In the third system of grant awards, there exists a normalizing constant S > 0 s.t., for any dispensation in which i professors receive 0,

j professors receive \$ 1,

k professors receive \$10,

the probability of that dispensation is $p^i q^j r^k / S$,

which is equal to $C^{i+j+k} \cdot e^{-\beta \cdot (j+10k)}/S$. That probability, then, depends only on

i+j+k, which is the number of professors, which is the total dollar payout. and i + 10k, the number of professors is = NSo, since the total dollar payout is also = N, and each award-dispensation has probability $C^N \cdot e^{-\beta \cdot N}/S$, we conclude: so they are all equally likely, which exactly describes the first system. Therefore, under the Boltzmann assumption,

the first and third systems are equivalent.

In $\S15$, we expose the inequitablity of the first system.

In fact, assuming N is sufficiently large, we show, in $\S15$, that:

with probability > 99%, over half of the professors receive \$0.

Thanks to V. Reiner for suggesting

applying Chebyshev's inequality to a sum of indicator variables,to transition from individual statistics to population statistics.

In §16 and §17 and §18, we extend the theory to handle cases where the award-sets are arbitrary finite sets of rational numbers,

not necessarily equal to $\{0, 1, 10\}$.

In $\S19$, we show that

irrational award amounts can lead to non-Boltzmann statistics. In §20 and §21 and §22, we extend our earlier results to include

degenerate energy-levels, with a finite set of states.

In §23 through §27, we extend these results further to include cases that involve a countably infinite set of states.

Thanks to C. Prouty for help with many calculations. For some of his Python code, see §28.

2. Some notation

A box around an expression indicates that it is global,

meaning that it is fixed (or "bound") to the end of these notes. Unboxed variables are freed at the end of each section, if not earlier.

Let $\mathbb{R}^* := \{-\infty\} \bigcup \mathbb{R} \bigcup \{\infty\}, \qquad \mathbb{Z}^* := \{-\infty\} \bigcup \mathbb{Z} \bigcup \{\infty\}.$ For any $s, t \in \mathbb{R}^*, \quad \text{let}$

Let $|\mathbb{N}| := [1..\infty)$ be the set of positive integers. For any finite set F, let |#F| be the number of elements in F. For any infinite set F, let $|\#F| := \infty$. Then $\#\mathbb{Z} = \infty = \#\mathbb{R}$. For any set F, we have: $\#F \in [0..\infty].$ For all $t \in \mathbb{R}$, let $||t|| := \max\{n \in \mathbb{N} \mid n \leq t\}$ be the floor of tFor any sets S, T, for any function $f: S \to T$, the **image** of f is: $|\mathbb{I}_f| := \{ f(x) \mid x \in S \} \subseteq T.$ For any sets S, T, for any function $f: S \to T$, we define: $f^*A := \{x \in S \mid f(x) \in A\}.$ for any set A, By convention, in these notes, we define $|0^0| := 1$. By " C^{ω} " we mean: "real-analytic". Fix an element of $\{z \in \mathbb{C} \mid z^2 = -1\}$ and **denote** it by $\sqrt{-1}$. $\Re : \mathbb{C} \to \mathbb{R}$ and $\Im : \mathbb{C} \to \mathbb{R}$ by: Define $\forall x, y \in \mathbb{R}, \quad \Re(x + y\sqrt{-1}) = x \text{ and } \Im(x + y\sqrt{-1}) = y.$

3. FIRST SYSTEM OF GRANT AWARDS

Let $[N] \in \mathbb{N}$. Think of N as large. Whenever we need to

formulate and prove precise mathematical statements, we will "pass to the thermodynamic limit", which means:

In pass to the thermodynamic limit, which means.

we replace N by a variable $n \in \mathbb{N}$, and let $n \to \infty$. ((Alternatively, within nonstandard analysis,

N could be defined as an *infinite* integer,

and the various approximations involving N,

could be defined as equality-modulo-infinitesimals.)) Suppose there are N professors, numbered 1 to N,

who apply, once per year, to the GFA (Grant Funding Agency), seeking funding for the very important work they are doing.

Each year, Congress authorizes N for the GFA to dispense

to the N professors.

The GFA has the rule: every award is 0 or 1 or 10 dollars.

The set of grant-dispensations is represented by:

 $\Omega := \left\{ \omega : [1..N] \to \{0, 1, 10\} \mid \sum_{\ell=1}^{N} [\omega(\ell)] = N \right\}.$ The GFA has set aside # Ω pieces of paper, and has written down all possible dispensations, one on each piece of paper. So, for example, there is a piece of paper that says: Professors 1 to N each get \$1. Another piece of paper says: Professors 1 to N - 10 each get \$1 and Professors N - 9 to N - 1 each get \$0 and Professor N gets \$10. Since N is large, it follows that $\#\Omega$ is large, and so there are many, many, many other pieces of paper. a GFA bureaucrat Each year, places all the pieces of paper in a big bin, then selects one at random and makes the awards as indicated on that piece of paper. Under this **first system** of awarding grants, we have: $\forall \omega \in \Omega$, the probability that the selected grant-dispensation is ω is equal to $1/(\#\Omega).$ Suppose I am one of the professors. Here is our **main problem**: Calculate my probability of getting \$0. Then calculate my probability of getting \$1. Then calculate my probability of getting \$10. Approximate answers are acceptable. In $\S5$ to $\S13$ of this note, we reformulate and then solve this problem. Spoiler: It's a Boltzmann distribution, approximately.

4. Particles and energy

Recall that $N \in \mathbb{N}$. Think of N as large.

Suppose there are N particles, numbered 1 to N,

each of which has a certain amount of energy.

Suppose the total energy is N, dispensed among the N particles.

Suppose physicists have somehow determined that, for any particle,

its possible energy-levels are: 0 or 1 or 10.

Recall: $\Omega = \left\{ \omega : [1..N] \to \{0, 1, 10\} \mid \sum_{\ell=1}^{N} [\omega(\ell)] = N \right\}.$ Then Ω represents the set of energy-dispensations. Assume that physicists have somehow determined

distant physicists have somehow determined

that this system of particles has a random energy-dispensation and that all energy-dispensations in Ω are equally probable. That is, physicists tell us:

 $\forall \omega \in \Omega$, the probability that

the energy-dispensation is ω

is equal to $1/(\#\Omega)$.

The equal probability of all energy-dispensations

is a recurring theme in microcanonical-ensemble thermodynamics, and can often be motivated through

rules of random energy transfer between random pairs of particles. For examples of this, either see $\S20$ below or

search for "Coconuts and Islanders" by B. Zhang,

and, in particular, see the work leading up to

the last paragraph of $\S3.2$ therein.

In $\S 20$ below,

instead of particles exchanging energy,

there are professors exchanging dollars,

but the principle is exactly the same.

In Zhang's exposition,

instead of particles exchanging energy,

there are islanders exchanging coconuts,

but the principle is exactly the same.

Returing to our N particles, pick any one of them.

Problem: Calculate its probability of having energy-level 0.

Then calculate its probability of having energy-level 1.

Then calculate its probability of having energy-level 10.

Approximate answers are acceptable.

Spoiler: It's a Boltzmann distribution, approximately.

Except for terminology, this problem is the same as

the main problem (end of $\S3$) about professors and grants.

We will go back to professors and grants.

Mathematically it makes no difference, but it's more fun.

5. Second and third systems of grant awards

In an effort to go paperless, the GFA changes to a new system: In this **second system**, instead of all those pieces of paper,

the GFA chooses p, q, r > 0 s.t. p + q + r = 1, and then, for each of the N professors,

awards \$ 0 with probability p,

\$ 1 with probability q,

\$10 with probability r.

No professor's award depends in any way on any other professor's; the awards are independent.

The expected payout, for any professor, is $p \cdot 0 + q \cdot 1 + r \cdot 10$ dollars. Under this second system,

there is no guarantee that the total payout will be N,

which is a difficulty that we will discuss later.

However, recognizing that the average award is *intended* to be \$1,

the GFA chooses the numbers p, q, r subject to the constraint that

 $p \cdot 0 + q \cdot 1 + r \cdot 10 = 1, \quad i.e.,$ q + 10r = 1.For each function $\omega : [1..N] \rightarrow \{0, 1, 10\},\$ let i_{ω} := #{ $\ell \in [1..N] \mid \omega(\ell) = 0$ }, $\begin{array}{ll} j_{\omega} & := & \#\{ \ \ell \in [1..N] \ | \ \omega(\ell) = & 1 \ \}, \\ k_{\omega} & := & \#\{ \ \ell \in [1..N] \ | \ \omega(\ell) = & 10 \ \}; \end{array}$ that is, i_ω is the number of professors awarded \$ 0 and j_{ω} is the number of professors awarded \$ 1 and k_{ω} is the number of professors awarded \$10. $\forall \omega : [1..N] \rightarrow \{0, 1, 10\}, \text{ we have:}$ Then, the total number of awards is $i_{\omega} + j_{\omega} + k_{\omega}$ $i_{\omega} \cdot 0 + j_{\omega} \cdot 1 + k_{\omega} \cdot 10,$ and the total dollar payout is $j_{\omega} + 10k_{\omega}.$ i.e., $\forall \omega : [1..N] \rightarrow \{0, 1, 10\}, \text{ we have:}$ Then, $i_{\omega} + j_{\omega} + k_{\omega} = N \quad \text{and} \quad j_{\omega} + 10k_{\omega} = \sum_{\ell=1}^{N} [\omega(\ell)].$ Recall: $\Omega = \left\{ \omega : [1..N] \rightarrow \{0, 1, 10\} \mid \sum_{\ell=1}^{N} [\omega(\ell)] = N \right\}.$ That is, Ω is the set of all payout functions $\omega : [1..N] \to \{0, 1, 10\}$ the total dollar payout is N. s.t. $\forall \omega : [1..N] \rightarrow \{0, 1, 10\},$ we have: Then: $\Leftrightarrow \qquad j_{\omega} + 10k_{\omega} = N.$ $\omega \in \Omega$ For every $i, j, k \in [0..N]$,

i+j+k=N and j+10k=N, if $\exists \omega \in \Omega \quad \text{s.t.} \quad (i, j, k) = (i_{\omega}, j_{\omega}, k_{\omega});$ then indeed, one such $\omega : [1..N] \rightarrow \{0, 1, 10\}$ is described by: $\omega = 0$ on [1..*i*], $\omega = 1$ on (i..i + j], $\omega = 10$ on (i + j..N]. Let $A := \{(i_{\omega}, j_{\omega}, k_{\omega}) \mid \omega \in \Omega\}.$ Then A is the set of all (i, j, k) s.t. $i, j, k \in [0..N]$ and i+j+k=N and j+10k=N. Under the second system, each 0 award happens with probability p and each \$ 1 award happens with probability qand each \$10 award happens with probability r. So, $\forall \omega : [1..N] \rightarrow \{0, 1, 10\},$ under the second system, the probability that the grant-dispensation is equal to ω $p^{i_{\omega}}a^{j_{\omega}}r^{k_{\omega}}.$ is $S := \sum_{\omega \in \Omega} p^{i_{\omega}} q^{j_{\omega}} r^{k_{\omega}}.$ Let Then S is the probability (using the second system) that $\omega \in \Omega$, the probability that the total payout is exactly N dollars. i.e.,Assuming N is large, it turns out that S is close to zero. under this second system, So, the probability of paying out exactly N dollars is very small. Congress only allocates N per year for the N professors. So, using this second system, each year, with probability $1-S \approx 1$, the GFA will run a surplus or a deficit. On the other hand, since q + 10r = 1, we see that, each year, the expected payout is\$1 per professor, so, each year, the expected total payout is N.So these surpluses and deficits should, over time, cancel one another. Unfortunately, Congress is a paragon of fiscal responsibility, and. as soon as it finds out about the GFA's second system, it insists that the GFA never again underspend or overspend. So the GFA changes its system one more time, as follows. Under its third system, each year, before announcing any of the awards publicly, the GFA writes out, in an *internal* memo, a *tentative* proposal of awards that, independently, for each of the N professors, awards 0 with probability p,

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\$ 1 with probability q,

\$10 with probability r.

If the memo's total award payout is NOT equal to N, the GFA deems the memo as unacceptable, deletes it, and starts over, making memo after memo,

until an acceptable one (meaning payout exactly N) appears.

Each memo has a probability S of being acceptable, so, each year, the GFA will likely need to repeat the memo process many times to get to a memo with total payout exactly equal to N. However, as soon as that happens,

the GFA uses that first acceptable memo,

and publicizes its dispensation of awards.

Mathematically, we are conditioning on the event $\omega \in \Omega$. So, using the third system, the probability that $\omega \notin \Omega$ is 0. Also, for this third system, $\forall \omega \in \Omega$, the probability of ω is $p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}}/S$.

The sum of these probabilities is 1:

$$\sum_{\omega \in \Omega} \frac{p^{i_{\omega}} q^{j_{\omega}} r^{k_{\omega}}}{S} = \frac{1}{S} \cdot \sum_{\omega \in \Omega} p^{i_{\omega}} q^{j_{\omega}} r^{k_{\omega}} = \frac{1}{S} \cdot S = 1$$

This third system is not necessarily equivalent to the first, because

in the first system, all the probabilities were $1/(\#\Omega)$, whereas, in the third system, they are $p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}}/S$. So a **new question** arises:

Is it possible to choose p, q, r > 0 in such a way that

p+q+r=1 and q+10r=1 and $\forall \omega \in \Omega$, $p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}}/S = 1/(\#\Omega)$?

If yes, then, using that (p, q, r),

the first and third systems are equivalent.

We will see that the answer to this new question, in fact, is yes. In the next two sections, assuming $N \ge 10$,

we will show how to compute the only (p, q, r) that works. Spoiler: It's a Boltzmann distribution, exactly.

6. Computing p, q, r à la Boltzmann

We will prove that, if $N \ge 10$, then

there is at most one (p, q, r) that satisfies these conditions,

specifically,
$$(p,q,r) = \frac{(1,9^{-1/10},9^{-1})}{1+9^{-1/10}+9^{-1}}$$

Define the dot product, \odot , on \mathbb{R}^3 , by:

 $\forall x, y, z, X, Y, Z \in \mathbb{R}, \ (x, y, z) \odot (X, Y, Z) = xX + yY + zZ.$ For all $u \in \mathbb{R}^3$, let $u^{\perp} := \{ v \in \mathbb{R}^3 \mid u \odot v = 0 \};$ then u^{\perp} is a vector subspace of \mathbb{R}^3 . $u \in u^{\perp \perp}$. Also, $\forall u \in \mathbb{R}^3$, let $U^{\perp} := \{ v \in \mathbb{R}^3 \mid \forall u \in U, u \odot v = 0 \};$ For all $U \subseteq \mathbb{R}^3$, then U^{\perp} is a vector subspace of \mathbb{R}^3 . $\forall t \in \mathbb{R}^3, \forall U \subseteq \mathbb{R}^3,$ $(t \in U) \Rightarrow (t^{\perp} \supseteq U^{\perp}).$ Also, $(T \subseteq U) \implies (T^{\perp} \supseteq U^{\perp}).$ $\forall T, U \subseteq \mathbb{R}^3,$ Also, For all $u, v \in \mathbb{R}^3$, let $\langle u, v \rangle_{\text{span}}$ denote the \mathbb{R} -span of $\{u, v\}$, *i.e.*, let $\langle u, v \rangle_{\text{span}} := \{ su + tv \mid s, t \in \mathbb{R} \};$ then $\langle u, v \rangle_{\text{span}}$ is a vector subspace of \mathbb{R}^3 . Recall (§5): $A = \{(i_{\omega}, j_{\omega}, k_{\omega}) \mid \omega \in \Omega\}.$ Recall (§5): A is the set of all (i, j, k) $i, j, k \in [0..N]$ and s.t. i + j + k = Nand j + 10k = N. Then: A is the set of all (i, j, k)s.t. $i, j, k \in [0..N]$ and and $(0, 1, 10) \odot (i, j, k) = N$. $(1, 1, 1) \odot (i, j, k) = N$ For all $a, b \in A$, we have $(1,1,1) \odot a = N = (1,1,1) \odot b$ and $(0, 1, 10) \odot a = N = (0, 1, 10) \odot b,$ so we get $(1,1,1) \odot (a-b) = 0$ and $(0,1,10) \odot (a-b) = 0$, so $a-b \in (1,1,1)^{\perp} \cap (0,1,10)^{\perp}$. Let $V := (1, 1, 1)^{\perp} \cap (0, 1, 10)^{\perp}$. Then: $\forall a, b \in A, a - b \in V.$ Let $D := \{a - b \mid a, b \in A\}.$ Then $D \subseteq V$. Also, we have: $V \subseteq (1, 1, 1)^{\perp}$ and $V \subseteq (0, 1, 10)^{\perp}$. $V^{\perp} \supseteq (1, 1, 1)^{\perp \perp}$ and $V^{\perp} \supseteq (0, 1, 10)^{\perp \perp}$. Then: Since $(1,1,1) \in (1,1,1)^{\perp \perp} \subseteq V^{\perp}$ and $(0,1,10) \in (0,1,10)^{\perp \perp} \subseteq V^{\perp}$, we get: $\langle (1, 1, 1), (0, 1, 10) \rangle_{\text{span}} \subseteq V^{\perp}$. Then: $W \subseteq V^{\perp}$. $W := \langle (1, 1, 1), (0, 1, 10) \rangle_{\text{span}}.$ Let Let $a_1 := (0, N, 0), a_2 := (9, N - 10, 1).$ Assume $N \ge 10$. Then $a_1, a_2 \in A$. Let $d_1 := a_2 - a_1$. Then $d_1 \in D$.

we get: $\dim d_1^{\perp} = 2.$ Since $d_1 \neq (0, 0, 0),$ Since $W = \langle (1, 1, 1), (0, 1, 10) \rangle_{\text{span}}$, we get: dim W = 2. Since $d_1 \in D \subseteq V$ and $W \subseteq V^{\perp}$, we get: $d_1^{\perp} \supseteq D^{\perp} \supseteq V^{\perp} \supseteq W$. So, since $\dim d_1^{\perp} = 2 = \dim W$, we get: $d_1^{\perp} = D^{\perp} = V^{\perp} = W$. Then $D^{\perp} = W$. Recall: $\forall \omega \in \Omega, \quad p^{i_{\omega}} q^{j_{\omega}} r^{k_{\omega}} / S = 1/(\#\Omega)$. since $A = \{(i_{\omega}, j_{\omega}, k_{\omega}) \mid \omega \in \Omega\},$ we get: So. $\forall (i,j,k) \in A, \qquad p^i q^j r^k \,/\, S \;=\; 1/(\#\Omega).$ Equivalently, $\forall (i, j, k) \in A$, $i \cdot (\ln p) + j \cdot (\ln q) + k \cdot (\ln r) - (\ln S) = -(\ln(\#\Omega)).$ Equivalently, $\forall (i, j, k) \in A$, $(i, j, k) \odot (\ln p, \ln q, \ln r) = (\ln S) - (\ln(\#\Omega)).$ Then: $\forall a, b \in A,$ $a \odot (\ln p, \ln q, \ln r) = (\ln S) - (\ln(\#\Omega)) = b \odot (\ln p, \ln q, \ln r),$ so we get: $(a-b) \odot (\ln p, \ln q, \ln r) = 0.$ $\forall d \in D, \qquad d \quad \odot \left(\ln p, \ln q, \ln r \right) = 0.$ Then: $(\ln p, \ln q, \ln r) \in D^{\perp}.$ Then: $(\ln p, \ln q, \ln r) \in D^{\perp} = W = \langle (1, 1, 1), (0, 1, 10) \rangle_{\text{span}},$ Since **choose** a real number C > 0 and $\beta \in \mathbb{R}$ s.t. $(\ln p, \ln q, \ln r) = (\ln C) \cdot (1, 1, 1) - \beta \cdot (0, 1, 10).$ $(\ln p, \ln q, \ln r) = (\ln C, (\ln C) - \beta, (\ln C) - 10\beta).$ Then $(p,q,r) = (C, Ce^{-\beta}, Ce^{-10\beta}).$ Then $(p,q,r) = C \cdot (1, e^{-\beta}, e^{-10\beta}).$ Then Then $(p,q,r) = C \cdot (1, e^{-\beta}, e^{-10\beta}).$ So, since p + q + r = 1, we get: $C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1.$ Then $C = \frac{1}{1 + e^{-\beta} + e^{-10\beta}}.$ Then $(p,q,r) = \frac{(1, e^{-\beta}, e^{-10\beta})}{1 + e^{-\beta} + e^{-10\beta}}.$ So, since q + 10r = 1, we get: $\frac{e^{-\beta} + 10e^{-10\beta}}{1 + e^{-\beta} + e^{-10\beta}} = 1.$ Then $e^{-\beta} + 10e^{-10\beta} = 1 + e^{-\beta} + e^{-10\beta}.$ Then $9e^{-10\beta} = 1.$ Then $e^{-10\beta} = 9^{-1}.$ Then $e^{-\beta} = 9^{-1/10}.$ Then $(p,q,r) = \frac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}.$

So this is the only (p, q, r) that can possibly work. In the next section, we show that it *does* work.

7. Showing the Boltzmann p, q, r work

In this section, we prove

the converse of the result from the preceding section. That is, we let $(p,q,r) := \frac{(1,9^{-1/10},9^{-1})}{1+9^{-1/10}+9^{-1}}$ and $S := \sum_{\omega \in \Omega} p^{i_\omega} q^{j_\omega} r^{k_\omega}$, and we wish to show: p + q + r = 1 and q + 10r = 1 and $\forall \omega \in \Omega, \quad p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}}/S = 1/(\#\Omega).$

 $\begin{array}{lll} Claim: & \forall \omega \in \Omega, \qquad p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}} = K. \\ Proof of Claim: & \textbf{Given } \omega \in \Omega, \qquad \textbf{want: } p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}} = K. \\ \text{Recall (§5): } & i_{\omega} + j_{\omega} + k_{\omega} = N \quad \text{and} \quad j_{\omega} + 10k_{\omega} = \sum_{\ell=1}^{N} [\omega(\ell)]. \\ \text{By definition of } \Omega, \quad \text{since } \omega \in \Omega, \quad \text{we get: } \sum_{\ell=1}^{N} [\omega(\ell)] = N. \\ \text{Then: } & j_{\omega} + 10k_{\omega} = N. \qquad \text{Recall: } (p,q,r) = (C, Ce^{-\beta}, Ce^{-10\beta}). \\ \text{Then: } & p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}} = C^{i_{\omega}} \cdot (Ce^{-\beta})^{j_{\omega}} \cdot (Ce^{-10\beta})^{k_{\omega}} \\ & = C^{i_{\omega} + j_{\omega} + k_{\omega}} \cdot e^{-\beta \cdot (j_{\omega} + 10k_{\omega})} = C^{N} \cdot e^{-\beta \cdot N} = K. \end{array}$

End of proof of Claim.

By definition of S, we have: $S = \sum_{\omega \in \Omega} p^{i_{\omega}} q^{j_{\omega}} r^{k_{\omega}}$. So, by the Claim, we get: $S = (\#\Omega) \cdot K$. Then $K/S = 1/(\#\Omega)$. 10/9 = 1 + (1/9). That is, $10 \cdot 9^{-1} = 1 + 9^{-1}.$ We have $10e^{-10\beta} = 1 + e^{-10\beta}$ $e^{-10\beta} = 9^{-1}$. we get: So, since $e^{-\beta} + 10e^{-10\beta} = 1 + e^{-\beta} + e^{-10\beta}.$ Then: $(p,q,r) = C \cdot (1, e^{-\beta}, e^{-10\beta}).$ Recall: By definition of C, we get: $C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1.$ $p + q + r = C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1$ Since $q + 10r = C \cdot (e^{-\beta} + 10e^{-10\beta})$ and since $= C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1,$ it remains only to show: $\forall \omega \in \Omega, \quad p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}}/S = 1/(\#\Omega).$ $p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}}/S = 1/(\#\Omega).$ Given $\omega \in \Omega$, want: $p^{i_{\omega}} q^{j_{\omega}} r^{k_{\omega}} = K.$ By the Claim, we get: $K/S = 1/(\#\Omega).$ Recall: $p^{i_{\omega}}q^{j_{\omega}}r^{k_{\omega}}/S = K/S = 1/(\#\Omega).$ Then:

8. INFINITE SUMMATION

DEFINITION 8.1. Let *S* be a set, and let $f: S \to [0; \infty]$. Let $\mathcal{F} := \{A \subseteq S \mid \#A < \infty\}$. Then: $\sum [f(x)] := \sup \sum [f(x)] \in [0; \infty]$.

Then:
$$\sum_{x \in S} [f(x)] := \sup_{A \in \mathcal{F}} \sum_{x \in A} [f(x)] \in [0; \infty]$$

DEFINITION 8.2. Let *S* be a set, and let $f: S \to \mathbb{R}$. Assume: $\sum_{x \in S} |f(x)| < \infty$.

Then:
$$\sum_{x \in S} [f(x)] := \left(\sum_{x \in S} |f(x)| \right) - \left(\sum_{x \in S} [|f(x)| - (f(x))] \right) \in \mathbb{R}.$$

Recall (§2): the notations $\Re(z)$ and $\Im(z)$.

DEFINITION 8.3. Let *S* be a set, and let $f: S \to \mathbb{C}$. Assume: $\sum_{x \in S} |f(x)| < \infty$.

Then:
$$\sum_{x \in S} [f(x)] := \left(\sum_{x \in S} [\Re(f(x))] \right) - \left(\sum_{x \in S} [\Im(f(x))] \right) \cdot \sqrt{-1} \in \mathbb{C}.$$

NOTE: For any set S, for any $f: S \to \mathbb{C}$, we have:

 $(\sum_{x\in S} |f(x)| < \infty) \Rightarrow \{x \in S | f(x) \neq 0\}$ is countable and, by subadditivity of absolute value, we get:

 $\left|\sum_{x\in S} [f(x)]\right| \leq \sum_{x\in S} |f(x)|.$

THEOREM 8.4. Let *S* be a set. Let $S_1, S_2, ... \subseteq S$. Assume: $S_1 \subseteq S_2 \subseteq \cdots$ and $S_1 \bigcup S_2 \bigcup \cdots = S$. Let $f: S \to \mathbb{C}$. Assume: $\sum_{x \in S} |f(x)| < \infty$. Then: as $n \to \infty$, $\sum_{x \in S_n} [f(x)] \to \sum_{x \in S} [f(x)]$.

The preceding is basic. We omit proof.

THEOREM 8.5. Let S and T be sets, $f: S \to \mathbb{R}$, $g: S \to T$. Then: $\sum_{y \in T} \sum_{x \in g^*\{y\}} |f(x)| = \sum_{x \in S} |f(x)|.$

THEOREM 8.6. Let S and T be sets, $f: S \to \mathbb{R}$, $g: S \to T$. Assume: $\sum_{x \in S} |f(x)| < \infty$. Then: $\forall y \in T$, $\sum_{x \in g^*\{y\}} |f(x)| < \infty$ and $\sum_{y \in T} |\sum_{x \in g^*\{y\}} [f(x)] | < \infty$ and $\sum_{y \in T} \sum_{x \in g^*\{y\}} [f(x)] = \sum_{x \in S} [f(x)]$.

The preceding two theorems are

elementary versions of Fubini's Theorem. We omit proofs.

9. Countable measure theory

By convention, in this note,

any countable set is given its discrete Borel structure.

Let Θ be a countable set. Let \mathcal{B} be the set of subsets of Θ . A measure on Θ is a function $\mu : \mathcal{B} \to [0; \infty]$ such that, \forall pairwise-disjoint $\Theta_1, \Theta_2, \ldots \subseteq \Theta$, we have: $\mu(\Theta_1 \mid \Theta_2 \mid \cdots) = (\mu(\Theta_1)) + (\mu(\Theta_2)) + \cdots$ A measure μ on a countable set Θ is completely determined by the function $t \mapsto \mu\{t\} : \Theta \to [0; \infty],$ $\forall \Theta_0 \subseteq \Theta$, we have $\mu(\Theta_0) = \sum_{t \in \Theta_0} \left[\mu\{t\} \right]$. because: **DEFINITION 9.1.** Let Θ be a countable set. Then \mathcal{M}_{Θ} denotes the set of measures on Θ , $\begin{array}{|c|c|c|} \hline \mathcal{F}\mathcal{M}_{\Theta} & := & \{\mu \in \mathcal{M}_{\Theta} \,|\, \mu(\Theta) < \infty\}, \\ \hline \mathcal{F}\mathcal{M}_{\Theta}^{\times} & := & \{\mu \in \mathcal{M}_{\Theta} \,|\, 0 < \mu(\Theta) < \infty\}, \end{array}$ and and $:= \{ \mu \in \mathcal{M}_{\Theta} \, | \, \mu(\Theta) = 1 \}.$ and \mathcal{M}_{Θ} is the set of measures on Θ Then \mathcal{FM}_{Θ} is the set of finite measures on Θ and $\mathcal{FM}_{\Theta}^{\times}$ is the set of nonzero finite measures on Θ and and \mathcal{P}_{Θ} is the set of probability measures on Θ . The only measure on \emptyset is the zero measure. $\mathcal{FM}_{\varnothing}^{\times} = \varnothing = \mathcal{P}_{\varnothing}.$ Therefore:

DEFINITION 9.2. Let Θ be a countable set, $\mu \in \mathcal{FM}_{\Theta}$. Let $n \in \mathbb{N}$. Then $\mu^n \in \mathcal{FM}_{\Theta^n}$ is defined by: $\forall x \in \Theta^n, \quad \mu^n \{x\} = (\mu\{x_1\}) \cdots (\mu\{x_n\}).$

The following is a basic fact, whose proof we omit:

 Θ be a countable set, $\mu \in \mathcal{FM}_{\Theta}$, $n \in [2..\infty)$. Let $Z \subseteq \Theta^n, \quad X \subseteq \Theta^{n-1}, \quad Y \subseteq \Theta.$ Assume that: Let under the standard bijection Θ^n \longleftrightarrow Θ^{n-1} × Θ, we have: Z \longleftrightarrow XХ Y_{\cdot} $\mu^n(Z) = (\mu^{n-1}(X)) \cdot (\mu(Y)).$ Then: It is common to identify Z with $X \times Y$, in which case we have: $\mu^n(X \times Y) = (\mu^{n-1}(X)) \cdot (\mu(Y)).$ We also omit proof of:

Let Θ be a countable set, $\mu \in \mathcal{FM}_{\Theta}$, $n \in \mathbb{N}$. Then: $\mu^n(\Theta^n) = (\mu(\Theta))^n$. In particular, $(\mu \in \mathcal{P}_{\Theta}) \Rightarrow (\mu^n \in \mathcal{P}_{\Theta^n})$.

The countable sets that are of interest in this note

all carry the discrete topology. We therefore define:

DEFINITION 9.3. Let Θ be a countable set, $\mu \in \mathcal{M}_{\Theta}$. Then the support of μ is: S_{μ} := { $t \in \Theta \mid \mu\{t\} \neq 0$ }.

DEFINITION 9.4. Let $\Theta \subseteq \mathbb{R}$ be countable, $\mu \in \mathcal{M}_{\Theta}$. Let $\rho \ge 1$ be real. Then: $[\mu|_{\rho}] := (\sum_{t \in \Theta} [|t|^{\rho} \cdot (\mu\{t\})])^{1/\rho}$.

Note: $\forall \text{countable } \Theta \subseteq \mathbb{R}, \quad \forall \mu \in \mathcal{FM}_{\Theta},$ if $\#S_{\mu} < \infty$, then: $\forall \text{real } \rho \ge 1, \ |\mu|_{\rho} < \infty.$

DEFINITION 9.5. Let $\Theta \subseteq \mathbb{R}$ be countable.

Let $\mu \in \mathcal{P}_{\Theta}$. Assume: $|\mu|_1 < \infty$. Then the mean of μ is: M_{μ} := $\sum_{t \in \Theta} [t \cdot (\mu\{t\})]$. Also, the variance of μ is: V_{μ} := $\sum_{t \in \Theta} [(t - M_{\mu})^2 \cdot (\mu\{t\})]$.

Let $\Theta \subseteq \mathbb{R}$ be countable, $\mu \in \mathcal{P}_{\Theta}$. Assume: $|\mu|_1 < \infty$. Then, by subadditivity of absolute value, we get $|M_{\mu}| \leq |\mu|_1$. In particular, $|M_{\mu}| < \infty$, *i.e.*, $-\infty < M_{\mu} < \infty$. Also, by expanding the square in the formula for V_{μ} ,

we get $V_{\mu} = |\mu|_2^2 - M_{\mu}^2$. In particular, $(V_{\mu} < \infty) \Leftrightarrow (|\mu|_2 < \infty)$.

Let $\Theta \subseteq \mathbb{R}$ be countable and let X be a Θ -valued random-variable. Let μ denote the distribution on Θ of X,

i.e., **define** $\mu \in \mathcal{P}_{\Theta}$ by: $\forall t \in \Theta, \quad \mu\{t\} = \Pr[X = t].$ Then, $\forall \text{real } \rho \ge 1$, we have: $|\mu|_{\rho}$ is the L^{ρ} -norm of X. Then, $\forall \text{real } \rho \ge 1$, we have: $(|\mu|_{\rho} < \infty) \Leftrightarrow (X \text{ is } L^{\rho}).$ In particular, $(|\mu|_{1} < \infty) \Leftrightarrow (X \text{ is } L^{1}).$ Also, if X is L^{1} , then $M_{\mu} = \mathbb{E}[X]$ and $V_{\mu} = \operatorname{Var}[X].$ That is, if X is L^{1} , then

 M_{μ} is the mean (aka expected value, aka average value) of X and V_{μ} is the variance of X. **THEOREM 9.6.** Let $\Theta \subseteq \mathbb{R}$ be countable, $\mu \in \mathcal{P}_{\Theta}$. Assume: $|\mu|_1 < \infty$. Then: $(\#S_{\mu} \ge 2) \Leftrightarrow (V_{\mu} > 0)$.

The preceding result is a measure-theoretic analogue of the statement:

An L^1 random-variable is not deterministic iff its variance is > 0. We omit proof.

Because $\forall t \in \mathbb{Z}, |t| \leq t^2$, we conclude:

for any \mathbb{Z} -valued random-variable X, $\mathbb{E}[|X|] \leq \mathbb{E}[X^2]$. It follows that for any \mathbb{Z} -valued L^2 random-variable X, we have: X is L^1 , and so $\mathbb{E}[X]$ is defined and finite.

Because $\forall t \in \mathbb{Z}, |t| \leq t^2$, we conclude: $\forall \Theta \subseteq \mathbb{Z}, \forall \mu \in \mathcal{M}_{\Theta}, |\mu|_1 \leq |\mu|_2^2$; it follows that if $|\mu|_2 < \infty$, then $|\mu|_1 < \infty$, and so M_{μ} is defined and finite.

DEFINITION 9.7. Let Θ be a countable set.

Let $\mu_1, \mu_2, \ldots \in \mathcal{P}_{\Theta}$ and let $\lambda \in \mathcal{P}_{\Theta}$. $By[\mu_1, \mu_2, \ldots \to \lambda]$, we mean: $\forall \Theta_0 \subseteq \Theta, \ \mu_1(\Theta_0), \mu_2(\Theta_0), \ldots \to \lambda(\Theta_0)$. Recall (§2): \forall function f, the notation: \mathbb{I}_f . Recall (§2): \forall function f, \forall set A, the notation: f^*A .

For any countable set S, for any set T, for any function $f: S \to T$, for any $\mu \in \mathcal{M}_S$, we **define** $f_*\mu \in \mathcal{M}_{\mathbb{I}_f}$ by: $\forall A \subseteq \mathbb{I}_f, \quad (f_*\mu)(A) = \mu(f^*A).$

Let S be a countable set, T a set, $f: S \to T$. Let $n \in \mathbb{N}$. Define $f^n: S^n \to T^n$ by: $\forall x \in S^n$, $f^n(x) = (f(x_1), \ldots, f(x_n))$. Then: $(f^n)_*(\mu^n) = (f_*\mu)^n$.

For any nonempty countable set Θ , for any $\mu \in \mathcal{FM}_{\Theta}^{\times}$,

let $\mathcal{N}(\mu)$:= $\frac{\mu}{\mu(\Theta)} \in \mathcal{P}_{\Theta}$; then $\forall \Theta_0 \subseteq \Theta$, $(\mathcal{N}(\mu))(\Theta_0) = \frac{\mu(\Theta_0)}{\mu(\Theta)}$, and $\mathcal{N}(\mu)$ is called the **normalization** of μ .

Let
$$\widehat{\Theta}$$
 be a countable set. Let $\mu \in \mathcal{M}_{\widehat{\Theta}}$. Let $\Theta \subseteq \widehat{\Theta}$.
Then the restriction of μ to Θ , denoted $\mu | \Theta \rangle \in \mathcal{M}_{\Theta}$,
is defined by: $\forall \Theta_0 \subseteq \Theta$, $(\mu | \Theta)(\Theta_0) = \mu(\Theta_0)$.

NOTE: We have $(\mu|\Theta)(\Theta) = \mu(\Theta)$. So, if $0 < \mu(\Theta) < \infty$, then:

$$\mu | \Theta \in \mathcal{FM}_{\Theta}^{\times} \quad \text{and} \quad \mathcal{N}(\mu | \Theta) = \frac{\mu | \Theta}{\mu(\Theta)}$$

and $\forall \Theta_0 \subseteq \Theta, \quad (\mathcal{N}(\mu | \Theta))(\Theta_0) = \frac{\mu(\Theta_0)}{\mu(\Theta)}.$

DEFINITION 9.8. Let F be a nonempty finite set.

Then we define $\nu_F \in \mathcal{P}_F$ by: $\forall f \in F, \quad \nu_F\{f\} = 1/(\#F).$ Also, we define $\nu_{\varnothing} : \{\emptyset\} \to \{-1\}$ by: $\nu_{\varnothing}(\emptyset) = -1.$

THEOREM 9.9. Let *F* be a nonempty finite set. Let $\theta \in \mathcal{P}_F$. Assume: $\forall f, g \in F, \ \theta\{f\} = \theta\{g\}.$ Then: $\theta = \nu_F$.

10. The Discrete Local Limit Theorem

DEFINITION 10.1. Let $E \subseteq \mathbb{Z}$. By E is residue-constrained, we mean: $\exists m \in [2..\infty), \exists n \in \mathbb{Z} \quad s.t. \quad E \subseteq m\mathbb{Z} + n.$ By E is residue-unconstrained, we mean: E is not residue-constrained.

Since $\emptyset \subseteq 2 \cdot \mathbb{Z} + 1$, we get: \emptyset is residue-constrained. For all $b \in \mathbb{Z}$, since $\{b\} \subseteq 2 \cdot \mathbb{Z} + b$, we get: $\{b\}$ is residue-constrained. Then: \forall residue-unconstrained $E \subseteq \mathbb{Z}$, $\#E \ge 2$. We have: $\{0,3,9\} \subseteq 3\mathbb{Z} + 0$ and $\{2,5,11\} \subseteq 3\mathbb{Z} + 2$,

so $\{0,3,9\}$ and $\{2,5,11\}$ are both residue-constrained. Here is a test for residue-unconstrainedness:

Let $E \subseteq \mathbb{Z}$. Assume $\#E \ge 2$. Let $\varepsilon_0 \in E$.

Then: (E is residue-unconstrained) iff ($\gcd(E-\varepsilon_0)=1$). By this test, we see that:

 $\{0, 1, 10\}$ and $\{2, 4, 8, 9\}$ and $\{3, 9, 13, 18\}$ are all residue-unconstrained.

DEFINITION 10.2. For all $\alpha \in \mathbb{R}$, for all real v > 0, **define** Φ_{α}^{v} : $\mathbb{R} \to (0; \infty)$ by: $\forall t \in \mathbb{R}$, $\Phi_{\alpha}^{v}(t) = \frac{\exp(-(t-\alpha)^{2} / (2v))}{\sqrt{2\pi v}}$. Note: Φ^v_{α} is a PDF of a normal variable with mean α and variance v.

The next result is a version of the Discrete Local Limit Theorem; this version is stated in probability-theoretic terms:

THEOREM 10.3. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let X_1, X_2, \ldots be an iid sequence of \mathbb{Z} -valued L^2 random-variables. Assume: $\forall n \in \mathbb{N}, \quad \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E.$ Let $\alpha \in \mathbb{R}$, $v \in [0; \infty]$. Assume: $\forall n \in \mathbb{N}$, $\mathbb{E}[X_n] = \alpha$ and $\operatorname{Var}[X_n] = v$. $0 < v < \infty$, and, $\forall t_1, t_2, \ldots \in \mathbb{Z}$, Then: as $n \to \infty$, $\sqrt{n} \cdot \left[\left(\Pr[X_1 + \dots + X_n = t_n] \right) - \left(\Phi_{n\alpha}^{nv}(t_n) \right) \right] \to 0.$ For a good exposition of this theorem and its proof, search on "Terence Tao Local Limit Theorem". Visit the website, and then expand "read the rest of this entry", and then scroll down to "-2. Local limit theorems -". In Theorem 10.3, since $E \subseteq \mathbb{Z}$, we have, for each $n \in \mathbb{N}$,
$$\begin{split} |X_n| \leqslant X_n^2 \text{ a.s.}, & \text{so} \quad \mathrm{E}[|X_n|] \leqslant \mathrm{E}[X_n^2], \\ & \text{so}, & \text{since } X_n \text{ is } L^2, \text{ we get } X_n \text{ is } L^1, \end{split}$$
and so $E[X_n]$ and $Var[X_n]$ are both defined. Moreover, in Theorem 10.3, $\forall n \in \mathbb{N}$, since $|\mathbf{E}[X_n]| \leq \mathbf{E}[|X_n|] \leq \mathbf{E}[X_n^2] < \infty$, we get: $\mathbf{E}[X_n]$ is finite. In Theorem 10.3, the proof that v > 0 is relatively simple: Since E is residue-unconstrained, we get: $\#E \ge 2.$ $\forall n \in \mathbb{N}, \quad \#\{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} \ge 2,$ Then, so X_n is not deterministic, which implies that $\operatorname{Var}[X_n] > 0$, and so v > 0. In Theorem 10.3, the proof that $v < \infty$ is relatively simple: $\forall n \in \mathbb{N}, \quad \operatorname{Var}[X_n] = \operatorname{E}[X_n^2] - (\operatorname{E}[X_n])^2 \leqslant \operatorname{E}[X_n^2] < \infty,$ and so $v < \infty$.

Next is another version of the Discrete Local Limit Theorem; this version is stated in measure-theoretic terms:

THEOREM 10.4. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let $\mu \in \mathcal{P}_E$. Assume: $S_{\mu} = E$. Assume: $|\mu|_2 < \infty$. Let $\alpha := M_{\mu}$, $v := V_{\mu}$. Then: $0 < v < \infty$, and, $\forall t_1, t_2, \ldots \in \mathbb{Z}$, as $n \to \infty$, $\sqrt{n} \cdot [(\mu^n \{f \in E^n \mid f_1 + \cdots + f_n = t_n\}) - (\Phi_{n\alpha}^{nv}(t_n))] \to 0$. In Theorem 10.4, since $S_{\mu} = E \subseteq \mathbb{Z}$ we get: $|\mu|_1 \leq |\mu|_2^2$. Since $|\mu|_1 \leq |\mu|_2^2 < \infty$, we get: M_{μ} and V_{μ} are both defined. Moreover, since $|M_{\mu}| \leq |\mu|_1 \leq |\mu|_2^2 < \infty$, we get: M_{μ} is finite. In Theorem 10.4, the proof that v > 0 is relatively simple:

Since *E* is residue-unconstrained, we get: $\#E \ge 2$. Since $\#S_{\mu} = \#E \ge 2$, by Theorem 9.6, we get: v > 0. In Theorem 10.4, the proof that $v < \infty$ is relatively simple: $v = V_{\mu} = |\mu|_2^2 - M_{\mu}^2 \le |\mu|_2^2 < \infty$.

Here is an application of Theorem 10.3:

THEOREM 10.5. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let X_1, X_2, \ldots be an iid sequence of \mathbb{Z} -valued L^2 random-variables. Assume: $\forall n \in \mathbb{N}, \quad \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E.$ Let $\alpha \in \mathbb{R}$, $v \in [0; \infty]$. Assume: $\forall n \in \mathbb{N}$, $\mathbb{E}[X_n] = \alpha$ and $\operatorname{Var}[X_n] = v$. Then: $0 < v < \infty$. Also, $\forall t_1, t_2, \ldots \in \mathbb{Z}$, $\{t_n - n\alpha \mid n \in \mathbb{N}\}\$ is bounded, if then, as $n \to \infty$, $\sqrt{n} \cdot (\Pr[X_1 + \dots + X_n = t_n]) \to 1/\sqrt{2\pi v}$. *Proof.* By Theorem 10.3, we get $0 < v < \infty$. **Given** $t_1, t_2, \ldots \in \mathbb{Z}$, assume $\{t_n - n\alpha \mid n \in \mathbb{N}\}$ is bounded, want: as $n \to \infty$, $\sqrt{n} \cdot (\Pr[X_1 + \dots + X_n = t_n]) \to 1/\sqrt{2\pi v}$. By Theorem 10.3, it suffices to show: as $n \to \infty$, $\sqrt{n} \cdot (\Phi_{n\alpha}^{nv}(t_n)) \to 1/\sqrt{2\pi v}$. We have: $\forall n \in \mathbb{N}$, $\Phi_{n\alpha}^{nv}(t_n) = \frac{\exp(-(t_n - n\alpha)^2 / (2nv))}{\sqrt{2\pi nv}}$. Since $\{t_n - n\alpha \mid n \in \mathbb{N}\}$ is bounded and since $0 < v < \infty$, we get: $-(t_n - n\alpha)^2 / (2nv) \rightarrow 0.$ as $n \to \infty$, as $n \to \infty$, $\exp(-(t_n - n\alpha)^2 / (2nv)) \to 1$. Then: $\sqrt{n} \cdot (\Phi_{n\alpha}^{nv}(t_n)) \longrightarrow 1/\sqrt{2\pi v}.$ Then: as $n \to \infty$, We record a measure-theoretic version of Theorem 10.5:

THEOREM 10.6. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let $\mu \in \mathcal{P}_E$. Let $\alpha := M_{\mu}, \quad v := V_{\mu}$. Assume: $S_{\mu} = E$ and $|\mu|_2 < \infty$. Let $\alpha := M_{\mu}, \quad v := V_{\mu}$. Then: $0 < v < \infty$. Also, $\forall t_1, t_2, \ldots \in \mathbb{Z}$, if $\{t_n - n\alpha \mid n \in \mathbb{N}\}$ is bounded, then, as $n \to \infty$, $\sqrt{n} \cdot (\mu^n \{f \in E^n \mid f_1 + \cdots + f_n = t_n\}) \to 1/\sqrt{2\pi v}$.

We also record the $t_n = t_0 + n\alpha$ special case of the past two theorems:

THEOREM 10.7. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained.

Let X_1, X_2, \ldots be an iid sequence of \mathbb{Z} -valued L^2 random-variables. Assume: $\forall n \in \mathbb{N}, \quad \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E.$ Let $t_2 \in \mathbb{Z}, v \in [0, \infty]$ Assume: $\forall n \in \mathbb{N}, \mathbb{E}[X] = \alpha$ and $\operatorname{Var}[X] = 0$

Let $t_0, \alpha \in \mathbb{Z}, v \in [0; \infty]$. Assume: $\forall n \in \mathbb{N}, E[X_n] = \alpha$ and $Var[X_n] = v$. Then: $0 < v < \infty$, and,

as
$$n \to \infty$$
, $\sqrt{n} \cdot \left(\Pr[X_1 + \dots + X_n = t_0 + n\alpha] \right) \to 1/\sqrt{2\pi v}.$

THEOREM 10.8. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let $\mu \in \mathcal{P}_E$. Assume: $S_{\mu} = E$. Assume: $|\mu|_2 < \infty$. Let $\alpha := M_{\mu}, v := V_{\mu}$. Assume: $\alpha \in \mathbb{Z}$. Let $t_0 \in \mathbb{Z}$. Then: $0 < v < \infty$, and, as $n \to \infty$, $\sqrt{n} \cdot (\mu^n \{ f \in E^n \mid f_1 + \dots + f_n = t_0 + n\alpha \}) \to 1/\sqrt{2\pi v}$.

We also record the $t_0 = 0$ special case of the past two theorems:

THEOREM 10.9. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let X_1, X_2, \ldots be an iid sequence of \mathbb{Z} -valued L^2 random-variables. Assume: $\forall n \in \mathbb{N}, \quad \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E.$ Let $\alpha \in \mathbb{Z}, v \in [0; \infty]$. Assume: $\forall n \in \mathbb{N}, E[X_n] = \alpha$ and $\operatorname{Var}[X_n] = v$. Then: $0 < v < \infty$, and, $as \ n \to \infty, \quad \sqrt{n} \cdot (\Pr[X_1 + \cdots + X_n = n\alpha]) \to 1/\sqrt{2\pi v}.$

THEOREM 10.10. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let $\mu \in \mathcal{P}_E$. Assume: $S_{\mu} = E$. Assume: $|\mu|_2 < \infty$. Let $\alpha := M_{\mu}, \quad v := V_{\mu}$. Assume: $\alpha \in \mathbb{Z}$. Then: $0 < v < \infty$, and, $as \ n \to \infty, \quad \sqrt{n} \cdot (\mu^n \{ f \in E^n \mid f_1 + \dots + f_n = n\alpha \}) \to 1/\sqrt{2\pi v}.$

11. Average events have low information, particular case

Suppose, in secret, I flip a coin 1000 times,

then reveal to you that

the total number of heads was 1000,

and then ask you to guess the last flip.

The answer is that, since *all* the coin flips were heads,

the last flip must have been a head.

Similarly, if I had told you that

the total number of heads was 0,

then you would have known that the last flip was a tail.

By contrast, if I had told you that

the total number of heads was 500,

it seems intuitively clear that

you'd have had very little information about the last flip. We wish to generalize and formalize that intuition,

and then provide rigorous proof of the resulting formal statement. Our main theorem is Theorem 12.3, in the next section. In this section, we go carefully through a special case:

Let $X_1, X_2...$ be \mathbb{Z} -valued iid random-variables s.t.,

 $\forall n \in \mathbb{N}, \quad \Pr[X_n = -1] = 1/2,$ $\Pr[X_n = 0] = 1/3,$ $\Pr[X_n = 3] = 1/6.$ X_n is L^1 and X_n is L^2 . $\forall n \in \mathbb{N},$ Then, $\operatorname{E}[X_n] = 0$ and $\operatorname{Var}[X_n] = 2$. $\forall n \in \mathbb{N},$ Also, $\forall n \in \mathbb{N},$ $-1 \leqslant X_n \leqslant 3$ a.s. Also, $T_n := X_1 + \dots + X_n.$ For all $n \in \mathbb{N}$, let Then: $\forall n \in \mathbb{N},$ $-n \leqslant T_n \leqslant 3n$ a.s. Then: $-1000 \leq T_{1000} \leq 3000$ a.s. $[T_{1000} = -1000] \Rightarrow [X_1 = \cdots = X_{1000} = -1],$ Also, and so $\Pr[X_{1000} = -1 \mid T_{1000} = -1000] = 1.$ $\Pr[X_{1000} = 3 | T_{1000} =$ [3000] = 1.Similarly, By contrast, the event $T_{1000} = 0$ would seem to give very little information about X_{1000} . It therefore seems reasonable to expect that $\Pr[X_{1000} = -1 \mid T_{1000} = 0] \approx 1/2$ and $\Pr[X_{1000} = 0 | T_{1000} = 0] \approx 1/3$ and $\Pr[X_{1000} = 3 \mid T_{1000} = 0] \approx 1/6.$ To make this precise, we will work "in the thermodynamic limit",

which means: we replace 1000 by a variable $n \in \mathbb{N}$, and let $n \to \infty$. That is, more precisely, we expect that, as $n \to \infty$, $\Pr[X_n = -1 | T_n = 0] \to 1/2$ and $\Pr[X_n = 0 | T_n = 0] \to 1/3$ and $\Pr[X_n = 3 | T_n = 0] \to 1/6$.

We will focus on proving the third of these limits;

proofs of the other two are similar.

By definition of conditional probability,

we wish to prove: As
$$n \to \infty$$
, $\frac{\Pr[(X_n = 3)\&(T_n = 0)]}{\Pr[T_n = 0]} \to 1/6.$

Claim: Let $n \in [2..\infty).$ $\Pr[(X_n = 3)\&(T_n = 0)] = (1/6) \cdot (\Pr[T_{n-1} = -3]).$ Then: Proof of Claim: We have: $T_n = X_1 + \cdots + X_{n-1} + X_n$. $\Pr[(X_n = 3)\&(T_n = 0)]$ Since $= \Pr[(X_n = 3)\&(X_1 + \dots + X_{n-1} + X_n = 0)]$ $= \Pr[(X_n = 3)\&(X_1 + \dots + X_{n-1} + 3 = 0)]$ $= \Pr[(X_n = 3)\&(X_1 + \dots + X_{n-1} = -3)],$ it follows, from independence of X_1, \ldots, X_n , that $\Pr[(X_n = 3)\&(T_n = 0)]$ $= (\Pr[X_n = 3]) \cdot (\Pr[X_1 + \dots + X_{n-1} = -3]).$ $\Pr[X_n = 3] = 1/6 \text{ and } X_1 + \dots + X_{n-1} = T_{n-1},$ So, since we get: $\Pr[(X_n = 3)\&(T_n = 0)] = (1/6) \cdot (\Pr[T_{n-1} = -3])$ End of proof of Claim.

By the claim, we wish to prove:

As
$$n \to \infty$$
, $\frac{(1/6) \cdot (\Pr[T_{n-1} = -3])}{\Pr[T_n = 0]} \to 1/6$
As $n \to \infty$, $\frac{\Pr[T_{n-1} = -3]}{\Pr[T_{n-1} = -3]} \to 1.$

We wish to prove: As $n \to \infty$

That is, we wish to prove:

As $n \to \infty$, $\Pr[T_{n-1} = -3]$ is asymptotic to $\Pr[T_n = 0]$. So the question becomes:

How do we get a handle on the asymptotics, as $n \to \infty$, of both $\Pr[T_{n-1} = -3]$ and $\Pr[T_n = 0]$? The Discrete Local Limit Theorem turns out to be just what we need.

 $\forall n \in \mathbb{N}, \quad \mathbb{E}[X_n] = 0 \text{ and } \operatorname{Var}[X_n] = 2.$ Recall: Let $\alpha := 0$ and v := 2. Then: $(\forall n \in \mathbb{N}, n\alpha = 0)$ and $(2\pi v = 4\pi)$. $\forall n \in \mathbb{N}, \quad \mathbb{E}[X_n] = \alpha \quad \text{and} \quad \operatorname{Var}[X_n] = v.$ Also, Let $E := \{-1, 0, 3\}$. Then E is residue-unconstrained. Also, we have: $\forall n \in \mathbb{N}, \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E.$ By Theorem 10.9, as $n \to \infty$, $\sqrt{n} \cdot (\Pr[X_1 + \dots + X_n = n\alpha]) \rightarrow 1/\sqrt{2\pi v},$ as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = 0]) \to 1/\sqrt{4\pi}$, Then: $\Pr[T_n = 0]$ is asymptotic to $1/\sqrt{4\pi n}$. so, as $n \to \infty$, as $n \to \infty$, $\Pr[T_{n-1} = -3]$ is asymptotic to $1/\sqrt{4\pi n}$. Want: Then, Let $t_0 := -3$. $\forall n \in \mathbb{N}, \quad t_0 + n\alpha = -3.$ By Theorem 10.7, as $n \to \infty$,

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$$\begin{array}{rcl} & \sqrt{n} \cdot \left(\Pr[X_1 + \dots + X_n &= t_0 + n\alpha] \right) & \rightarrow & 1/\sqrt{2\pi v}. \end{array}$$
Recall: $\forall n \in \mathbb{N}, \ T_n = X_1 + \dots + X_n.$
Then: as $n \to \infty$, $\sqrt{n} \cdot \left(\Pr[T_n &= -3] \right) & \rightarrow & 1/\sqrt{4\pi}.$
Then, as $n \to \infty$, $\sqrt{n-1} \cdot \left(\Pr[T_{n-1} = -3] \right) & \rightarrow & 1/\sqrt{4\pi}.$
Then, as $n \to \infty$, $\Pr[T_{n-1} = -3]$ is asymptotic to $1/\sqrt{4\pi(n-1)},$
which is asymptotic to $1/\sqrt{4\pi n}.$

12. Average events have low information, general result

We now seek to generalize our work in \$11;

in the example at the end of this section, we show that Theorem 12.3 reproduces the result of §11.

THEOREM 12.1. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let X_1, X_2, \ldots be an iid sequence of \mathbb{Z} -valued L^2 random-variables. Assume: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E$. Let $\alpha, P \in \mathbb{R}$. Assume: $\forall n \in \mathbb{N}$, $E[X_n] = \alpha$ and $\Pr[X_n = \varepsilon_0] = P$. Let $\varepsilon_0 \in E$. Let $t_1, t_2, \ldots \in \mathbb{Z}$. Assume: $\{t_n - n\alpha \mid n \in \mathbb{N}\}$ is bounded. Then: as $n \to \infty$, $\Pr[X_n = \varepsilon_0 \mid X_1 + \cdots + X_n = t_n] \to P$.

I don't know whether " L^{2} " can be replaced by " L^{1} ".

Part of the content of Theorem 12.1 is:

 $\forall \text{sufficiently large } n \in \mathbb{N}, \quad \Pr[X_1 + \dots + X_n = t_n] > 0$ since, otherwise, $\Pr[X_n = \varepsilon_0 | X_1 + \dots + X_n = t_n] \text{ would not be defined.}$

Proof. Since X_1, X_2, \ldots are all \mathbb{Z} -valued and L^2 ,

and since $\forall t \in \mathbb{Z}$, $|t| \leq t^2$ we get: X_1, X_2, \ldots are all L^1 . So, since X_1, X_2, \ldots is an identically distributed sequence,

choose $v \in [0; \infty]$ s.t., $\forall n \in \mathbb{N}$, $\operatorname{Var}[X_n] = v$. By Theorem 10.5, we have: $0 < v < \infty$ and

 $\begin{array}{rl} \operatorname{as} n \to \infty, & \sqrt{n} \cdot \left(\Pr[X_1 + \dots + X_n = t_n] \right) \to 1/\sqrt{2\pi v}. \\ \text{For all } n \in \mathbb{N}, & \operatorname{let} T_n := X_1 + \dots + X_n. \\ \text{Then:} & \operatorname{as} n \to \infty, & \sqrt{n} \cdot \left(\Pr[& T_n & = t_n] \right) \to 1/\sqrt{2\pi v}. \\ \text{Want:} & \operatorname{as} n \to \infty, & \Pr[X_n = \varepsilon_0 \mid T_n = t_n] \to P. \\ \text{Let } D_1 := \{t_n - n\alpha \mid n \in \mathbb{N}\}. & \text{By hypothesis, } D_1 \text{ is bounded.} \\ \text{Let } D_2 := \{t_n - n\alpha \mid n \in [2..\infty)\}. & \text{Then } D_2 \subseteq D_1. \\ \text{Let } D_3 := \{t_{n+1} - (n+1) \cdot \alpha \mid n \in \mathbb{N}\}. & \text{Then } D_3 = D_2. \\ \text{For all } n \in \mathbb{N}, & \operatorname{let} & \widetilde{t}_n := t_{n+1} - \varepsilon_0. \\ \text{Let } & D_4 & := \{ & \widetilde{t}_n & - n\alpha & \mid n \in \mathbb{N} \}. \end{array}$

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we get: as $n \to \infty$, $\Pr[X_n = \varepsilon_0 | T_n = t_n] \to P.$

Recall (§9): \forall countable set Θ ,

 $\begin{array}{c} \mathcal{FM}_{\Theta}^{\times} \text{ is the set of nonzero finite measures on } \Theta \\ \text{and} \qquad \mathcal{P}_{\Theta} \text{ is the set of probability measures on } \Theta. \\ \text{Recall (\$9): } \forall \text{nonempty countable set } \Theta, \quad \forall \mu \in \mathcal{FM}_{\Theta}^{\times}, \\ \mathcal{N}(\mu) \text{ is the normalization of } \mu. \end{array}$

Here is a measure-theoretic version of the preceding theorem:

THEOREM 12.2. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let $\mu \in \mathcal{P}_E$. Assume: $S_{\mu} = E$. Assume: $|\mu|_2 < \infty$. Let $\alpha := M_{\mu}$. Let $\varepsilon_0 \in E$, $P := \mu \{\varepsilon_0\}$. Let $t_1, t_2, \ldots \in \mathbb{Z}$. Assume: $\{t_n - n\alpha \mid n \in \mathbb{N}\}$ is bounded. For all $n \in \mathbb{N}$, let $\Omega_n := \{f \in E^n \mid f_1 + \cdots + f_n = t_n\}$. Then: as $n \to \infty$, $(\mathcal{N}(\mu^n \mid \Omega_n))\{f \in \Omega_n \mid f_n = \varepsilon_0\} \to P$.

I don't know whether " $|\mu|_2 < \infty$ " can be replaced by " $|\mu|_1 < \infty$ ".

Part of the content of Theorem 12.2 is:

 $\begin{array}{ll} \forall \text{sufficiently large } n \in \mathbb{N}, & \mu^n(\Omega_n) > 0, \\ \text{since, otherwise,} & \mu^n | \Omega_n & \text{would be the zero measure on } \Omega_n, \\ \text{and so} & \mathcal{N}(\mu^n | \Omega_n) & \text{would not be defined.} \end{array}$

We record the $t_n = n\alpha$ special case of the past two theorems:

THEOREM 12.3. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let X_1, X_2, \ldots be an iid sequence of \mathbb{Z} -valued L^2 random-variables. Assume: $\forall n \in \mathbb{N}, \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E$. Let $\alpha \in \mathbb{Z}, P \in \mathbb{R}$. Let $\varepsilon_0 \in E$. Assume: $\forall n \in \mathbb{N}, E[X_n] = \alpha$ and $\Pr[X_n = \varepsilon_0] = P$. Then: as $n \to \infty$, $\Pr[X_n = \varepsilon_0 \mid X_1 + \cdots + X_n = n\alpha] \to P$.

THEOREM 12.4. Let $E \subseteq \mathbb{Z}$ be residue-unconstrained. Let $\mu \in \mathcal{P}_E$. Let $\alpha := M_{\mu}$. Assume: $\alpha \in \mathbb{Z}$ and $S_{\mu} = E$ and $|\mu|_2 < \infty$. For all $n \in \mathbb{N}$, let $\Omega_n := \{f \in E^n \mid f_1 + \dots + f_n = n\alpha\}$. Let $\varepsilon_0 \in E$. Let $P := \mu\{\varepsilon_0\}$. Then: $as \ n \to \infty$, $(\mathcal{N}(\mu^n \mid \Omega_n))\{f \in \Omega_n \mid f_n = \varepsilon_0\} \to P$. Example: Let $E := \{-1, 0, 3\}$. Then: $E \subseteq \mathbb{Z}$ and E is residue-unconstrained. Let $X_1, X_2 \dots$ be \mathbb{Z} -valued iid random-variables s.t., $\forall n \in \mathbb{N}$, $\Pr[X_n = -1] = 1/2$, $\Pr[X_n = 0] = 1/3$.

 $\Pr[X_n = 3] = 1/6.$ $\forall n \in \mathbb{N}, \qquad \{ t \in \mathbb{Z} \mid \Pr[X_n = t] > 0 \} = E.$ Then: Let $\varepsilon_0 = 3$, P := 1/6. $\Pr[X_n = \varepsilon_0] = P.$ Then: $\forall n \in \mathbb{N},$ $\mathbf{E}[X_n] = 0.$ Let $\alpha := 0$. We have: $\forall n \in \mathbb{N}$, $\forall n \in \mathbb{N}.$ $\mathrm{E}[X_n] = \alpha.$ Then, Then, by Theorem 12.3, we have: as $n \to \infty$, $\Pr[X_n = \varepsilon_0 | X_1 + \dots + X_n = n\alpha] \to P$. as $n \to \infty$, $\Pr[X_n = 3 | X_1 + \dots + X_n = 0] \to 1/6.$ Then: let $T_n := X_1 + \cdots + X_n$. For all $n \in \mathbb{N}$, Then: as $n \to \infty$, $\Pr[X_n = 3 | T_n = 0] \to 1/6.$ Thus Theorem 12.3 reproduces the result of $\S11$.

13. Solving the main problem

Again, let's say I am one of the professors applying to the GFA. We will show: Under the GFA's *first* system $(\S 3)$,

my probability of getting 0 is p, approximately and my probability of getting 1 is q, approximately and my probability of getting 10 is r, approximately.

So, since q + 10r = 1, we get:

 $\forall n \in \mathbb{N}, \quad \mathrm{E}[X_n] = 1.$ We model the GFA's second system (§5) by: $\forall \ell \in [1..N]$, Professor# ℓ receives X_{ℓ} dollars. For all $n \in \mathbb{N}$, let $T_n := X_1 + \cdots + X_n$. We model the GFA's *third* system (§5) by: $\forall \ell \in [1..N]$, Professor# ℓ receives X_{ℓ} dollars, conditioned on $T_N = N$. $\forall \omega \in \Omega, \quad p^{i_{\omega}} q^{j_{\omega}} r^{k_{\omega}} / S = 1 / (\#\Omega),$ Since it follows that: the third system is equivalent to the first. For definiteness, let's assume that I am Professor#N. Then, assuming N is large, we wish to show: $\Pr[X_N = 0 | T_N = N] \approx p$ and $\Pr[X_N = 1 | T_N = N] \approx q$ and $\Pr[X_N = 10 \mid T_N = N] \approx r.$ To be more precise, we wish to show: as $n \to \infty$, $\Pr[X_n = 0 | T_n = n] \rightarrow p$ and $\Pr[X_n = 1 | T_n = n] \rightarrow q$ and $\Pr[X_n = 10 \mid T_n = n] \rightarrow r.$ Let $E := \{0, 1, 10\}.$ Then: E is residue-unconstrained. Given $\varepsilon_0 \in E$, let $P := \begin{cases} p, & \text{if } \varepsilon_0 = 0 \\ q, & \text{if } \varepsilon_0 = 1 \\ r, & \text{if } \varepsilon_0 = 10, \end{cases}$ as $n \to \infty$, $\Pr[X_n = \varepsilon_0 | T_n = n] \to P$. want: By definition of X_1, X_2, \ldots , we get: $\forall n \in \mathbb{N}, Pr[X_n = \varepsilon_0] = P$. Then: $\alpha \in \mathbb{Z}$ and $\forall n \in \mathbb{N}, E[X_n] = \alpha$. Let $\alpha := 1$. $\forall n \in \mathbb{N}, \ \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E.$ Also, by Theorem 12.3, we have: Then, as $n \to \infty$, $\Pr[X_n = \varepsilon_0 | X_1 + \dots + X_n = n\alpha] \to P$. as $n \to \infty$, $\Pr[X_n = \varepsilon_0]$ $T_n = n] \to P$. Then: 14. Probability of two professors getting zero Under the GFA's first system, since N is large, one would expect:

the award amounts of two different professors are almost independent. Then, for example, one would expect: the probability that two professors both receive zero dollars

should be very close to the square of the probability that one professor receives zero dollars. We will formalize this statement and prove it, below. For definiteness, we will assume that

the two professors are Professor #(N-1) and Professor #N.

 $(p,q,r) := \frac{(1,9^{-1/10},9^{-1})}{1+9^{-1/10}+9^{-1}}.$ Let Then (§7): p + q + r = 1. Let X_1, X_2, \ldots be \mathbb{Z} -valued iid random-variables s.t., $\forall n \in \mathbb{N}$, $\Pr[X_n = 0] = p,$ $\Pr[X_n = 1] = q,$ $\Pr[X_n = 10] = r.$ Then X_1, X_2, \ldots is a sequence of L^2 random-variables. For all $n \in \mathbb{N}$, let $T_n := X_1 + \cdots + X_n$. Assuming N is large, our goal is to prove: Pr $[X_{N-1} = 0 = X_N | T_N = N] \approx p^2$. To be more precise, we will prove: $\Pr [X_{n-1} = 0 = X_n | T_n = n] \to p^2.$ as $n \to \infty$, For all $n \in \mathbb{N}$, define $\psi_n : \mathbb{Z} \to \mathbb{R}$ by: $\forall t \in \mathbb{Z}, \quad \psi_n(t) = \Pr[T_n = t].$ $a_n := \psi_n(n+2), \quad z_n := \psi_n(n).$ For all $n \in \mathbb{N}$, \mathbf{let} Since, $\forall n \in \mathbb{N}$, we have $\psi_n(n) = \Pr[T_n = n] = \Pr[X_1 + \dots + X_n = n]$ $\geq \Pr[X_1 = \dots = X_n = 1] = q^n > 0,$ we conclude: $\forall n \in \mathbb{N}, \quad z_n > 0.$ Claim: Let $n \in [3..\infty)$. Then $\Pr[X_{n-1} = 0 = X_n | T_n = n] = p^2 \cdot \frac{a_{n-2}}{z_n}$. Proof of Claim: We have $T_n = X_1 + \cdots + X_{n-2} + X_{n-1} + X_n$. $\Pr[(X_{n-1} = 0 = X_n)\&(T_n = n)]$ Since $= \Pr[(X_{n-1} = 0 = X_n)\&(X_1 + \dots + X_{n-2} + X_{n-1} + X_n = n)]$ $= \Pr[(X_{n-1} = 0 = X_n)\&(X_1 + \dots + X_{n-2} + 0 + 0 = n)]$ $= \Pr[(X_{n-1} = 0 = X_n)\&(X_1 + \dots + X_{n-2})]$ = nit follows, from independence of X_1, \ldots, X_n , that $\Pr[(X_{n-1} = 0 = X_n)\&(T_n = n)]$ $= (\Pr[X_{n-1} = 0]) \cdot (\Pr[X_n = 0]) \cdot (\Pr[X_1 + \dots + X_{n-2} = n]).$ So, since $\Pr[X_{n-1} = 0] = p = \Pr[X_n = 0]$ and since $X_1 + \dots + X_{n-2} = T_{n-2}$, we get: $\Pr[(X_{n-1} = 0 = X_n)\&(T_n = n)] = p^2 \cdot (\Pr[T_{n-2} = n]).$ Then $\Pr[X_{n-1} = 0 = X_n | T_n = n] = \frac{\Pr[(X_{n-1} = 0 = X_n)\&(T_n = n)]}{\Pr[T_n = n]}$

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$$= \frac{p^2 \cdot (\Pr[T_{n-2} = n])}{\Pr[T_n = n]} = p^2 \cdot \frac{\psi_{n-2}(n)}{\psi_n(n)} = p^2 \cdot \frac{a_{n-2}}{z_n}.$$

End of proof of Claim.

Because of the Claim, we want to show: as $n \to \infty$, $p^2 \cdot \frac{a_{n-2}}{z_n} \to p^2$. Want: as $n \to \infty$, $\frac{a_{n-2}}{z_n} \to 1$. We compute: $\forall n \in \mathbb{N}, \quad \mathbf{E}[X_n] = q + 10r.$ Then: $\mathrm{E}[X_n] = 1.$ Recall (§7): q + 10r = 1. $\forall n \in \mathbb{N},$ $\forall n \in \mathbb{N}, \quad \operatorname{Var}[X_n] = q + 100r - 1.$ We compute: Let v := q + 100r - 1. Then: $\forall n \in \mathbb{N}, \quad \operatorname{Var}[X_n] = v.$ Since v = (q + 10r - 1) + 90r = 0 + 90r = 90r, and since $0 < r < \infty$, we get: $0 < v < \infty$. Let $\tau := 1/\sqrt{2\pi v}$. Then: $0 < \tau < \infty$. Then: $(\alpha \in \mathbb{Z})$ and $(\forall n \in \mathbb{N}, E[X_n] = \alpha)$. Let $\alpha := 1$. Then, $\forall n \in \mathbb{N}, \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E.$ Let $E := \{0, 1, 10\}.$ Also, E is residue-unconstrained. By Theorem 10.9, as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = n\alpha]) \to 1/\sqrt{2\pi v}$. as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = n]) \to \tau$. Then: as $n \to \infty$, $\sqrt{n} \cdot (\psi_n(n)) \to \tau$. Then: as $n \to \infty$, $\sqrt{n} \cdot z_n \to \tau$. Then: Then $t_0 \in \mathbb{Z}$ and $\forall n \in \mathbb{N}, t_0 + n\alpha = n + 2.$ Let $t_0 := 2$. By Theorem 10.7, as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = t_0 + n\alpha]) \to 1/\sqrt{2\pi v}$. as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = n+2]) \to \tau$. Then: as $n \to \infty$, $\sqrt{n} \cdot (\psi_n(n+2)) \to \tau$. Then: as $n \to \infty$, $\sqrt{n} \cdot a_n \to \tau$. Then: as $n \to \infty$, $\sqrt{n-2} \cdot a_{n-2} \to \tau$. Then: $\sqrt{n} \cdot z_n \quad \to \tau.$ Recall: as $n \to \infty$, Dividing the last two limits, we get: as $n \to \infty$, $\frac{\sqrt{n-2} \cdot a_{n-2}}{\sqrt{n} \cdot z_n} \to 1$. as $n \to \infty$, $\frac{\sqrt{n}}{\sqrt{n-2}} \to 1$. Also, Multiplying these last two limits, we get: as $n \to \infty$, $\frac{a_{n-2}}{z_n} \to 1$. Also,

15. Fraction of professors getting a zero award

 $(p,q,r) := \frac{(1,9^{-1/10},9^{-1})}{1+9^{-1/10}+9^{-1}}$ Let We compute $(p,q,r) \approx (0.5225, 0.4194, 0.0581),$ all accurate to four decimal places. Let X_1, X_2, \ldots be \mathbb{Z} -valued iid random-variables s.t., $\forall n \in \mathbb{N}$, $\Pr[X_n = 0] = p,$ $\Pr[X_n = 1] = q,$ $\Pr[X_n = 10] = r.$ For all $n \in \mathbb{N}$, let $T_n := X_1 + \cdots + X_n$. For all $n \in \mathbb{N}$, let I_n be the indicator variable of the event: $X_n = 0$. For all $n \in \mathbb{N}$, let $J_n := (I_1 + \cdots + I_n)/n$. Using the GFA's first (or third) awards system, the random-variable conditioned on $T_N = N$ J_N the fraction of professors receiving a \$0 award. represents In this section, we will prove the following: Claim: $\forall \delta > 0$, as $n \to \infty$, Pr $[p - \delta < J_n < p + \delta | T_n = n] \to 1$. for a moment, that this Claim is true. Assume, as $n \to \infty$, Pr $[p - 0.02 < J_n < p + 0.02 | T_n = n] \to 1$. Then: From this, it follows that, if N is sufficiently large, then Pr [$p - 0.02 < J_N < p + 0.02 | T_N = N] > 0.99,$ $\Pr[p - 0.02 < J_N \qquad | T_N = N] > 0.99,$ \mathbf{SO} $\Pr[J_N > p - 0.02]$ $|T_N = N| > 0.99.$ \mathbf{SO} $p \approx 0.5225$, accurate to four decimal places, we get Since p - 0.02> 0.5. $[J_N > p - 0.02] \quad \Rightarrow \quad [J_n > 0.5],$ \mathbf{SO} $\Pr\left[J_N > p - 0.02 \qquad | T_N = N\right]$ \mathbf{SO} \leq Pr [$J_N > 0.5$ $|T_N = N|$. if N is sufficiently large, then, Therefore, since $|T_N = N|$ $\Pr\left[J_N > 0.5\right]$ $|T_N = N| > 0.99,$ \geq Pr [$J_N > p - 0.02$ we conclude: under the GFA's first system, with probability > 99%, over 50% of the professors receive \$0.

Proof of Claim: **Given** $\delta > 0$, **want:** as $n \to \infty$, $\Pr[p - \delta < J_n < p + \delta | T_n = n] \to 1$.

Let $E := \{0, 1, 10\}.$ Then E is residue-unconstrained. $\forall n \in \mathbb{N}, \ \{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E.$ Also, Then: $\alpha \in \mathbb{Z}$ and $\forall n \in \mathbb{N}, E[X_n] = \alpha$. Let $\alpha := 1$. let $\kappa_n := E \begin{bmatrix} I_n & | T_n = n \end{bmatrix}$. For all $n \in \mathbb{N},$ $\kappa_n = \Pr[X_n = 0 \mid T_n = n].$ Then: $\forall n \in \mathbb{N},$ By Theorem 12.3, we get: as $n \to \infty$, $\Pr[X_n = 0 | X_1 + \dots + X_n = n\alpha] \to p$. as $n \to \infty$, $\Pr[X_n = 0]$ That is, $T_n = n] \rightarrow p.$ $\rightarrow p.$ Then: as $n \to \infty$, κ_n So, $\exists n_0 \in \mathbb{N}$ s.t., $\forall n \in [n_0 .. \infty)$, $p - (\delta/2) < \kappa_n < p + (\delta/2),$ we have both $p - \delta < \kappa_n - (\delta/2)$ and $\kappa_n + (\delta/2) ,$ and so $\left[\kappa_n - \left(\delta/2\right) < J_n < \kappa_n + \left(\delta/2\right)\right] \implies \left[p - \delta < J_n < p + \delta\right],$ and so $\Pr[\kappa_n - (\delta/2) < J_n < \kappa_n + (\delta/2) | T_n = n]$ and so $< J_n < p + \delta \quad |T_n = n|.$ $\Pr[p-\delta]$ \leq It therefore suffices to show: $\Pr\left[\kappa_n - (\delta/2) < J_n < \kappa_n + (\delta/2) \mid T_n = n\right] \rightarrow 1.$ as $n \to \infty$, We have: $\forall n \in \mathbb{N}, T_n$ is invariant under permutation of X_1, \ldots, X_n , as is the joint-distribution of X_1, \ldots, X_n . Then: $\forall n \in \mathbb{N}, \forall i \in [1..n],$ $\mathbf{E} \begin{bmatrix} I_i \mid T_n = n \end{bmatrix} = \mathbf{E} \begin{bmatrix} I_n \mid T_n = n \end{bmatrix}.$ Then: $\forall n \in \mathbb{N}, \forall i \in [1..n], \quad \mathbf{E} \begin{bmatrix} I_i \mid T_n = n \end{bmatrix} = \kappa_n.$ Since, $\forall n \in \mathbb{N}, \quad J_n = (I_1 + \dots + I_n)/n$, we get: $\forall n \in \mathbb{N}, \quad \mathbf{E} \begin{bmatrix} J_n \mid T_n = n \end{bmatrix} = \left(\sum_{i=1}^n \mathbf{E} \begin{bmatrix} I_i \mid T_n = n \end{bmatrix} \right) / n.$ Then: $\forall n \in \mathbb{N}, \quad \mathbf{E} \begin{bmatrix} J_n \mid T_n = n \end{bmatrix} = \left(\sum_{i=1}^n \kappa_n \right) / n.$ Then: $\forall n \in \mathbb{N}, \quad \mathbb{E} \left[J_n \mid T_n = n \right] = ($) / n. $n\kappa_n$ Then: $\forall n \in \mathbb{N}$, $\mathbb{E} \begin{bmatrix} J_n \mid T_n = n \end{bmatrix} = \kappa_n$. For all $n \in \mathbb{N}, \text{ let } v_n := \operatorname{Var} [J_n | T_n = n].$ by Chebyshev's inequality, we have: $\forall n \in \mathbb{N}$, Then, $\Pr[\kappa_n - (\delta/2) < J_n < \kappa_n + (\delta/2) | T_n = n] \ge 1 - (v_n/(\delta/2)^2).$ It therefore suffices to show: as $n \to \infty$, $v_n \to 0$. Recall: as $n \to \infty$, $\kappa_n \to p$. $v_n = \operatorname{Var} [J_n | T_n = n]$ Since $\forall n \in \mathbb{N}$, $= (E[J_n^2 | T_n = n]) - (E[J_n | T_n = n])^2$ $= \left(\mathbf{E} \left[J_n^2 | T_n = n \right] \right) - \kappa_n^2.$ $\kappa_n^2 \to p^2,$ and since, as $n \to \infty$, we want: as $n \to \infty$, $E[J_n^2 | T_n = n] \to p^2$. For all $n \in [2..\infty)$, let $\lambda_n := \mathbb{E} [I_{n-1} \cdot I_n | T_n = n].$ $\lambda_n = \Pr\left[X_{n-1} = 0 = X_n \mid T_n = n\right].$ Then: $\forall n \in [2..\infty),$

we get: as $n \to \infty$, $\lambda_n \to p^2$. So, by the result of $\S14$, For all $n \in \mathbb{N}$, since I_n is an indicator variable, we get: $I_n \in \{0, 1\}$ a.s. = I_n^2 a.s. Then: $\forall n \in \mathbb{N},$ I_n $\mathbf{E}\left[I_n \mid T_n = n \right] = \mathbf{E}\left[I_n^2 \mid T_n = n \right].$ Then: $\forall n \in \mathbb{N},$ Recall: $\forall n \in \mathbb{N},$ $\mathbf{E}\left[I_n \mid T_n = n \right] = \kappa_n.$ $\kappa_n = \mathbf{E} \left[I_n^2 \mid T_n = n \right].$ Then: $\forall n \in \mathbb{N},$ For all $n \in \mathbb{N}$, for all $i, j \in [1..n]$, let $c_{ijn} := \mathbb{E} [I_i \cdot I_j | T_n = n]$. We have: $\forall n \in \mathbb{N}, T_n \text{ is invariant under permutation of } X_1, \ldots, X_n,$ as is the joint-distribution of X_1, \ldots, X_n . $\forall n \in \mathbb{N}, \forall i \in [1..n], \quad \mathcal{E} \left[I_i^2 \mid T_n = n \right] = \mathcal{E} \left[I_n^2, \mid T_n = n \right],$ Then $\forall n \in \mathbb{N}, \forall i \in [1..n], \quad \mathbb{E} \left[I_i^2 \mid T_n = n \right] = \kappa_n,$ so, $\forall n \in \mathbb{N}, \forall i \in [1..n],$ so, $= \kappa_n.$ c_{iin} $\forall n \in [2..\infty), \forall i, j \in [1..n], \text{ if } i \neq j, \text{ then}$ Similarly, $\mathbf{E} \begin{bmatrix} I_i \cdot I_j \mid T_n = n \end{bmatrix} = \mathbf{E} \begin{bmatrix} I_{n-1} \cdot I_n \mid T_n = n \end{bmatrix},$ $\forall n \in [2..\infty), \forall i, j \in [1..n], \text{ if } i \neq j, \text{ then}$ so, $\mathbf{E}\left[I_i \cdot I_j \mid T_n = n \right] = \lambda_n.$ $\forall n \in [2..\infty), \forall i, j \in [1..n], \text{ if } i \neq j,$ then so, $c_{ijn} = \lambda_n.$ $C_{ijn} = \lambda_n.$ $C_{ijn} = \lambda_n.$ $(n \in \mathbb{N}, \forall i, j \in [1..n], \quad c_{ijn} = \begin{cases} \kappa_n, & \text{if } i = j \\ \lambda_n, & \text{if } i \neq j. \end{cases}$ $(n \in \mathbb{N}, \quad \sum_{i=1}^n \sum_{j=1}^n c_{ijn} = n \cdot \kappa_n + (n^2 - n) \cdot \lambda_n.$ $(n \in \mathbb{N}, \quad \sum_{i=1}^n \sum_{j=1}^n c_{ijn} = n \cdot \kappa_n + (n^2 - n) \cdot \lambda_n.$ $(n \in \mathbb{N}, \quad J_n = (I_1 + \dots + I_n)/n,$ $(n \in \mathbb{N}, \quad J_n^2 = (\sum_{i=1}^n \sum_{j=1}^n [I_i \cdot I_j]) / n^2.$ $(n \in \mathbb{N}, \quad E[J_n^2 | T_n = n] = (\sum_{i=1}^n \sum_{j=1}^n c_{ijn}) / n^2.$ $(n \in \mathbb{N}, \quad E[J_n^2 | T_n = n] = (1/n) \cdot \kappa_n + (1 - (1/n)) \cdot \lambda_n.$ $(n \in \mathbb{N}, \quad E[I_n^2 | T_n = n] = (0 + n + (1 - (1/n)) \cdot \lambda_n.$ Then: $\forall n \in \mathbb{N}, \forall i, j \in [1..n],$ Then: $\forall n \in \mathbb{N}$, Recall: as $n \to \infty$, Since we get: $\forall n \in \mathbb{N}$, Then: Then: as $n \to \infty$, $\mathbf{E} \begin{bmatrix} J_n^2 \mid T_n = n \end{bmatrix} \to 0 \cdot p + 1 \cdot p^2$. Then: as $n \to \infty$, $\mathbb{E} \begin{bmatrix} J_n^2 \mid T_n = n \end{bmatrix} \to p^2$. Then: End of proof of Claim.

16. Boltzmann distributions on nonempty finite sets

Recall (§9): \forall countable set Θ ,

 $\mathcal{M}_{\Theta} \text{ is the set of measures on } \Theta$ and $\mathcal{F}\mathcal{M}_{\Theta}^{\times}$ is the set of nonzero finite measures on Θ and \mathcal{P}_{Θ} is the set of probability measures on Θ . Recall (§9): $\forall \text{nonempty countable set } \Theta, \quad \forall \mu \in \mathcal{F}\mathcal{M}_{\Theta}^{\times},$ $\mathcal{N}(\mu)$ is the normalization of μ . **DEFINITION 16.1.** Let $E \subseteq \mathbb{R}$ be nonempty and finite, $\beta \in \mathbb{R}$. The unnormalized- β -Boltzmann distribution on E is

the measure
$$\widehat{B}_{\beta}^{E} \in \mathcal{FM}_{E}^{\times}$$
 defined by:
 $\forall \varepsilon \in E, \ \widehat{B}_{\beta}^{E} \{\varepsilon\} = e^{-\beta \cdot \varepsilon}.$
Also, the β -Boltzmann distribution on E is
$$B_{\beta}^{E} := \mathcal{N}(\widehat{B}_{\beta}^{E}) \in \mathcal{P}_{E}.$$
Then: $\forall \varepsilon \in E$ we have: $B^{E}(\varepsilon) = (\widehat{B}^{E}(\varepsilon))/(\widehat{B}^{E}(E))$

Then: $\forall \varepsilon \in E$, we have: $B^E_\beta\{\varepsilon\} = (\widehat{B}^E_\beta\{\varepsilon\}) / (\widehat{B}^E_\beta(E)).$

Example: Let
$$E := \{0, 1, 10\}$$
 and let $\beta \in \mathbb{R}$.
Then: $\hat{B}^E_{\beta}\{0\} = 1$, $\hat{B}^E_{\beta}\{1\} = e^{-\beta}$, $\hat{B}^E_{\beta}\{10\} = e^{-10\beta}$.
Let $C := 1/(1 + e^{-\beta} + e^{-10\beta})$.
Then: $B^E_{\beta}\{0\} = C$, $B^E_{\beta}\{1\} = Ce^{-\beta}$, $B^E_{\beta}\{10\} = Ce^{-10\beta}$

$$\begin{array}{ll} Example: \mbox{ Let } E := \{2,4,8,9\} \mbox{ and } \mbox{ let } \beta \in \mathbb{R}.\\ \mbox{Then:} & \widehat{B}^E_{\beta}\{2\} = e^{-2\beta}, & \widehat{B}^E_{\beta}\{4\} = e^{-4\beta},\\ & \widehat{B}^E_{\beta}\{8\} = e^{-8\beta}, & \widehat{B}^E_{\beta}\{9\} = e^{-9\beta}.\\ \mbox{Let } C := 1/(e^{-2\beta} + e^{-4\beta} + e^{-8\beta} + e^{-9\beta}).\\ \mbox{Then:} & B^E_{\beta}\{2\} = Ce^{-2\beta}, & B^E_{\beta}\{4\} = Ce^{-4\beta},\\ & B^E_{\beta}\{8\} = Ce^{-8\beta}, & B^E_{\beta}\{9\} = Ce^{-9\beta}. \end{array}$$

Recall (§9): For any countable set Θ , for any $\mu \in \mathcal{M}_{\Theta}$, S_{μ} is the support of μ . Note: \forall nonempty finite $E \subseteq \mathbb{R}, \forall \beta \in \mathbb{R}$, we have: $S_{\hat{B}_{\beta}^{E}} = E = S_{B_{\beta}^{E}}$. **THEOREM 16.2. Let** $E \subseteq \mathbb{R}$ be nonempty and finite. Let $\varepsilon_{0} \in E, \quad \beta, \xi \in \mathbb{R}$. Then: $B_{\beta}^{E-\xi} \{\varepsilon_{0} - \xi\} = B_{\beta}^{E} \{\varepsilon_{0}\}.$ Proof. We have: $B_{\beta}^{E-\xi} \{\varepsilon_{0} - \xi\} = \frac{e^{-\beta \cdot (\varepsilon_{0} - \xi)}}{\sum_{\varepsilon \in E} [e^{-\beta \cdot (\varepsilon - \xi)}]}$ $e^{-\beta \cdot \varepsilon_{0}} \cdot e^{\beta \cdot \xi}$

$$= \frac{e^{-\beta \cdot \varepsilon_{0}} \cdot e^{\beta \cdot \xi}}{\sum_{\varepsilon \in E} \left[e^{-\beta \cdot \varepsilon} \cdot e^{\beta \cdot \xi}\right]}$$
$$= \frac{e^{\beta \cdot \xi} \cdot e^{-\beta \cdot \varepsilon_{0}}}{e^{\beta \cdot \xi} \cdot \sum_{\varepsilon \in E} \left[e^{-\beta \cdot \varepsilon}\right]}$$
$$= \frac{e^{-\beta \cdot \varepsilon_{0}}}{\sum_{\varepsilon \in E} \left[e^{-\beta \cdot \varepsilon}\right]} = B_{\beta}^{E} \{\varepsilon_{0}\}. \quad \Box$$

Recall (§9): Let $\Theta \subseteq \mathbb{R}$ be countable, $\mu \in \mathcal{P}_{\Theta}$. Assume $\#S_{\mu} < \infty$. Then $|\mu|_1 < \infty$ and M_{μ} is the mean of μ and V_{μ} is the variance of μ . Let $E \subseteq \mathbb{R}$ be nonempty and finite. Let $\beta \in \mathbb{R}$. We define: $\begin{array}{l} \boxed{\Gamma_{\beta}^{E}} &:= \sum_{\varepsilon \in E} \left[\varepsilon \cdot e^{\beta \cdot \varepsilon} \right], \\ \hline \Delta_{\beta}^{E} &:= \sum_{\varepsilon \in E} \left[e^{\beta \cdot \varepsilon} \right], \\ \hline A_{\beta}^{E} &:= \Gamma_{\beta}^{E} / \Delta_{\beta}^{E}. \\ \hline \Gamma_{\beta}^{E} &= \sum_{\varepsilon \in E} \left[\varepsilon \cdot \left(\widehat{B}_{\beta}^{E} \{ \varepsilon \} \right) \right]. \end{array}$ Then: $\Delta_{\beta}^{E} = \sum_{\varepsilon \in E} [\hat{B}_{\beta}^{E}\{\varepsilon\}], \quad \text{and so} \quad \Delta_{\beta}^{E} = \hat{B}_{\beta}^{E}(E).$ $\frac{\Gamma_{\beta}^{E}}{\Delta_{\beta}^{E}} = \frac{\sum_{\varepsilon \in E} [\varepsilon \cdot (\hat{B}_{\beta}^{E}\{\varepsilon\})]}{\hat{B}_{\beta}^{E}(E)} = \sum_{\varepsilon \in E} [\varepsilon \cdot (B_{\beta}^{E}\{\varepsilon\})],$ Also, Since we conclude:

 A^E_β is the average value of any *E*-valued random-variable Then: whose distribution in E is B^E_β .

THEOREM 16.3. Let $E \subseteq \mathbb{R}$ be nonempty and finite. Let $\beta, \xi \in \mathbb{R}$. Then: $A_{\beta}^{E-\xi} = A_{\beta}^E - \xi.$

Want: $M_{B_{\beta}^{E-\xi}} = M_{B_{\beta}^{E}} - \xi.$ Proof. Let $\lambda := B_{\beta}^{E-\xi}, \quad \mu := B_{\beta}^{E}.$ Want: $M_{\lambda} = M_{\mu} - \xi$. We have: $\lambda \in \mathcal{P}_{E-\xi}$ and $\mu \in \mathcal{P}_{E}$. By Theorem 16.2, we have: $\forall \varepsilon \in E, \quad B_{\beta}^{E-\xi} \{\varepsilon - \xi\} = B_{\beta}^{E} \{\varepsilon\}.$ Then: $\forall \varepsilon \in E, \quad \lambda \{\varepsilon - \xi\} = \mu \{\varepsilon\}.$ we get: $\mu(E) = 1$. Since $\mu \in \mathcal{P}_E$, $M_{\lambda} = \sum_{\varepsilon \in E} \left[(\varepsilon - \xi) \cdot (\lambda \{\varepsilon - \xi\}) \right]$ $= \sum_{\varepsilon \in E} \left[(\varepsilon - \xi) \cdot (\mu \{\varepsilon\}) \right]$ Then: $\begin{aligned} &= \sum_{\varepsilon \in E} \left[\varepsilon \cdot (\mu\{\varepsilon\}) - \xi \cdot (\mu\{\varepsilon\}) \right] \\ &= \left(\sum_{\varepsilon \in E} \left[\varepsilon \cdot (\mu\{\varepsilon\}) \right] \right) - \left(\sum_{\varepsilon \in E} \left[\xi \cdot (\mu\{\varepsilon\}) \right] \right) \\ &= \left(\sum_{\varepsilon \in E} \left[\varepsilon \cdot (\mu\{\varepsilon\}) \right] \right) - \xi \cdot \left(\sum_{\varepsilon \in E} \left[\mu\{\varepsilon\} \right] \right) \\ &= M_{\mu} - \xi \cdot (\mu(E)) = M_{\mu} - \xi \cdot 1 = M_{\mu} - \xi. \end{aligned}$

THEOREM 16.4. Let $E \subseteq \mathbb{R}$ be nonempty and finite. Then: $as \ \beta \to \infty, \quad A^E_\beta \to \min E$ $as \ \beta \to -\infty, \quad A^E_\beta \to \max E.$ and

The proof is a matter of bookkeeping, best explained by example:

Let $E := \{2, 4, 8, 9\}$. Then $\min E = 2$ and $\max E = 9$. Since, $\forall \beta \in \mathbb{R}, \quad A_{\beta}^{E} = \frac{2e^{-2\beta} + 4e^{-4\beta} + 8e^{-8\beta} + 9e^{-9\beta}}{e^{-2\beta} + e^{-4\beta} + e^{-8\beta} + e^{-9\beta}},$ we get as $\beta \to \infty, \quad A_{\beta}^{E} \to 2/1$

and as $\beta \to -\infty$, $A_{\beta}^{E} \to 9/1$, and so as $\beta \to \infty$, $A_{\beta}^{E} \to \min E$ and as $\beta \to -\infty$, $A_{\beta}^{E} \to \max E$.

For all nonempty, finite $E \subseteq \mathbb{R}$, **define** $A^{E}_{\bullet} : \mathbb{R} \to \mathbb{R}$ by: $\forall \beta \in \mathbb{R}, \quad A^{E}_{\bullet}(\beta) = A^{E}_{\beta}.$

Recall (§2): " C^{ω} " means "real-analytic".

THEOREM 16.5. Let $E \subseteq \mathbb{R}$. Assume: $2 \leq \#E < \infty$. Then: A^{E}_{\bullet} is a strictly-decreasing C^{ω} -diffeomorphism from \mathbb{R} onto $(\min E; \max E)$.

Proof. Let $\kappa := \#E$. Choose $\varepsilon_1, \ldots, \varepsilon_{\kappa} \in \mathbb{R}$ s.t. $E = \{\varepsilon_1, \ldots, \varepsilon_{\kappa}\}.$ $2 \leq \kappa < \infty \quad \text{and} \quad \varepsilon_1, \dots, \varepsilon_{\kappa} \text{ are distinct.}$ $\forall \beta \in \mathbb{R}, A^E_{\bullet}(\beta) = \frac{\sum_{i=1}^{\kappa} [\varepsilon_i \cdot e^{-\beta \cdot \varepsilon_i}]}{\sum_{j=1}^{\kappa} [e^{-\beta \cdot \varepsilon_j}]}. \quad \text{Then } A^E_{\bullet} : \mathbb{R} \to \mathbb{R} \text{ is } C^{\omega}.$ Theorem 16.4 and the C^{ω} -Inverse Function Theorem and Then: Then: So, by the Mean Value Theorem, it suffices to show: $(A^E_{\bullet})' < 0$ on \mathbb{R} . $\begin{array}{l} \textbf{Given } \beta \in \mathbb{R}, \qquad \textbf{want: } (A^E_{\bullet})'(\beta) < 0. \\ P := \sum_{i=1}^{\kappa} [\varepsilon_i \cdot e^{-\beta \cdot \varepsilon_i}], \qquad P' := \sum_{i=1}^{\kappa} [(-\varepsilon_i^2) \cdot e^{-\beta \cdot \varepsilon_i}]. \\ Q := \sum_{j=1}^{\kappa} [e^{-\beta \cdot \varepsilon_j}], \qquad Q' := \sum_{j=1}^{\kappa} [(-\varepsilon_j) \cdot e^{-\beta \cdot \varepsilon_j}]. \end{array}$ Let Let Then Q > 0. Also, by the Quotient Rule, $(A^{E}_{\bullet})'(\beta) = [QP' - PQ']/Q^2$. Want: QP' - PQ' < 0. $QP' = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left[\left(-\varepsilon_{i}^{2} \right) \cdot e^{-\beta \cdot (\varepsilon_{i} + \varepsilon_{j})} \right].$ $PQ' = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left[\left(-\varepsilon_{i}^{2} \right) \cdot e^{-\beta \cdot (\varepsilon_{i} + \varepsilon_{j})} \right].$ $QP' - PQ' = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left[\left(-\varepsilon_{i}^{2} + \varepsilon_{i}\varepsilon_{j} \right) \cdot e^{-\beta \cdot (\varepsilon_{i} + \varepsilon_{j})} \right].$ We have: QP'We have: Then: Interchanging i and j, we get: $QP' - PQ' = \sum_{j=1}^{\kappa} \sum_{i=1}^{\kappa} \left[\left(-\varepsilon_j^2 + \varepsilon_j \varepsilon_i \right) \cdot e^{-\beta \cdot (\varepsilon_j + \varepsilon_i)} \right].$ By commutativity of addition and multiplication. adding the last two equations gives: $2 \cdot (QP' - PQ') = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left[\left(-\varepsilon_i^2 - \varepsilon_j^2 + 2\varepsilon_i \varepsilon_j \right) \cdot e^{-\beta \cdot (\varepsilon_i + \varepsilon_j)} \right].$ Then: $2 \cdot (QP' - PQ') = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left[-(\varepsilon_i - \varepsilon_j)^2 \cdot e^{-\beta \cdot (\varepsilon_i + \varepsilon_j)} \right].$ Then: $2 \cdot (QP' - PQ') < 0.$ Then: QP' - PQ' < 0.**DEFINITION 16.6.** Let $E \subseteq \mathbb{R}$. Assume: $2 \leq \#E < \infty$. Let $\alpha \in (\min E; \max E)$.

The α -Boltzmann-parameter on E is: $BP^E_{\alpha} := (A^E_{\bullet})^{-1}(\alpha).$

So the α -Boltzmann-parameter on E is the unique $\beta \in \mathbb{R}$ s.t. $A_{\beta}^{E} = \alpha$.

Example: Computations in §7 show:

$$\begin{aligned} \forall \beta \in \mathbb{R}, & \text{if } \beta = (\ln 9)/10, \text{ then } \frac{e^{-\beta} + 10e^{-10\beta}}{1 + e^{-\beta} + 10e^{-10\beta}} = 1. \\ \text{Then, } & \forall \beta \in \mathbb{R}, & \text{if } \beta = (\ln 9)/10, \text{ then } (A_{\bullet}^{\{0,1,10\}})(\beta) = 1. \\ & \text{Then: } (A_{\bullet}^{\{0,1,10\}})((\ln 9)/10) = 1. \\ & \text{Then: } (A_{\bullet}^{\{0,1,10\}})^{-1}(1) = (\ln 9)/10. \\ & \text{Then: } BP_1^{\{0,1,10\}} = (\ln 9)/10. \end{aligned}$$

Example: Let $E := \{2, 4, 8, 9\}, \quad \alpha := 5, \quad \beta := BP_{\alpha}^{E}$. To compute β , we need to solve $A_{\bullet}^{E}(\beta) = 5$ for β . Since A_{\bullet}^{E} is strictly-decreasing, there are iterative methods of solution, and we get: $\beta \approx 0.0918$, accurate to four decimal places.

(Thanks to C. Prouty for these calculations. See §28.)

17. Residue-unconstrained finite sets

In the next three theorems, we generalize our work in $\S13$

from $\{0, 1, 10\}$ to arbitrary finite residue-unconstrained sets. In the example at the end of this section,

we show that Theorem 17.3 below reproduces the result of \$13.

Recall (§9): \forall countable set Θ ,

 $\mathcal{FM}_{\Theta} \text{ is the set of finite measures on } \Theta$ and $\mathcal{FM}_{\Theta}^{\times} \text{ is the set of nonzero finite measures on } \Theta$ and $\mathcal{P}_{\Theta} \text{ is the set of probability measures on } \Theta.$ Recall (§9): \forall nonempty finite set F, $\forall f \in F$, $\nu_F\{f\} = 1/(\#F)$. Recall (Definition 9.2): \forall countable set Θ , $\forall \mu \in \mathcal{FM}_{\Theta}, \forall n \in \mathbb{N},$ $\forall x \in \Theta^n, \quad \mu^n\{x\} = (\mu\{x_1\}) \cdots (\mu\{x_n\}).$
THEOREM 17.1. Let $E \subseteq \mathbb{Z}$ be finite and residue-unconstrained. Let $\alpha \in (\min E; \max E)$. Let $\beta := BP_{\alpha}^{E}$. Let $t_{1}, t_{2}, \ldots \in \mathbb{Z}$. Assume: $\{t_{n} - n\alpha \mid n \in \mathbb{N}\}$ is bounded. For all $n \in \mathbb{N}$, let $\Omega_{n} := \{f \in E^{n} \mid f_{1} + \cdots + f_{n} = t_{n}\}$. Let $\varepsilon_{0} \in E$. Then: as $n \to \infty$, $\nu_{\Omega_{n}} \{f \in \Omega_{n} \mid f_{n} = \varepsilon_{0}\} \to B_{\beta}^{E} \{\varepsilon_{0}\}$.

Recall (§9): $\nu_{\emptyset}(\emptyset) = -1.$

So, since $B_{\beta}^{E}{\{\varepsilon_{0}\}} > 0$, part of the content of this theorem is: \forall sufficiently large $n \in \mathbb{N}$, $\Omega_{n} \neq \emptyset$;

see Claim 1 in the proof below.

Proof. Since E is residue-unconstrained, we get: $E \neq \emptyset$. $E \subseteq \mathbb{Z}$ and E is finite. By hypothesis, Then: $E \subseteq \mathbb{R}$ and E is nonempty and finite. Let $\mu := B_{\beta}^{E}$. Then: $\mu \in \mathcal{P}_{E}$ and $S_{\mu} = E$. So, since $\mu \in \mathcal{P}_E \subseteq \mathcal{FM}_E$, we get: $|\mu|_1 < \infty$ and $|\mu|_2 < \infty$. Since $\beta = BP_{\alpha}^{E} = (A_{\bullet}^{E})^{-1}(\alpha)$, we get: $(A_{\bullet}^{E})(\beta) = \alpha$. So, since $(A_{\bullet}^{E})(\beta) = A_{\beta}^{E} = M_{B_{\beta}^{E}} = M_{\mu}$, we get: $M_{\mu} = \alpha$. For all $n \in \mathbb{N}$, define $\psi_n : \mathbb{Z} \to \mathbb{R}$ by: $\forall t \in \mathbb{Z}, \quad \psi_n(t) = \mu^n \{ f \in E^n \mid f_1 + \dots + f_n = t \}.$ $\forall n \in \mathbb{N}, \ \psi_n(t_n) = \mu^n(\Omega_n).$ Then: Let $v := V_{\mu}$. By Theorem 10.6, we get: $0 < v < \infty$. Then: $0 < \tau < \infty$. Let $\tau := 1/\sqrt{2\pi v}$. By Theorem 10.6, we get: as $n \to \infty$, $\sqrt{n} \cdot (\mu^n \{ f \in E^n \mid f_1 + \dots + f_n = t_n \}) \to 1/\sqrt{2\pi v}$. Then: as $n \to \infty$, $\sqrt{n} \cdot (\psi_n(t_n)) \to \tau$. So, since $\tau > 0$, choose $n_0 \in \mathbb{N}$ s.t.: $\forall n \in [n_0 \dots \infty), \sqrt{n} \cdot (\psi_n(t_n)) > 0$.

Claim 1: Let $n \in [n_0..\infty)$. Then: $\mu^n(\Omega_n) > 0$. Proof of Claim 1: Recall: $\psi_n(t_n) = \mu^n(\Omega_n)$. Want: $\psi_n(t_n) > 0$. By the choice of n_0 , we get: $\sqrt{n} \cdot (\psi_n(t_n)) > 0$. Then: $\psi_n(t_n) > 0$. End of proof of Claim 1.

 $\mu \in \mathcal{P}_E.$ Recall: $\mu^n(\Omega_n) \leqslant 1.$ $\forall n \in \mathbb{N}, \ \mu^n \in \mathcal{P}_{E^n},$ \mathbf{SO} Then: $0 < \mu^n(\Omega_n) \leqslant 1.$ So, by Claim 1, $\forall n \in [n_0..\infty)$, $(\mu^n | \Omega_n)(\Omega_n) = \mu^n(\Omega_n).$ Also, we have: $\forall n \in \mathbb{N},$ $\forall n \in [n_0..\infty), \quad 0 < (\mu^n | \Omega_n)(\Omega_n) \leq 1.$ Then: $\mu^n \mid \Omega_n \in \mathcal{FM}_{\Omega}^{\times}$. $\forall n \in [n_0 \dots \infty),$ Then:

Then: $\forall n \in [n_0..\infty), \qquad \mathcal{N}(\mu^n \mid \Omega_n) \in \mathcal{P}_{\Omega_n}.$

Claim 2: Let $n \in [n_0..\infty)$. Then: $\mathcal{N}(\mu^n \mid \Omega_n) = \nu_{\Omega_n}$. Proof of Claim 2: Let $\theta := \mathcal{N}(\mu^n | \Omega_n), \quad F := \Omega_n.$ Then $\theta \in \mathcal{P}_F.$ Want: $\theta = \nu_F$. By Theorem 9.9, given $f, g \in F$, want: $\theta\{f\} = \theta\{g\}$. By Claim 1, we have: $\mu^n(\Omega_n) > 0.$ $\theta = \frac{\mu^n |\Omega_n|}{\mu^n (\Omega_n)}.$ Since $(\mu^n | \Omega_n)(\Omega_n) = \mu^n(\Omega_n)$ and $\theta = \mathcal{N}(\mu^n | \Omega)$, we get: $\frac{(\mu^n | \Omega_n)\{f\}}{\mu^n(\Omega_n)} = \frac{(\mu^n | \Omega_n)\{g\}}{\mu^n(\Omega_n)}.$ Want: $(\mu^n | \hat{\Omega}_n) \{ f \} = (\mu^n | \hat{\Omega}_n) \{ g \}.$ Want: Since $f, g \in F = \Omega_n$, we get: $(\mu^n | \Omega_n) \{f\} = \mu^n \{f\}$ and $(\mu^n | \Omega_n) \{g\} = \mu^n \{g\}.$ $\mu^n\{f\} = \mu^n\{g\}.$ Want: Since $E \subseteq \mathbb{R}$ is nonempty and finite, we get: $\hat{B}^{E}_{\beta}(E) > 0.$ Let $C := 1/(\hat{B}^E_\beta(E))$. Then $\mathcal{N}(\hat{B}^E_\beta) = C \cdot \hat{B}^E_\beta$ $\forall \varepsilon \in E, \quad \widehat{B}^E_\beta \{\varepsilon\} = e^{-\beta \cdot \varepsilon}.$ By definition of \hat{B}^{E}_{β} , we have: So, since $\mu = B_{\beta}^{E} = \mathcal{N}(\hat{B}_{\beta}^{E}) = C \cdot \hat{B}_{\beta}^{E}$, $\forall \varepsilon \in E, \qquad \mu\{\varepsilon\} = Ce^{-\beta \cdot \varepsilon}.$ we get: Since $f \in F = \Omega_n$, by definition of Ω_n , we get: $f_1 + \cdots + f_n = t_n$. Since $g \in F = \Omega_n$, by definition of Ω_n , we get: $g_1 + \cdots + g_n = t_n$. $f_1 + \dots + f_n = t_n = g_1 + \dots + g_n,$ Since $C^n e^{-\beta \cdot (f_1 + \dots + f_n)} = C^n e^{-\beta \cdot (g_1 + \dots + g_n)}$ we get: Then: $(Ce^{-\beta \cdot f_1}) \cdots (Ce^{-\beta \cdot f_n}) = (Ce^{-\beta \cdot g_1}) \cdots (Ce^{-\beta \cdot g_n}).$ Then: $(\mu\{f_1\}) \cdots (\mu\{f_n\}) = (\mu\{g_1\}) \cdots (\mu\{g_n\}).$ = $\mu^n \{q\}.$ Then: $\mu^n{f}$ End of proof of Claim 2.

By hypothesis, E is residue-unconstrained and $\varepsilon_0 \in E$ and $t_1, t_2, \ldots \in \mathbb{Z}$ and $\{t_n - n\alpha \mid n \in \mathbb{N}\}$ is bounded. Recall: $\mu \in \mathcal{P}_E$ and $S_\mu = E$ and $|\mu|_2 < \infty$ and $M_\mu = \alpha$. Let $P := \mu\{\varepsilon_0\}$. Then, since $\mu = B_\beta^E$, we get: $P = B_\beta^E\{\varepsilon_0\}$. We want: as $n \to \infty$, $\nu_{\Omega_n} \{f \in \Omega_n \mid f_n = \varepsilon_0\} \to P$. By Theorem 12.2, as $n \to \infty$, $(\mathcal{N}(\mu^n \mid \Omega_n))\{f \in \Omega_n \mid f_n = \varepsilon_0\} \to P$. So, by Claim 2, as $n \to \infty$, $\nu_{\Omega_n} \{f \in \Omega_n \mid f_n = \varepsilon_0\} \to P$. Recall (§2): $\forall t \in \mathbb{R}$, |t| is the floor of t.

We record the $t_n = |n\alpha|$ version of the preceding theorem:

THEOREM 17.2. Let $E \subseteq \mathbb{Z}$ be finite and residue-unconstrained. Let $\alpha \in (\min E; \max E)$. Let $\beta := BP_{\alpha}^{E}$. For all $n \in \mathbb{N}$, let $\Omega_{n} := \{f \in E^{n} \mid f_{1} + \cdots + f_{n} = \lfloor n\alpha \rfloor\}$. Let $\varepsilon_{0} \in E$. Then: as $n \to \infty$, $\nu_{\Omega_{n}} \{f \in \Omega_{n} \mid f_{n} = \varepsilon_{0}\} \to B_{\beta}^{E} \{\varepsilon_{0}\}$.

We record the $\alpha \in \mathbb{Z}$ special case of the preceding theorem:

THEOREM 17.3. Let $E \subseteq \mathbb{Z}$ be finite and residue-unconstrained. Let $\alpha \in (\min E; \max E)$. Let $\beta := BP_{\alpha}^{E}$. Assume $\alpha \in \mathbb{Z}$. For all $n \in \mathbb{N}$, let $\Omega_{n} := \{f \in E^{n} | f_{1} + \dots + f_{n} = n\alpha\}$. Let $\varepsilon_{0} \in E$. Then: as $n \to \infty$, $\nu_{\Omega_{n}} \{f \in \Omega_{n} | f_{n} = \varepsilon_{0}\} \to B_{\beta}^{E} \{\varepsilon_{0}\}$.

Example: Suppose $E = \{0, 1, 10\}$ and $\alpha = 1$. Then $\Omega_N = \{f \in E^N \mid f_1 + \dots + f_N = N\},\$ so Ω_N represents the set of all GFA dispensations,

as described in $\S3$.

The measure ν_{Ω_N} gives equal probability to each dispensation,

so ν_{Ω_N} represents the GFA's first system for awarding grants, also described in §3.

Since $\beta = BP_{\alpha}^{E} = BP_{1}^{\{0,1,10\}}$, we calculate: $\beta = (\ln 9)/10$. More calculation gives: $(B_{\beta}^{E}\{0\}, B_{\beta}^{E}\{1\}, B_{\beta}^{E}\{10\}) = \frac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}$. Since N is large, by Theorem 17.3, we get: $\nu_{\Omega_{N}}\{f \in \Omega_{N} \mid f_{N} = \varepsilon_{0}\} \approx B_{\beta}^{E}\{\varepsilon_{0}\}$. So, if I am the Nth professor, then, under the first system,

my probability of receiving ε_0 dollars

is approximately equal to $B^E_\beta\{\varepsilon_0\}.$

Thus Theorem 17.3 reproduces the result of $\S13$.

18. RATIONAL AWARD SETS

In this section, we investigate what happens if

the set of awards is an arbitrary set of rational numbers.

Recall that, on our Earth, which is Earth-1218,

grants are 0, 1, 10, with average grant 1.

Example: In a parallel universe, on Earth-googol-plex, there are N_0 professors applying for grants from its GFA.

By GFA rules, grant amounts are \$10, \$14.45, \$54,

and Congress allocates \$13.37 per professor.

Earth-googol-plex has its own GFA.

That GFA is using the "first system" for awarding grants,

in which every dispensation is equally likely.

Question: For any professor,

what is the approximate probability of receiving \$10? \$14.45? \$54? To simplify this problem, we can imagine that

that GFA makes two rounds of awards.

In the first round, it simply dispenses \$10 to each professor.

In the second round, using the first system, it dispenses

additional grants of \$0, \$4.45, \$44, with average grant \$3.37. We seek the approximate probability of the additional grant being

each of the numbers 0, 4.45, 44.

To simplify this problem still more, we can

change monetary units so that the grant amounts are all integers: Additional grants, in pennies, are 0, 445, 4400, with average grant 337,

and we seek the approximate probability of receiving 0, 445, 4400. Unfortunately, $\{0, 445, 4400\} \subseteq 5\mathbb{Z} + 0$,

so $\{0, 445, 4400\}$ is not residue-unconstrained,

making it difficult to apply the Discrete Local Limit Theorem. Since $gcd\{0, 445, 4400\} = 5$, we can change monetary units again: Additional grants, in nickels, are 0, 89, 880, with average grant 337/5,

and we seek the approximate probability of receiving 0, 89, 880.

Let $E := \{0, 89, 880\}$ and let $\alpha := 337/5$.

Since $0 \in E$ and gcd(E) = 1, we get: E is residue-unconstrained. The amount of money (in nickels) allocated by Congress is $N_0\alpha$,

to be dispensed among the N_0 professors.

Unfortunately, a census reveals that: N_0 is not divisible by 5. Recall: $\alpha = 337/5$. Then $N_0 \alpha \notin \mathbb{Z}$, while $0, 89, 880 \in \mathbb{Z}$. It is therefore improvible to dispense the grant mapping

It is therefore *impossible* to dispense the grant money.

The bureaucracy seizes up, there is pandemonium in the streets, and the military steps in to impose order.

The superheroes of Earth-googol-plex are committed to democracy, and so they reverse time and select a different time-line.

On this new time-line, E and α are unchanged, but

there is a new number, N_1 , of professors,

and N_1 is blissfully divisible by 5. Then: $N_1 \alpha \in \mathbb{Z}$. Let $\varepsilon_0 \in E$ be given.

We want: the approximate probability of receiving ε_0 nickels. Recall (§2): $\forall t \in \mathbb{R}, |t|$ is the floor of t. For all $n \in \mathbb{N}$, let $\Omega_n := \{f \in E^n \mid f_1 + \dots + f_n = \lfloor n\alpha \rfloor\}$. Since $N_1 \alpha \in \mathbb{Z}$, we get: $\Omega_{N_1} = \{f \in E^{N_1} \mid f_1 + \dots + f_{N_1} = N_1 \alpha\}$. We want: an approximation to $\nu_{\Omega_{N_1}} \{f \in \Omega_{N_1} \mid f_{N_1} = \varepsilon_0\}$. Recall: E is residue-unconstrained. Let $\beta := BP_{\alpha}^E$. By Theorem 17.2, we have: as $n \to \infty$, $\nu_{\Omega_n} \{f \in \Omega_n \mid f_n = \varepsilon_0\} \to B_{\beta}^E \{\varepsilon_0\}$. So, assuming N_1 is large, we get $\nu_{\Omega_{N_1}} \{f \in \Omega_{N_1} \mid f_{N_1} = \varepsilon_0\} \approx B_{\beta}^E \{\varepsilon_0\}$. For each $\varepsilon_0 \in \{0, 89, 880\}$, we want to compute $B_{\beta}^E \{\varepsilon_0\}$. We therefore want to compute $(B_{\beta}^E \{0\}, B_{\beta}^E \{89\}, B_{\beta}^E \{880\})$. Since $\beta = BP_{\alpha}^E = BP_{337/5}^{\{0,89,880\}}$, we see that: to evaluate β , we must solve $A_{\bullet}^{\{0,89,880\}}(\beta) = 337/5$ for β . Since, by Theorem 16.5, $A_{\bullet}^{\{0,89,880\}}$ is strictly-decreasing,

there are simple iterative methods to do this.

We calculate $\beta=0.003144,$ accurate to six decimal places. We also calculate

 $(B_{\beta}^{E}\{0\}, B_{\beta}^{E}\{89\}, B_{\beta}^{E}\{880\}) = (0.5498, 0.4156, 0.0345),$

all accurate to four decimal places.

(Thanks to C. Prouty for this calculation. See §28.)

Recall $(\S3)$: N is a large positive integer.

More generally: Imagine a parallel universe with N professors.

Let E_0 denote the set of grant-awards.

Assume $E_0 \subseteq \mathbb{Q}$ and $2 \leq \#E_0 < \infty$.

Let $\alpha_0 \in (\min E_0; \max E_0)$ denote the average award.

Since $\#E_0 \ge 2$, we get: $E_0 \ne \emptyset$. Choose $\varepsilon_0 \in E_0$. Then $\varepsilon_0 \in \mathbb{Q}$. Let $E_1 := E_0 - \varepsilon_0$, $\alpha_1 := \alpha_0 - \varepsilon_0$. Then $\alpha_1 \in (\min E_1; \max E_1)$. Also, $0 \in E_1$.

So, by giving out awards in two rounds (first ε_0 , then the remainder), we are reduced to a case where 0 is a possible grant-award.

Since $E_1 = E_0 - \varepsilon_0 \subseteq \mathbb{Q}$, **choose** $m \in \mathbb{N}$ s.t. $mE_1 \subseteq \mathbb{Z}$. Let $E_2 := mE_1$, $\alpha_2 := m\alpha_1$. Also, $0 \in E_2 \subseteq \mathbb{Z}$. Then $\alpha_2 \in (\min E_2; \max E_2)$.

So, by change of monetary unit,

we are reduced to a case where every grant-award is an integer and where 0 is a possible grant-award.

Let $g := \operatorname{gcd}(E_2), E := E_2/g, \alpha := \alpha_2/g$. Then $\alpha \in (\min E; \max E)$.

Also, $0 \in E \subseteq \mathbb{Z}$ and gcd(E) = 1, so E is residue-unconstrained. So, by change of monetary unit, we are reduced to a case where

the set of grant-awards is a residue-unconstrained set of integers. If $N\alpha \notin \mathbb{Z}$, then, since every grant-award is an integer,

no dispensation is possible, leading to

your typical military dictatorship and superhero intervention. If $N\alpha \in \mathbb{Z}$, then, using Theorem 17.2,

we can compute the approximate probability of each award.

19. IRRATIONAL AWARDS

In this section, we briefly discuss what can happen if

NOT every grant award is a rational number.

Here, we only present an example to show that

the award probabilities may NOT follow a Boltzmann distribution.

Example: On Earth-aleph-1, the GFA gives

grants of 0, $\sqrt{2}$, $\sqrt{3}$, $10 - \sqrt{2} - \sqrt{3}$ dollars,

with an average grant of 1 dollar,

giving equal probability to every possible dispensation.

Let K be the number of professors. Assume: K is divisible by 10. Let M := K/10. Then $M \in \mathbb{N}$ and there are 10M professors.

Moreover, since the average grant is 1 dollar, we conclude:

there are 10M dollars to dispense among the 10M professors.

Claim: On Earth-aleph-1, every dispensation of awards has

7M	grants of	0	dollars,				
M	grants of	$\sqrt{2}$	dollars,				
M	grants of	$\sqrt{3}$	dollars,				
M	grants of	$10 - \sqrt{2} - \sqrt{3}$	dollars.				
Proof of Claim:	Given a dis	pensation,					
let w be the	ne number of	0	dollar grants	and			
let x be the	ne number of	$\sqrt{2}$	dollar grants	and			
let y be the	ne number of	$\sqrt{3}$	dollar grants	and			
let z be the	ne number of	$10 - \sqrt{2} - \sqrt{3}$	dollar grants,				
	want: $w =$	= 7M and $x =$	y = z = M.				
Because the total money dispensed is $10M$ dollars, we get:							
$w \cdot 0 + x \cdot \sqrt{2} + y \cdot \sqrt{3} + z \cdot (10 - \sqrt{2} - \sqrt{3}) = 10M.$							
Then: $(10z - 10M) \cdot 1 + (x - z) \cdot \sqrt{2} + (y - z) \cdot \sqrt{3} = 0.$							

So, since $1, \sqrt{2}, \sqrt{3}$ are linearly indpendent over \mathbb{Q} , we get:

10z - 10M = 0 and x - z = 0 and y - z = 0. Then z = M and x = z and y = z. Then x = y = z = M. It remains only to show: w = 7M.

Because there are 10*M* professors, we get: w + x + y + z = 10M. Then: w + M + M + M = 10M. Then: w = 7M. End of proof of Claim.

By the Claim, in each dispensation, there are exactly *M* grants of $10 - \sqrt{2} - \sqrt{3}$ dollars. Of the four grant amounts, the largest is $10 - \sqrt{2} - \sqrt{3}$. So, if I am one of the 10M professors, then I would hope to be among the lucky M receiving $10 - \sqrt{2} - \sqrt{3}$ dollars. M/(10M), i.e.,My probability of being so lucky is: 10%. That is, we obtain a probabily of: $10 - \sqrt{2} - \sqrt{3}$ 10% for dollars. Extending this reasoning, we obtain probabilies of: 70%for 0 dollars, $\sqrt{2}$ 10%for dollars. 1007 c J _ 11

10%	IOT	$\sqrt{3}$	doll	$\operatorname{ars},$		
10%	for	$10 - \sqrt{2} - \sqrt{3}$	doll	ars.		
In a Boltzmann dis	stribution,	depending on whe	ether	$\beta = 0$	or	$\beta \neq 0,$

either the probabilities are all equal

or the probabilities are all distinct from one another. The numbers 70,10,10,10 are neither all equal nor all distinct. Thus, the 70-10-10 distribution above is NOT Boltzmann.

20. Earth-minimum-Mahlo-cardinal and the BUA

Next, we wish to handle thermodynamic systems in which many states may have a single energy-level.
One says that such an energy-level is "degenerate".
In this section, we develop a whimsical example.
In §21 and §22, we will develop a general theory.

Recall that $N \in \mathbb{N}$ is large.

In a parallel universe, on Earth-minimum-Mahlo-cardinal,

the BUA (Best University Anywhere) employs N professors. Each professor has a number, from 1 to N. Each professor wanders the campus,

carrying two bags: one red, one blue. Each bag is closed from view, but has money in it or is empty. The "state" of a professor is the pair $\sigma = (\sigma_1, \sigma_2)$ such that σ_1 is the number of dollars in the professor's red bag, σ_2 is the number of dollars in the professor's blue bag; the professor's "wealth" is $\sigma_1 + \sigma_2$ dollars. So, if I am one of the professors, and if my state is (3, 2), then I have: \$3 in my red bag and \$2 in my blue bag, and my wealth is \$5. By BUA rules, the amount of money in any bag is always \$0 or \$1 or \$2 or \$3 or \$4, each professor's wealth is always \leq \$7. and Therefore, the set of allowable states is $([0..4] \times [0..4]) \setminus \{(4,4)\}.$ Let $\Sigma := ([0..4] \times [0..4]) \setminus \{(4,4)\}.$ Since $\#([0..4] \times [0..4]) = 5 \cdot 5 = 25$, we get: $\#\Sigma = 24$. **Define** $\varepsilon : \Sigma \to [0..7]$ by: $\forall \sigma \in \Sigma, \ \varepsilon(\sigma) = \sigma_1 + \sigma_2.$ For convenience of notation, $\forall \sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. If I am one of the professors, and if my state is $\sigma = (\sigma_1, \sigma_2) \in \Sigma$, then I have: σ_1 in my red bag and σ_2 in my blue bag, and my wealth is ε_{σ} . Since $\varepsilon_{(3,2)} = 5 = \varepsilon_{(1,4)}$, we see that ε is not one-to-one, and we have a so-called "degeneracy" at 5. This function ε has many other degeneracies, as well. Recall: The professors are numbered, from 1 to N. At random moments, random pairs of wandering professors cross paths, and interact. Each interaction involves three steps: and then a game and then a verbal offer a rejection or a money transfer. The first step, the game, is played as follows: one of the two professors flips a fair coin and if heads, then the lower-numbered professor wins and if tails, then the higher-numbered professor wins.

Next, without touching any money, the losing professor verbally offers \$1 to the winning profes	sor			
The loging professor then fling a fair coin				
if heads, then the loser's red has is enough	and			
if tails, then the loser's blue bag is opened	anu			
If the locer's open bag is empty then				
then the winner collectly rejects the \$1 offer				
then the winner gallantly rejects the \$1 offer				
the professors continue their wonderings	and			
On the other hand, if the loger's open has is NOT events, then				
both of the winner's base are opened	-,			
Dotti of the willier's bags are opened.	$\Phi 7$			
Recall that, by DOA rules, every professor's wearth must be \leq	Φ1.			
If the winner s weath is 57 ,	and			
then the winner rejects the 51 oner	and			
the opened bags are closed, the interaction is over,	and			
the professors continue their wanderings.				
On the other hand, if the winner's wealth is \leq \$6,	,			
then the winner flips a fair coin,				
if heads, then the winner's red bag is closed	and			
if tails, then the winner's blue bag is closed.				
At this point, the winner has one open bag, as does the lose	r.			
Moreover, the loser's open bag is NOT empty.				
Recall that no bag may have more than \$4.				
If the winner's open bag has \$4,				
then the winner rejects the \$1 offer				
the opened bags are closed, the interaction is over,	and			
the professors continue their wanderings.				
On the other hand, if the winner's open bag has \leq \$3,				
then \$1 is transferred				
from the losing professor's open bag				
to the winning professor's open bag;				
then the opened bags are closed, the interaction is over,	and			
the professors continue their wanderings.				
Because of these interactions,				
the wealth of an individual professor may change over time	,			
but the sum of the wealths of all of them is constant;				

there is "conservation of (total) wealth".

An audit reveals that, at the BUA, that total wealth is always N.

Recall: $\Sigma = ([0..4] \times [0..4]) \setminus \{(4,4)\}$ is the set of states. A "state-dispensation" is a function $[1..N] \rightarrow \Sigma$,

representing the states of all N professors.

So, if, at some point in time, the state-dispensation is $\omega : [1..N] \to \Sigma$, then, for every $\ell \in [1..N]$, the state of Professor $\#\ell$ is $\omega(\ell)$, and the wealth of Professor $\#\ell$ is $\varepsilon_{\omega(\ell)}$; therefore, the total wealth of all the professors is $\sum_{\ell=1}^{N} \varepsilon_{\omega(\ell)}$. As we mentioned, at the BUA, that total wealth is N. Let $\Omega^* := \left\{ \omega : [1..N] \to \Sigma \mid \sum_{\ell=1}^{N} \varepsilon_{\omega(\ell)} = N \right\}$. Then Ω^* represents the set of all state-dispensations at the BUA.

The random interactions, described above,

induce a discrete Markov-chain on Ω^* . This, in turn, induces a map $\Pi : \mathcal{P}_{\Omega^*} \to \mathcal{P}_{\Omega^*}$.

Let $T := \#\Omega^*$. Fix an ordering of Ω^* , *i.e.*, a bijection $[1..T] \leftrightarrow \Omega^*$. The Markov-chain then has a $T \times T$ transition-matrix Φ ,

with entries in [0, 1], whose column-sums are all = 1. For every $\phi, \psi \in \Omega^*$, the probability of transitioning from ϕ to ψ is equal to

the probability of transitioning from ψ to ϕ .

That is, the transition-matrix Φ is symmetric.

So, since the column-sums of Φ are all 1,

we get: the row-sums of Φ are all 1.

Let v be a $T \times 1$ column vector whose entries are all 1. Then $\Phi v = v$. Let w := v/T. Then: all the entries of w are 1/T and $\Phi w = w$. Recall that the probability-distribution $\nu_{\Omega^*} \in \mathcal{P}_{\Omega^*}$

assigns equal probability to each state-dispensation in Ω^* . That is, $\forall \omega \in \Omega^*$, $\nu_{\Omega^*} \{\omega\} = 1/T$.

Since the entries of w are equal to these ν_{Ω^*} -probabilities,

and since $\Phi w = w$, we get: $\Pi(\nu_{\Omega^*}) = \nu_{\Omega^*}$.

We will say that two state-dispensations $\phi, \psi \in \Omega^*$ are "adjacent",

 $\label{eq:formula} \mbox{if there is an interaction that carries ϕ to ψ}.$ For any $\phi,\psi\in\Omega^*,$

 $\exists a \text{ finite sequence of interactions that carries } \phi \text{ to } \psi.$

 $\forall \phi, \psi \in \Omega^*, \exists m \in \mathbb{N}, \exists \omega_0, \dots, \omega_m \in \Omega^*$ That is: $\phi = \omega_0$ and $\omega_m = \psi$ s.t. $\forall i \in [1..m], \quad \omega_{i-1} \text{ is adjacent to } \omega_i.$ and s.t. any two state-dispensations That is, are connected by an adjacency-path. the Markov-chain on Ω^* is irreducible. That is, Recall that some interactions result in a rejection; such interactions do not change the state-dispensation. So, a state-dispensation is sometimes adjacent to itself. That is, there are adjacency-cycles of length 1. the Markov-chain is aperiodic. It follows that So, since the Markov-chain is irreducible and since $\Pi(\nu_{\Omega^*}) = \nu_{\Omega^*}$, by the Perron-Frobenius Theorem, we get: $\forall \mu \in \mathcal{P}_{\Omega^*},$ $\mu, \Pi(\mu), \Pi(\Pi(\mu)), \Pi(\Pi(\Pi(\mu))), \ldots \rightarrow \nu_{\Omega^*}.$ That is, for any starting probability-distribution on Ω^* , after enough random interactions, the resulting probability-distribution on Ω^* will be approximately equal to $\nu_{\Omega^*},$ to any desired level of accuracy.

Problem: Suppose I am Professor #N at the BUA. Suppose that the probability-distribution μ of state-d

Suppose that the probability-distribution μ of state-dispensations is approximately equal to ν_{Ω^*} . For each $\sigma \in \Sigma$, compute my probability of being in state σ . That is, $\forall \sigma \in \Sigma$, compute $\mu \{ \omega \in \Omega^* | \omega(N) = \sigma \}$. Since $\#\Sigma = 24$, there will be 24 answers. Approximate answers are acceptable.

To make a precise mathematical problem,

we, in fact, assume that μ is *exactly* equal to ν_{Ω^*} , and we seek the exact "thermodynamic limit", meaning we replace N with a variable $n \in \mathbb{N}$, and let $n \to \infty$.

In the next two sections, we will develop a theory to solve problems like this one. We need only adapt our earlier methods to allow for degeneracies. Our main theorems are

Theorem 22.1 and Theorem 22.2 and Theorem 22.3, and the solution to the above "precise mathematical problem" appears in the example at the end of §22.

21. Boltzmann distributions on finite sets with Degeneracy

In this section, we adapt our earlier work (§16) on Boltzmann distributions to allow for degeneracies.

Recall (§9): \forall countable set Θ , $\mathcal{FM}_{\Theta}^{\times}$ is the set of nonzero finite measures on Θ and \mathcal{P}_{Θ} is the set of probability measures on Θ . Recall (§9): \forall nonempty countable set Θ , $\forall \mu \in \mathcal{FM}_{\Theta}^{\times}$, $\mathcal{N}(\mu)$ is the normalization of μ .

DEFINITION 21.1. Let Σ be a nonempty finite set.

Let $\varepsilon : \Sigma \to \mathbb{R}$. Let $\beta \in \mathbb{R}$. Then $\widehat{B}_{\beta}^{\varepsilon} \in \mathcal{FM}_{\Sigma}^{\times}$ is defined by: $\forall \sigma \in \Sigma, \ \widehat{B}_{\beta}^{\varepsilon} \{\sigma\} = e^{-\beta \cdot (\varepsilon(\sigma))}$. Also, we define: $\overline{B}_{\beta}^{\varepsilon} := \mathcal{N}(\widehat{B}_{\beta}^{\varepsilon}) \in \mathcal{P}_{\Sigma}$.

Then: $\forall \text{nonempty finite set } \Sigma, \forall \varepsilon : \Sigma \to \mathbb{R}, \forall \beta \in \mathbb{R},$ $\widehat{B}^{\varepsilon}_{\beta}(\Sigma) > 0 \text{ and } \forall \sigma \in \Sigma, B^{\varepsilon}_{\beta}\{\sigma\} = (\widehat{B}^{\varepsilon}_{\beta}\{\sigma\}) / (\widehat{B}^{\varepsilon}_{\beta}(\Sigma))$ and $S_{\widehat{B}^{\varepsilon}_{\beta}} = \Sigma = S_{B^{\varepsilon}_{\beta}}.$

 $\begin{array}{ll} Example: \ \mathbf{Let} \ \Sigma := \{0,1,10\} \ \mathrm{and} \ \mathbf{let} \ \beta \in \mathbb{R}.\\ \mathbf{Define} \ \varepsilon : \Sigma \to \mathbb{R} \ \mathrm{by:} & \forall \sigma \in \Sigma, \ \varepsilon(\sigma) = \sigma.\\ & \ \mathrm{Then:} \ \ \widehat{B}^{\varepsilon}_{\beta}\{0\} = 1, \ \ \widehat{B}^{\varepsilon}_{\beta}\{1\} = e^{-\beta}, \ \ \widehat{B}^{\varepsilon}_{\beta}\{10\} = e^{-10\beta}.\\ & \ \mathbf{Let} \ C := 1/(1 + e^{-\beta} + e^{-10\beta}).\\ & \ \mathrm{Then:} \ \ B^{\varepsilon}_{\beta}\{0\} = C, \ \ B^{\varepsilon}_{\beta}\{1\} = Ce^{-\beta}, \ B^{\varepsilon}_{\beta}\{10\} = Ce^{-10\beta}. \end{array}$

 $\begin{array}{ll} Example: \mbox{ Let } \Sigma := \{2,4,8,9\} \mbox{ and let } \beta \in \mathbb{R}.\\ \mbox{ Define } \varepsilon : \Sigma \to \mathbb{R} \mbox{ by: } & \forall \sigma \in \Sigma, \ \varepsilon(\sigma) = \sigma.\\ \mbox{ Then: } & \widehat{B}^{\varepsilon}_{\beta}\{2\} = e^{-2\beta}, \ & \widehat{B}^{\varepsilon}_{\beta}\{4\} = e^{-4\beta},\\ & & \widehat{B}^{\varepsilon}_{\beta}\{8\} = e^{-8\beta}, \ & \widehat{B}^{\varepsilon}_{\beta}\{9\} = e^{-9\beta}.\\ \mbox{ Let } C := 1/(e^{-2\beta} + e^{-4\beta} + e^{-8\beta} + e^{-9\beta}).\\ \mbox{ Then: } & & B^{\varepsilon}_{\beta}\{2\} = Ce^{-2\beta}, \ & B^{\varepsilon}_{\beta}\{4\} = Ce^{-4\beta}, \end{array}$

$$B^{\varepsilon}_{\beta}\{8\} = Ce^{-8\beta}, \quad B^{\varepsilon}_{\beta}\{9\} = Ce^{-9\beta}.$$

In the preceding three examples, ε is one-to-one. That is, ε has no degeneracies.

In the next, ε has one degeneracy, at energy-level 9.

 $\begin{array}{ll} Example: \mbox{ Let } \Sigma := \{1,2,3,4\} \mbox{ and define } \varepsilon : \Sigma \to \mathbb{R} \mbox{ by:} \\ \varepsilon(1) = 2, \quad \varepsilon(2) = 4, \quad \varepsilon(3) = 9, \quad \varepsilon(4) = 9. \\ \mbox{ Then:} \quad \hat{B}^{\varepsilon}_{\beta}\{1\} = e^{-2\beta}, \quad \hat{B}^{\varepsilon}_{\beta}\{2\} = e^{-4\beta}, \\ \quad \hat{B}^{\varepsilon}_{\beta}\{3\} = e^{-9\beta}, \quad \hat{B}^{\varepsilon}_{\beta}\{4\} = e^{-9\beta}. \\ \mbox{ Let } C := 1/(e^{-2\beta} + e^{-4\beta} + 2e^{-9\beta}). \\ \mbox{ Then:} \quad B^{\varepsilon}_{\beta}\{1\} = Ce^{-2\beta}, \quad B^{\varepsilon}_{\beta}\{2\} = Ce^{-4\beta}, \\ \quad B^{\varepsilon}_{\beta}\{3\} = Ce^{-9\beta}, \quad B^{\varepsilon}_{\beta}\{4\} = Ce^{-9\beta}. \end{array}$

In the next example, ε has many degeneracies.

 $\begin{array}{ll} Example: \mbox{ Let } \Sigma := \left(\begin{bmatrix} 0..4 \end{bmatrix} \times \begin{bmatrix} 0..4 \end{bmatrix} \right) \setminus \{ (4,4) \}. \\ \mbox{ Let } \beta \in \mathbb{R} \mbox{ and define } \varepsilon : \Sigma \to \mathbb{R} \mbox{ by: } \quad \forall \sigma \in \Sigma, \quad \varepsilon(\sigma) = \sigma_1 + \sigma_2. \\ \mbox{ Then: } & \widehat{B}^{\varepsilon}_{\beta}\{(3,2)\} = e^{-5\beta}, \quad \widehat{B}^{\varepsilon}_{\beta}\{(1,4)\} = e^{-5\beta}, \quad \widehat{B}^{\varepsilon}_{\beta}\{(0,0)\} = 1. \\ \mbox{ Generally, } & \forall \sigma \in \Sigma, \quad \widehat{B}^{\varepsilon}_{\beta}\{\sigma\} = e^{-(\sigma_1 + \sigma_2) \cdot \beta}. \\ \mbox{ Let } C := 1/(\sum_{\sigma \in \Sigma} \left[e^{-(\sigma_1 + \sigma_2) \cdot \beta} \right]). \\ \mbox{ Then: } & B^{\varepsilon}_{\beta}\{(3,2)\} = C e^{-5\beta}, \quad B^{\varepsilon}_{\beta}\{(1,4)\} = C e^{-5\beta}, \quad B^{\varepsilon}_{\beta}\{(0,0)\} = C. \\ \mbox{ Generally, } & \forall \sigma \in \Sigma, \quad B^{\varepsilon}_{\beta}\{\sigma\} = C e^{-(\sigma_1 + \sigma_2) \cdot \beta}. \end{array}$

THEOREM 21.2. Let Σ be a nonempty finite set. Let $\varepsilon: \Sigma \to \mathbb{R}, \quad \xi, \beta \in \mathbb{R}.$ Then: $B_{\beta}^{\varepsilon} = B_{\beta}^{\varepsilon-\xi}.$

Proof. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. Since, $\forall \sigma \in \Sigma$, $\hat{B}^{\varepsilon}_{\beta} \{\sigma\} = e^{-\beta \cdot \varepsilon_{\sigma}} = e^{-\beta \cdot \xi} \cdot e^{-\beta \cdot (\varepsilon_{\sigma} - \xi)} = e^{-\beta \cdot \xi} \cdot (\hat{B}^{\varepsilon - \xi}_{\beta} \{\sigma\})$,

we get:
$$\hat{B}^{\varepsilon}_{\beta} = e^{-\beta \cdot \xi} \cdot \hat{B}^{\varepsilon-\xi}_{\beta}.$$

Since $e^{-\beta \xi} > 0$, we get: $\mathcal{N}(e^{-\beta \cdot \xi} \cdot \hat{B}^{\varepsilon-\xi}_{\beta}) = \mathcal{N}(\hat{B}^{\varepsilon-\xi}_{\beta}).$
Then: $B^{\varepsilon}_{\beta} = \mathcal{N}(\hat{B}^{\varepsilon}_{\beta}) = \mathcal{N}(e^{-\beta \cdot \xi} \cdot \hat{B}^{\varepsilon-\xi}_{\beta}) = \mathcal{N}(\hat{B}^{\varepsilon-\xi}_{\beta}) = B^{\varepsilon-\xi}_{\beta}.$

DEFINITION 21.3. Let Σ be a nonempty finite set, $\varepsilon: \Sigma \to \mathbb{R}$. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma).$ $\begin{array}{c} \overline{\Gamma_{\beta}^{\varepsilon}} & := & \sum_{\sigma \in \Sigma} \left[\varepsilon_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \right], \\ \overline{\Delta_{\beta}^{\varepsilon}} & := & \sum_{\sigma \in \Sigma} \left[e^{-\beta \cdot \varepsilon_{\sigma}} \right], \\ \overline{A_{\beta}^{\varepsilon}} & := & \Gamma_{\beta}^{\varepsilon} / \Delta_{\beta}^{\varepsilon}. \end{array}$ For all $\beta \in \mathbb{R}$, \mathbf{let}

Let Σ be a nonempty finite set, $\varepsilon : \Sigma \to \mathbb{R}$. $\Gamma_{\beta}^{\varepsilon} = \sum_{\sigma \in \Sigma} \left[\varepsilon_{\sigma} \cdot \left(\widehat{B}_{\beta}^{\varepsilon} \{ \sigma \} \right) \right],$ Since $\Gamma^{\varepsilon}_{\beta}$ is the integral of ε wrt $\hat{B}^{\varepsilon}_{\beta}$. we get: $\begin{array}{lll} \Delta^{\varepsilon}_{\beta} & = & \sum_{\sigma \in \Sigma} \left[\hat{B}^{\varepsilon}_{\beta} \{\sigma\} \right], \\ \Delta^{\varepsilon}_{\beta} & = & \hat{B}^{\varepsilon}_{\beta}(\Sigma). \\ \frac{\Gamma^{\varepsilon}_{\beta}}{\Delta^{\varepsilon}_{\beta}} & = & \frac{\sum_{\sigma \in \Sigma} \left[\varepsilon_{\sigma} \cdot \left(\hat{B}^{\varepsilon}_{\beta} \{\sigma\} \right) \right]}{\hat{B}^{\varepsilon}_{\beta}(\Sigma)}, \end{array}$ Since we get: Since $A_{\beta}^{\varepsilon} = \sum_{\sigma \in \Sigma} \left[\varepsilon_{\sigma} \cdot \left(B_{\beta}^{\varepsilon} \{ \sigma \} \right) \right].$ we get: A^{ε}_{β} is the average value of ε wrt B^{ε}_{β} . Then:

Recall (§2) the notations \mathbb{I}_f , f^*A . Recall (§9) the notation $\varepsilon_*\mu$. Recall (Definition 9.5) the notation M_{μ} .

THEOREM 21.4. Let Σ be a nonempty finite set. Let $\varepsilon: \Sigma \to \mathbb{R}, \quad \beta \in \mathbb{R}.$ Then: $M_{\varepsilon_*B_{\alpha}^{\varepsilon}} = A_{\beta}^{\varepsilon}$.

Proof. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. Σ is the disjoint union, over $t \in \mathbb{I}_{\varepsilon}$, of $\varepsilon^* \{t\}$, Because $\sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^* \{t\}} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})] = \sum_{\sigma \in \Sigma} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})].$ $A_{\beta}^{\varepsilon} = \sum_{\sigma \in \Sigma} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})].$ $\sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^* \{t\}} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})] = A_{\beta}^{\varepsilon}.$ $\sum_{t \in \mathbb{I}_{\varepsilon}} [t \cdot ((\varepsilon_* B_{\beta}^{\varepsilon})\{t\})] = \sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^* \{t\}} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})].$ $\sum_{t \in \mathbb{I}_{\varepsilon}} [t \cdot ((\varepsilon_* B_{\beta}^{\varepsilon})\{t\})] = \sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^* \{t\}} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})].$ $\sum_{t \in \mathbb{I}_{\varepsilon}} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\tau\})] = \sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^* \{t\}} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})].$ we get: Also, Then: So, since we want: Want: $\forall t \in \mathbb{I}_{\varepsilon}$, Given $t \in \mathbb{I}_{\varepsilon}$, want: $t \cdot ((\varepsilon_* B_{\beta}^{\varepsilon}) \{t\}) = \sum_{\sigma \in \varepsilon^* \{t\}} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon} \{\sigma\})].$ For all $\sigma \in \varepsilon^* \{t\}$, since $\varepsilon_\sigma = \varepsilon(\sigma) \in \{t\}$, we get: $\varepsilon_\sigma = t$. Want: $t \cdot ((\varepsilon_* B_\beta^{\varepsilon}) \{t\}) = \sum_{\sigma \in \varepsilon^* \{t\}} [t \cdot (B_\beta^{\varepsilon} \{\sigma\})].$ $\varepsilon^*{t}$ is the disjoint union, over $\sigma \in \varepsilon^*{t}$, of $\{\sigma\}$, Because

we get:

$$B_{\beta}^{\varepsilon}(\varepsilon^{*}\{t\}) = \sum_{\sigma \in \varepsilon^{*}\{t\}} \begin{bmatrix} B_{\beta}^{\varepsilon}\{\sigma\} \end{bmatrix}.$$
Also,

$$(\varepsilon_{*}B_{\beta}^{\varepsilon})\{t\} = B_{\beta}^{\varepsilon}(\varepsilon^{*}\{t\}).$$
Then:

$$t \cdot ((\varepsilon_{*}B_{\beta}^{\varepsilon})\{t\}) = t \cdot (B_{\beta}^{\varepsilon}(\varepsilon^{*}\{t\})) = \sum_{\sigma \in \varepsilon^{*}\{t\}} \begin{bmatrix} t \cdot (B_{\beta}^{\varepsilon}\{\sigma\}) \end{bmatrix}.$$

THEOREM 21.5. Let Σ be a nonempty finite set. Let $\varepsilon: \Sigma \to \mathbb{R}, \quad \beta, \xi \in \mathbb{R}.$ Then: $A_{\beta}^{\varepsilon-\xi} = A_{\beta}^{\varepsilon} - \xi.$

THEOREM 21.6. Let Σ be a nonempty finite set, $\varepsilon : \Sigma \to \mathbb{R}$. Then: $as \ \beta \to \infty, \quad A_{\beta}^{\varepsilon} \to \min \mathbb{I}_{\varepsilon}$ $and \qquad as \ \beta \to -\infty, \quad A_{\beta}^{\varepsilon} \to \max \mathbb{I}_{\varepsilon}$.

The proof is a matter of bookkeeping, best explained by example: Let $\Sigma := \{1, 2, 3, 4\}$ and define $\varepsilon : \Sigma \to \mathbb{R}$ by:

$$\begin{split} \varepsilon(1) &= 2, \quad \varepsilon(2) = 4, \quad \varepsilon(3) = 9, \quad \varepsilon(4) = 9. \\ \text{Then } \mathbb{I}_{\varepsilon} &= \{2, 4, 9\}, \text{ so } \min \mathbb{I}_{\varepsilon} = 2 \text{ and } \max \mathbb{I}_{\varepsilon} = 9. \\ \text{Since} \qquad \forall \beta \in \mathbb{R}, \quad A_{\beta}^{\varepsilon} &= \frac{2e^{-2\beta} + 4e^{-4\beta} + 9e^{-9\beta} + 9e^{-9\beta}}{e^{-2\beta} + e^{-4\beta} + 18e^{-9\beta}}, \\ &= \frac{2e^{-2\beta} + 4e^{-4\beta} + 18e^{-9\beta}}{e^{-2\beta} + e^{-4\beta} + 2e^{-9\beta}}, \\ \text{we get} \qquad \text{as } \beta \to \infty, \quad A_{\beta}^{\varepsilon} \to 2/1 \\ \text{and } \text{as } \beta \to -\infty, \quad A_{\beta}^{\varepsilon} \to 18/2, \\ \text{and so} \qquad \text{as } \beta \to \infty, \quad A_{\beta}^{\varepsilon} \to \min \mathbb{I}_{\varepsilon} \end{split}$$

and as $\beta \to -\infty$, $A_{\beta}^{\varepsilon} \to \max \mathbb{I}_{\varepsilon}$.

For any nonempty finite set Σ , for any $\varepsilon : \Sigma \to \mathbb{R}$, **define** $A^{\varepsilon}_{\bullet}$: $\mathbb{R} \to \mathbb{R}$ by: $\forall \beta \in \mathbb{R}, A^{\varepsilon}_{\bullet}(\beta) = A^{\varepsilon}_{\beta}$.

Recall (§2): " C^{ω} " means "real-analytic".

THEOREM 21.7. Let Σ be a finite set. Let $\varepsilon: \Sigma \to \mathbb{R}$. Assume: $\#\mathbb{I}_{\varepsilon} \ge 2$. Then: $A^{\varepsilon}_{\bullet}$ is a strictly-decreasing C^{ω} -diffeomorphism from \mathbb{R} onto $(\min \mathbb{I}_{\varepsilon}; \max \mathbb{I}_{\varepsilon}).$

We have: $\forall \beta \in \mathbb{R}, A_{\bullet}^{\varepsilon}(\beta) = \frac{\sum_{\sigma \in \Sigma} [\varepsilon_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}}]}{\sum_{\tau \in \Sigma} [e^{-\beta \cdot \varepsilon_{\tau}}]}$. Then $A_{\bullet}^{\varepsilon} : \mathbb{R} \to \mathbb{R}$ is C^{ω} . *Proof.* For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. by Theorem 21.6 and the C^{ω} -Inverse Function Theorem and So, the Mean Value Theorem, it suffices to show: $(A_{\bullet}^{\varepsilon})' < 0$ on \mathbb{R} . Given $\beta \in \mathbb{R}$, want: $(A^{\varepsilon}_{\bullet})'(\beta) < 0$. $P := \sum_{\sigma \in \Sigma} \left[\varepsilon_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \right], \quad P' := \sum_{\sigma \in \Sigma} \left[\left(-\varepsilon_{\sigma}^2 \right) \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \right].$ $Q := \sum_{\tau \in \Sigma} \left[e^{-\beta \cdot \varepsilon_{\tau}} \right], \quad Q' := \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\tau} \right) \cdot e^{-\beta \cdot \varepsilon_{\tau}} \right].$ Let Let Then Q > 0. Also, by the Quotient Rule, $(A^{\varepsilon}_{\bullet})'(\beta) = [QP' - PQ']/Q^2$. Want: QP' - PQ' < 0. $QP' = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\sigma}^{2} \right) \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})} \right].$ $PQ' = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\sigma}^{2} \right) \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})} \right].$ $QP' - PQ' = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\sigma}^{2} + \varepsilon_{\sigma}\varepsilon_{\tau} \right) \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})} \right].$ We have: QP'We have: Then: Interchanging σ and τ , we get: $QP' - PQ' = \sum_{\tau \in \Sigma} \sum_{\sigma \in \Sigma} \left[\left(-\varepsilon_{\tau}^2 + \varepsilon_{\tau} \varepsilon_{\sigma} \right) \cdot e^{-\beta \cdot \left(\varepsilon_{\tau} + \varepsilon_{\sigma} \right)} \right].$ By commutativity of addition and multiplication, adding the last two equations gives: $2 \cdot (QP' - PQ') = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\sigma}^2 - \varepsilon_{\tau}^2 + 2\varepsilon_{\sigma}\varepsilon_{\tau} \right) \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})} \right].$ Then: $2 \cdot (QP' - PQ') = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\begin{array}{c} -(\varepsilon_{\sigma} - \varepsilon_{\tau})^2 & \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})} \right].$ Then: $2 \cdot (QP' - PQ') < 0.$ Then: QP' - PQ' < 0.**DEFINITION 21.8.** Let Σ be a finite set. Let $\varepsilon : \Sigma \to \mathbb{R}$. Let $\alpha \in (\min \mathbb{I}_{\varepsilon}; \max \mathbb{I}_{\varepsilon}).$ Assume: $\#\mathbb{I}_{\varepsilon} \ge 2$. $BP^{\varepsilon}_{\alpha} := (A^{\varepsilon}_{\bullet})^{-1}(\alpha).$ The α -Boltzmann-parameter on ε is: So the α -Boltzmann-parameter on ε is the unique $\beta \in \mathbb{R}$ s.t. $A_{\beta}^{\varepsilon} = \alpha$. *Example:* Let $\Sigma := \{0, 1, 10\}$ and define $\varepsilon : \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma, \ \varepsilon(\sigma) = \sigma.$ Then: $BP_1^{\varepsilon} = (\ln 9)/10.$ Computation shows: $A_{(\ln 9)/10}^{\varepsilon} = 1.$ *Example:* Let $\Sigma := \{2, 4, 8, 9\}$ and define $\varepsilon : \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma, \ \varepsilon(\sigma) = \sigma.$ To evaluate BP_5^{ε} , we must solve $A_{\bullet}^{\varepsilon}(\beta) = 5$ for β , and, since, by Theorem 21.7, $A^{\varepsilon}_{\bullet}$ is strictly-decreasing,

there are simple iterative methods to do this.

We compute: $BP_5^{\varepsilon} \approx 0.0918$, accurate to four decimal places.

(Thanks to C. Prouty for this calculation. See $\S28$.)

Next, let
$$\Sigma := \{1, 2, 3, 4\}$$
 and define $\overline{\varepsilon} : \Sigma \to \mathbb{R}$ by:
 $\overline{\varepsilon}(1) = 2, \quad \overline{\varepsilon}(2) = 4, \quad \overline{\varepsilon}(3) = 8, \quad \overline{\varepsilon}(4) = 9.$
Then $A_{\bullet}^{\overline{\varepsilon}} = A_{\bullet}^{\varepsilon}, \quad \text{so} \quad BP_{5}^{\overline{\varepsilon}} = BP_{5}^{\varepsilon}.$
Then $BP_{5}^{\overline{\varepsilon}} \approx 0.0918, \quad \text{accurate to four decimal places.}$

Example: Let $\Sigma := \{1, 2, 3, 4\}$ and define $\varepsilon : \Sigma \to \mathbb{R}$ by: $\varepsilon(1) = 2, \quad \varepsilon(2) = 4, \quad \varepsilon(3) = 9, \quad \varepsilon(4) = 9.$

To evaluate BP^{ε}₅, we must solve $A^{\varepsilon}_{\bullet}(\beta) = 5$ for β , and, since, by Theorem 21.7, $A^{\varepsilon}_{\bullet}$ is strictly-decreasing, there are simple iterative methods to do this.

We compute: $BP_5^{\varepsilon} \approx 0.1060$, accurate to four decimal places. (Thanks to C. Prouty for this calculation. See §28.)

Example: Let $\Sigma := ([0..4] \times [0..4]) \setminus \{(4,4)\}.$

- **Define** $\varepsilon : \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma, \ \varepsilon(\sigma) = \sigma_1 + \sigma_2.$
- To evaluate BP_1^{ε} , we must solve $A_{\bullet}^{\varepsilon}(\beta) = 1$ for β , and, since, by Theorem 21.7, $A_{\bullet}^{\varepsilon}$ is strictly-decreasing, there are simple iterative methods to do this.

We compute: $BP_1^{\varepsilon} \approx 1.0670$, accurate to four decimal places. (Thanks to C. Prouty for this calculation. See §28.)

THEOREM 21.9. Let Σ be a finite set.

Let $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\#\mathbb{I}_{\varepsilon} \ge 2$. Let $\alpha \in (\min \mathbb{I}_{\varepsilon}; \max \mathbb{I}_{\varepsilon})$. Let $\xi \in \mathbb{R}$. Then: $\mathrm{BP}_{\alpha-\xi}^{\varepsilon-\xi} = \mathrm{BP}_{\alpha}^{\varepsilon}$. Proof. Let $\beta := \mathrm{BP}_{\alpha}^{\varepsilon}$. Want: $\mathrm{BP}_{\alpha-\xi}^{\varepsilon-\xi} = \beta$. Since $\beta = \mathrm{BP}_{\alpha}^{\varepsilon} = (A_{\bullet}^{\varepsilon})^{-1}(\alpha)$, we get: $(A_{\bullet}^{\varepsilon})(\beta) = \alpha$. By Theorem 21.5, $A_{\beta}^{\varepsilon-\xi} = A_{\beta}^{\varepsilon} - \xi$. Since $(A_{\bullet}^{\varepsilon-\xi})(\beta) = A_{\beta}^{\varepsilon-\xi} = A_{\beta}^{\varepsilon} - \xi = ((A_{\bullet}^{\varepsilon})(\beta)) - \xi = \alpha - \xi$, we get: $\beta = (A_{\bullet}^{\varepsilon-\xi})^{-1}(\alpha - \xi)$. So, since $\mathrm{BP}_{\alpha-\xi}^{\varepsilon-\xi} = (A_{\bullet}^{\varepsilon-\xi})^{-1}(\alpha - \xi)$, we get: $\mathrm{BP}_{\alpha-\xi}^{\varepsilon-\xi} = \beta$. \Box

22. Degenerate energy levels

Recall (§2): the notations \mathbb{I}_f and f^*A . Recall (§9): the notation ν_F .

THEOREM 22.1. Let Σ be a finite set.

Let $\varepsilon: \Sigma \to \mathbb{Z}$. Assume \mathbb{I}_{ε} is residue-unconstrained.

Let $\alpha \in (\min \mathbb{I}_{\varepsilon}; \max \mathbb{I}_{\varepsilon}).$ Let $\beta := BP_{\alpha}^{\varepsilon}$. Assume: $\{t_n - n\alpha \mid n \in \mathbb{N}\}\$ is bounded. Let $t_1, t_2, \ldots \in \mathbb{Z}$. let $\Omega_n := \{ f \in \Sigma^n | (\varepsilon(f_1)) + \dots + (\varepsilon(f_n)) = t_n \}.$ For all $n \in \mathbb{N}$, Then: as $n \to \infty$, $\nu_{\Omega_n} \{ f \in \Omega_n \mid f_n = \sigma_0 \} \to B^{\varepsilon}_{\beta} \{ \sigma_0 \}.$ Let $\sigma_0 \in \Sigma$. Recall $(\S 9)$: $\nu_{\varnothing}(\varnothing) = -1.$ So, since $B^{\varepsilon}_{\beta}\{\sigma_0\} > 0$, part of the content of Theorem 22.1 is: $\forall \text{sufficiently large } n \in \mathbb{N},$ $\Omega_n \neq \emptyset;$ see Claim 1 in the proof below. *Proof.* Since \mathbb{I}_{ε} is residue-unconstrained, we get: $\mathbb{I}_{\varepsilon} \neq \emptyset$. So, since $\varepsilon : \Sigma \to \mathbb{Z}$, we conclude: $\Sigma \neq \emptyset$. By hypothesis, Σ is finite. Then: Σ is a nonempty finite set. Since $\beta = BP^{\varepsilon}_{\alpha} = (A^{\varepsilon}_{\bullet})^{-1}(\alpha),$ we get: $A^{\varepsilon}(\beta) = \alpha.$ By Theorem 21.4, we have: $M_{\varepsilon_*B^\varepsilon_\beta} = A^\varepsilon_\beta.$ since $A^{\varepsilon}_{\beta} = A^{\varepsilon}_{\bullet}(\beta) = \alpha$, So, we get: $M_{\varepsilon_*B^{\varepsilon}_{\alpha}} = \alpha.$ Then: $\mu \in \mathcal{P}_{\Sigma}$ Let $\mu := B_{\beta}^{\varepsilon}$. $M_{\varepsilon_*\mu} = \alpha.$ and Let $E := \mathbb{I}_{\varepsilon}, \quad \widetilde{\mu} := \varepsilon_* \mu.$ Then: $\widetilde{\mu} \in \mathcal{P}_E$ $M_{\tilde{\mu}}$ $= \alpha$. and By hypothesis, E is residue-unconstrained. Since $\varepsilon : \Sigma \to \mathbb{Z}$, we get: $E \subseteq \mathbb{Z}.$ Since Σ is finite, we get: E is finite. So, since $\widetilde{\mu} \in \mathcal{P}_E \subseteq \mathcal{FM}_E$, we get: $|\widetilde{\mu}|_1 < \infty$ and $|\widetilde{\mu}|_2 < \infty$. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. Then: $\forall n \in \mathbb{N}, \ \Omega_n = \{ f \in \Sigma^n \, | \, \varepsilon_{f_1} + \dots + \varepsilon_{f_n} = t_n \}.$ For all $n \in \mathbb{N}$, **define** $\varepsilon^n : \Sigma^n \to E^n$ by: $\forall f_1, \ldots, f_n \in \Sigma, \quad \varepsilon^n(f_1, \ldots, f_n) = (\varepsilon_{f_1}, \ldots, \varepsilon_{f_n}).$ since $\varepsilon_*\mu = \widetilde{\mu}$, we get: $\forall n \in \mathbb{N}, \ (\varepsilon^n)_*(\mu^n) = \widetilde{\mu}^n$. Then, $\widetilde{\Omega}_n := \{ \widetilde{f} \in E^n \mid \widetilde{f}_1 + \dots + \widetilde{f}_n = t_n \};$ let For all $n \in \mathbb{N}$, $(\varepsilon^n)^*\widetilde{\Omega}_n = \Omega_n.$ then $\mu^n((\varepsilon^n)^*\widetilde{\Omega}_n) = \mu^n(\Omega_n).$ Then: $\forall n \in \mathbb{N},$ $((\varepsilon^n)_*\mu^n)(\widetilde{\Omega}_n) = \mu^n(\Omega_n).$ Then: $\forall n \in \mathbb{N},$ $\widetilde{\mu}^n(\widetilde{\Omega}_n) = \mu^n(\Omega_n).$ Then: $\forall n \in \mathbb{N},$ For all $n \in \mathbb{N}$, **define** $\psi_n : \mathbb{Z} \to \mathbb{R}$ by: $\forall t \in \mathbb{Z}, \quad \psi_n(t) = \widetilde{\mu}^n \{ \widetilde{f} \in E^n \mid \widetilde{f_1} + \dots + \widetilde{f_n} = t \}.$ Then: $\forall n \in \mathbb{N}, \quad \psi_n(t_n) = \widetilde{\mu}^n(\Omega_n).$ Since E is finite and residue-unconstrained, we get: $2 \leq \#E < \infty$. Since $\varepsilon : \Sigma \to \mathbb{Z}$, we get: $S_{B^{\varepsilon}_{\beta}} = \Sigma.$ $S_{\mu}^{\beta} = \Sigma.$ So, since $\mu = B_{\beta}^{\varepsilon}$, we get: So, since $\varepsilon : \Sigma \to \mathbb{Z}$, $S_{\varepsilon \star \mu} = \mathbb{I}_{\varepsilon}.$ we get:

So, since $\varepsilon_*\mu = \widetilde{\mu}$ and $\mathbb{I}_{\varepsilon} = E$, we get: $S_{\widetilde{\mu}}$ = E.Since E is finite, we get: E is countable. Let $v := V_{\tilde{\mu}}$. By Theorem 10.6, we get: $0 < v < \infty$. Let $\tau := 1/\sqrt{2\pi v}$. Then: $0 < \tau < \infty$. By Theorem 10.6, we get: as $n \to \infty$, $\sqrt{n} \cdot (\mu^n \{ \widetilde{f} \in E^n \mid \widetilde{f}_1 + \dots + \widetilde{f}_n = t_n \}) \to 1/\sqrt{2\pi v}.$ Then: as $n \to \infty$, $\sqrt{n} \cdot ($ $\psi_n(t_n)$ $) \rightarrow \tau$. So, since $\tau > 0$, choose $n_0 \in [2..\infty)$ such that: $\forall n \in [n_0..\infty),$ $\sqrt{n} \cdot (\psi_n(t_n)) > 0.$

 $\begin{array}{lll} Claim \ 1: & \mbox{Let } n \in [n_0..\infty). & \mbox{Then:} & \mu^n(\Omega_n) > 0. \\ Proof \ of \ Claim \ 1: \ {\rm Recall:} & \widetilde{\mu}^n(\widetilde{\Omega}_n) = \mu^n(\Omega_n) & \mbox{and} & \psi_n(t_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n). \\ {\rm By \ the \ choice \ of \ } n_0, \ {\rm we \ get:} & \sqrt{n} \cdot (\psi_n(t_n)) > 0. & \mbox{Then:} & \psi_n(t_n) > 0. \\ & \mbox{Then:} & \mu^n(\Omega_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n) = \psi_n(t_n) > 0. \end{array}$

End of proof of Claim 1.

Then $\hat{B}^{\varepsilon}_{\beta}(\Sigma) > 0.$ Recall: $\Sigma \neq \emptyset$ and $\varepsilon : \Sigma \to \mathbb{Z}$. Let $C := 1/(\widehat{B}^{\varepsilon}_{\beta}(\Sigma)).$ Then $\mathcal{N}(\hat{B}^{\varepsilon}_{\beta}) = C \cdot \hat{B}^{\varepsilon}_{\beta}$ By definition of $\hat{B}^{\varepsilon}_{\beta}$, we have: $\forall \sigma \in \Sigma$, $\hat{B}^{\varepsilon}_{\beta} \{\sigma\} = e^{-\beta \cdot \varepsilon_{\sigma}}$. $\mu = B_{\beta}^{\varepsilon} = \mathcal{N}(\hat{B}_{\beta}^{\varepsilon}) = C \cdot \hat{B}_{\beta}^{\varepsilon},$ So, since $\mu\{\sigma\} = Ce^{-\beta \cdot \varepsilon_{\sigma}}.$ we get: $\forall \sigma \in \Sigma,$ Since $\mu \in \mathcal{P}_{\Sigma}$, we get: $\forall n \in \mathbb{N}, \ \mu^n \in \mathcal{P}_{\Sigma^n}$, so $\mu^n(\Omega_n) \leqslant 1.$ So, by Claim 1, $\forall n \in [n_0 \dots \infty)$, $0 < \mu^n(\Omega_n) \leq 1.$ Also, we have: $(\mu^n | \Omega_n)(\Omega_n) = \mu^n(\Omega_n).$ $\forall n \in \mathbb{N},$ $0 < (\mu^n | \Omega_n)(\Omega_n) \le 1.$ Then: $\forall n \in [n_0..\infty),$ $\forall n \in [n_0 \dots \infty),$ $\mu^n \mid \Omega_n \in \mathcal{FM}_{\Omega_-}^{\times}.$ Then: $\mathcal{N}(\mu^n \,|\, \Omega_n) \in$ $\forall n \in [n_0..\infty),$ Then: \mathcal{P}_{Ω_n} . $(\mu^n | \Omega_n)(S) = \mu^n(S).$ Also, $\forall n \in \mathbb{N}, \, \forall S \subseteq \Omega_n,$ $(\mu^n | \Omega_n)(\Omega_n) = \mu^n(\Omega_n).$ Then: $\forall n \in \mathbb{N},$ For all $n \in \mathbb{N},$ let $z_n := \mu^n(\Omega_n).$ $\forall n \in [n_0..\infty),$ $(\mu^n | \Omega_n)(\Omega_n) = z_n \text{ and } 0 < z_n \leq 1.$ Then: $n \in [n_0 \dots \infty),$ $\lambda_n := \mathcal{N}(\mu^n | \Omega_n).$ For all let $\lambda_n = (\mu^n | \Omega_n) / z_n.$ Then: $\forall n \in [n_0..\infty),$ $\lambda_n(S) = (\mu^n(S))/z_n.$ $\forall n \in [n_0 \dots \infty), \quad \forall S \subseteq \Omega_n,$ Then: Claim 2: Let $n \in [n_0..\infty)$. Then: $\lambda_n = \nu_{\Omega_n}$.

Since $\lambda_n = \mathcal{N}(\mu^n | \Omega_n) = \mathcal{N}(\mu^n | F)$, we get: $\lambda_n \in \mathcal{P}_F$. By Theorem 9.9, given $f, q \in F$, want: $\lambda_n \{f\} = \lambda_n \{q\}$. Want: $(\mu^n{f})/z_n = (\mu^n{g})/z_n$. Want: $\mu^n{f} = \mu^n{g}$. For all $i \in [1..n]$, let $\widetilde{f}_i := \varepsilon_{f_i}$ and $\widetilde{g}_i := \varepsilon_{g_i}$. $\forall \sigma \in \Sigma, \ \mu\{\sigma\} = Ce^{-\beta \cdot \varepsilon_{\sigma}}.$ Recall: Then: $\forall i \in [1..n], \ \mu\{f_i\} = Ce^{-\beta \cdot \tilde{f}_i} \text{ and } \mu\{g_i\} = Ce^{-\beta \cdot \tilde{g}_i}.$ Since $f \in F = \Omega_n$, we get: $\varepsilon_{f_1} + \cdots + \varepsilon_{f_n} = t_n$. Since $g \in F = \Omega_n$, we get: $\varepsilon_{g_1} + \dots + \varepsilon_{g_n} = t_n$. Since $\tilde{f}_1 + \dots + \tilde{f}_n = \varepsilon_{f_1} + \dots + \varepsilon_{f_n} = t_n$ $=\varepsilon_{g_1}+\cdots+\varepsilon_{g_n}=\widetilde{g}_1+\cdots+\widetilde{g}_n,\\ C^n e^{-\beta\cdot(\widetilde{g}_1+\cdots+\widetilde{g}_n)}=C^n e^{-\beta\cdot(\widetilde{g}_1+\cdots+\widetilde{g}_n)}.$ we get: Then: $(Ce^{-\beta \cdot \tilde{f}_1}) \cdots (Ce^{-\beta \cdot \tilde{f}_n}) = (Ce^{-\beta \cdot \tilde{g}_1}) \cdots (Ce^{-\beta \cdot \tilde{g}_n}).$ Then: $(\mu\{f_1\}) \cdots (\mu\{f_n\}) = (\mu\{g_1\}) \cdots (\mu\{g_n\}).$ Then: $\mu^n{f}$ $\mu^n \{q\}.$ =

End of proof of Claim 2.

Since $\varepsilon(\sigma_0) = \varepsilon_{\sigma_0} \in \{\varepsilon_{\sigma_0}\}$, we get: $\sigma_0 \in \varepsilon^* \{\varepsilon_{\sigma_0}\}$. Then $\varepsilon^* \{\varepsilon_{\sigma_0}\} \neq \emptyset$, so $\#(\varepsilon^* \{\varepsilon_{\sigma_0}\}) \ge 1$. Let $k := \#(\varepsilon^* \{\varepsilon_{\sigma_0}\})$. Then: $k \ge 1$.

 $\begin{array}{ll} Claim \ 4: & \mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = k \cdot (\mu\{\sigma_0\}). \\ Proof \ of \ Claim \ 4: & \text{Since} & \varepsilon^*\{\varepsilon_{\sigma_0}\} \ \text{is equal to} \\ & \text{the disjoint union, } & \text{over } \sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}, & \text{of } \{\sigma\}, \\ & \text{we get:} & \mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = \sum_{\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}} [\mu\{\sigma\}], \\ \text{So, by Claim 3, we get:} & \mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = \sum_{\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}} [\mu\{\sigma_0\}]. \\ \text{So, since } k = \#(\varepsilon^*\{\varepsilon_{\sigma_0}\}), & \text{we get:} & \mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = k \cdot (\mu\{\sigma_0\}). \\ End \ of \ proof \ of \ Claim \ 4. \end{array}$

Claim 5: Let $n \in [2..\infty)$. Let $\sigma \in \varepsilon^* \{\varepsilon_{\sigma_0}\}$. Then: $\mu^n \{f \in \Omega_n \mid f_n = \sigma\} = \mu^n \{f \in \Omega_n \mid f_n = \sigma_0\}$. Proof of Claim 5:

Let $X := \{f \in \Sigma^{n-1} \mid \varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} = t_n - \varepsilon_{\sigma}\}.$ Recall: $\Omega_n = \{f \in \Sigma^n \mid \varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} + \varepsilon_{f_n} = t_n\}.$ Since $\{f \in \Omega_n \mid f_n = \sigma\}$ $= \{f \in \Sigma^n \mid [\varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} + \varepsilon_{f_n} = t_n] \& [f_n = \sigma]\}$ $= \{f \in \Sigma^n \mid [\varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} + \varepsilon_{\sigma} = t_n] \& [f_n = \sigma]\}$ $= \{f \in \Sigma^n \mid [\varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} = t_n - \varepsilon_{\sigma}] \& [f_n = \sigma]\},$ it follows that we denote the sterm and bijection Σ^n .

it follows that, under the standard bijection $\Sigma^n \leftrightarrow \Sigma^{n-1} \times \Sigma$, we have:

 $\{f \in \Omega_n \mid f_n = \sigma\} \subseteq \Sigma^n$ $X \times \{\sigma\} \subseteq \Sigma^{n-1} \times \Sigma.$ corresponds to $\mu^{n} \{ f \in \Omega_{n} \mid f_{n} = \sigma \} = (\mu^{n-1}(X)) \cdot (\mu\{\sigma\}).$ Then: Want: $\mu^n \{ f \in \Omega_n \mid f_n = \sigma_0 \} = (\mu^{n-1}(X)) \cdot (\mu \{\sigma\}).$ By Claim 3, we have: $\mu\{\sigma\} = \mu\{\sigma_0\}.$ Want: $\mu^n \{ f \in \Omega_n \mid f_n = \sigma_0 \} = (\mu^{n-1}(X)) \cdot (\mu \{ \sigma_0 \}).$ Since $\sigma \in \varepsilon^* \{ \varepsilon_{\sigma_0} \}$, we get: $\varepsilon(\sigma) \in \{ \varepsilon_{\sigma_0} \}$. $\varepsilon_{\sigma} = \varepsilon(\sigma) \in \{\varepsilon_{\sigma_0}\}, \text{ we get: } \varepsilon_{\sigma} = \varepsilon_{\sigma_0}.$ Since $X = \{ f \in \Sigma^{n-1} \mid \varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} = t_n - \varepsilon_{\sigma_0} \}.$ Then $\begin{cases} f \in \Omega_n & | & f_n = \sigma_0 \\ f \in \Sigma^n & | [\varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} + \varepsilon_{f_n} = t_n] \& [f_n = \sigma_0] \\ f \in \Sigma^n & | \varepsilon_{f_1} - \varepsilon_{f_n} = t_n \end{bmatrix} \end{cases}$ Since $= \{ f \in \Sigma^n \quad | [\varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} + \varepsilon_{\sigma_0} = t_n] \& [f_n = \sigma_0] \}$ $= \{ f \in \Sigma^n \quad | [\varepsilon_{f_1} + \dots + \varepsilon_{f_{n-1}} = t_n - \varepsilon_{\sigma_0}] \& [f_n = \sigma_0] \},\$

it follows that, under the standard bijection $\Sigma^n \leftrightarrow \Sigma^{n-1} \times \Sigma$, we have:

 $\{f \in \Omega_n \mid f_n = \sigma_0\} \subseteq \Sigma^n$ corresponds to $X \times \{\sigma_0\} \subseteq \Sigma^{n-1} \times \Sigma$. Then: $\mu^n \{f \in \Omega_n \mid f_n = \sigma_0\} = (\mu^{n-1}(X)) \cdot (\mu \{\sigma_0\}).$ End of proof of Claim 5.

Claim 6: Let $n \in [2..\infty)$. $\widetilde{\mu}^n \{ \widetilde{\widetilde{f}} \in \widetilde{\Omega}_n \mid \widetilde{f}_n = \varepsilon_{\sigma_0} \} = k \cdot (\mu^n \{ f \in \Omega_n \mid f_n = \sigma_0 \}).$ Then: Proof of Claim 6: Recall: $(\varepsilon^n)^* \widetilde{\Omega}_n = \Omega_n.$ $(\varepsilon^n)^* \{ \widetilde{f} \in \widetilde{\Omega}_n \mid \widetilde{f}_n = \varepsilon_{\sigma_0} \} = \{ f \in \Omega_n \mid f_n \in \varepsilon^* \{ \varepsilon_{\sigma_0} \} \},\$ Then and so $\mu^n((\varepsilon^n)^*\{\widetilde{f}\in\widetilde{\Omega}_n \mid \widetilde{f}_n=\varepsilon_{\sigma_0}\})=\mu^n\{f\in\Omega_n \mid f_n\in\varepsilon^*\{\varepsilon_{\sigma_0}\}\}.$ Then: $((\varepsilon^n)_*(\mu^n))\{\widetilde{f}\in\widetilde{\Omega}_n\mid \widetilde{f}_n=\varepsilon_{\sigma_0}\}=\mu^n\{f\in\Omega_n\mid f_n\in\varepsilon^*\{\varepsilon_{\sigma_0}\}\}.$ Recall: $(\varepsilon^n)_*(\mu^n) = \widetilde{\mu}^n$. $\widetilde{\mu}^n \{ \widetilde{f} \in \widetilde{\Omega}_n \mid \widetilde{f}_n = \varepsilon_{\sigma_0} \} = \mu^n \{ f \in \Omega_n \mid f_n \in \varepsilon^* \{ \varepsilon_{\sigma_0} \} \}.$ Then: Want: $\mu^n \{ f \in \Omega_n \mid f_n \in \varepsilon^* \{ \varepsilon_{\sigma_0} \} \} = k \cdot (\mu^n \{ f \in \Omega_n \mid f_n = \sigma_0 \}).$ $\{f \in \Omega_n \mid f_n \in \varepsilon^* \{\varepsilon_{\sigma_0}\}\}$ Since is the disjoint union, over $\sigma \in \varepsilon^* \{ \varepsilon_{\sigma_0} \}$, of $\{f \in \Omega_n \mid f_n = \sigma\},\$

we get: $\mu^n \{ f \in \Omega_n \mid f_n \in \varepsilon^* \{ \varepsilon_{\sigma_0} \} \} = \sum_{\sigma \in \varepsilon^* \{ \varepsilon_{\sigma_0} \}} [\mu^n \{ f \in \Omega_n \mid f_n = \sigma \}].$ Then, by Claim 5, we conclude: $\mu^n \{ f \in \Omega_n \, | \, f_n \in \varepsilon^* \{ \varepsilon_{\sigma_0} \} \} = \sum_{\sigma \in \varepsilon^* \{ \varepsilon_{\sigma_0} \}} [\mu^n \{ f \in \Omega_n \, | \, f_n = \sigma_0 \}].$ since $k = #(\varepsilon^* \{ \varepsilon_{\sigma_0} \})$, we get: So, $\mu^n \{ f \in \Omega_n \mid f_n \in \varepsilon^* \{ \varepsilon_{\sigma_0} \} \} = k \quad \cdot \quad (\mu^n \{ f \in \Omega_n \mid f_n = \sigma_0 \}).$ End of proof of Claim 6. $\mu^n(\Omega_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n).$ Recall: $\forall n \in \mathbb{N}$, $0 < \mu^n(\Omega_n) \leqslant 1.$ $\forall n \in [n_0..\infty),$ $0 < \widetilde{\mu}^n(\widetilde{\Omega}_n) \leq 1.$ $\forall n \in [n_0..\infty),$ $(\widetilde{\mu}^n | \widetilde{\Omega}_n)(S) = \widetilde{\mu}^n(S).$ $\forall n \in \mathbb{N}, \, \forall S \subseteq \widetilde{\Omega}_n,$ $(\widetilde{\mu}^n | \widetilde{\Omega}_n) (\widetilde{\Omega}_n) = \widetilde{\mu}^n (\widetilde{\Omega}_n)$ $\forall n \in \mathbb{N},$

Recall: Then: Also. Then: By dividing the last two equations, we get: $\forall n \in [n_0..\infty), \forall S \subseteq \widetilde{\Omega}_n, \ (\mathcal{N}(\widetilde{\mu}^n | \widetilde{\Omega}_n))(S) = (\widetilde{\mu}^n(S))/(\widetilde{\mu}^n(\widetilde{\Omega}_n)).$ $n \in [n_0..\infty), \quad \text{let} \quad \widetilde{\lambda}_n := \mathcal{N}(\widetilde{\mu}^n | \widetilde{\Omega}_n).$ For all $\forall n \in [n_0 .. \infty), \forall S \subseteq \widetilde{\Omega}_n,$ $\widetilde{\lambda}_n(S) = (\widetilde{\mu}^n(S))/(\widetilde{\mu}^n(\widetilde{\Omega}_n)).$ Then: So, since $\forall n \in \mathbb{N}$, $z_n = \mu^n(\Omega_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n)$, we get: $\forall n \in [n_0..\infty), \forall S \subseteq \widetilde{\Omega}_n, \qquad \widetilde{\lambda}_n(S) = (\widetilde{\mu}^n(S))/z_n.$ Recall: $\forall n \in [n_0..\infty),$ $\lambda_n = \mathcal{N}(\mu^n | \Omega_n).$ Recall: $\forall n \in [n_0..\infty), \forall S \subseteq \Omega_n$, $\lambda_n(S) = (\mu^n(S))/z_n.$

 $\begin{array}{lll} Claim \ 7: \ \mathbf{Let} & n \in [n_0..\infty). \\ & \text{Then:} & \widetilde{\lambda}_n \{ \widetilde{f} \in \widetilde{\Omega}_n \mid \widetilde{f}_n = \varepsilon_{\sigma_0} \} = k \cdot (\lambda_n \{ f \in \Omega_n \mid f_n = \sigma_0 \}). \\ Proof \ of \ Claim \ 7: & \text{By choice of } n_0, \ \text{we have:} & n_0 \in [2..\infty). \\ \text{Then} \ [n_0..\infty) \subseteq [2..\infty), \ \text{so, since} \ n \in [n_0..\infty), \ \text{we get:} & n \in [2..\infty). \\ \text{Then, by Claim 6,} & \widetilde{\mu}^n \{ \widetilde{f} \in \widetilde{\Omega}_n \mid \widetilde{f}_n = \varepsilon_{\sigma_0} \} = k \cdot (\mu^n \{ f \in \Omega_n \mid f_n = \sigma_0 \}). \\ \text{Dividing this last equation by } z_n \ \text{yields} \\ & \widetilde{\lambda}_n \{ \widetilde{f} \in \widetilde{\Omega}_n \mid \widetilde{f}_n = \varepsilon_{\sigma_0} \} = k \cdot (\lambda_n \{ f \in \Omega_n \mid f_n = \sigma_0 \}). \\ \end{array}$

End of proof of Claim 7.

Let $P := \mu\{\sigma_0\}$ and $\widetilde{P} := \widetilde{\mu}\{\varepsilon_{\sigma_0}\}$. Recall: $k \ge 1$. By Claim 4, we have: $\mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = k \cdot (\mu\{\sigma_0\})$. Recall: $\widetilde{\mu} = \varepsilon_*\mu$. Since $\widetilde{P} = \widetilde{\mu}\{\varepsilon_{\sigma_0}\} = (\varepsilon_*\mu)\{\varepsilon_{\sigma_0}\} = \mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = k \cdot (\mu\{\sigma_0\}) = k \cdot P$, we get: $\widetilde{P}/k = P$. Recall: $M_{\widetilde{\mu}} = \alpha$ and $\widetilde{\mu} \in \mathcal{P}_E$ and $S_{\widetilde{\mu}} = E$.

Recall: *E* is residue-unconstrained and $|\tilde{\mu}|_2 < \infty$.

Since $\varepsilon_{\sigma_0} = \varepsilon(\sigma_0) \in \mathbb{I}_{\varepsilon} = E$, we get: $\varepsilon_{\sigma_0} \in E$. Let $\tilde{\varepsilon}_0 := \varepsilon_{\sigma_0}$. Then: $\tilde{\varepsilon}_0 \in E$ and $\tilde{P} = \tilde{\mu}\{\tilde{\varepsilon}_0\}$. Recall: $\forall n \in \mathbb{N}$, $\tilde{\Omega}_n := \{\tilde{f} \in E^n \mid \tilde{f}_1 + \dots + \tilde{f}_n = t_n\}$. By hypothesis, $t_1, t_2, \dots \in \mathbb{Z}$ and $\{t_n - n\alpha \mid n \in \mathbb{N}\}$ is bounded. By Theorem 12.2, as $n \to \infty$, $\mathcal{N}(\tilde{\mu}^n \mid \tilde{\Omega}_n) \{\tilde{f} \in \tilde{\Omega}_n \mid \tilde{f}_n = \tilde{\varepsilon}_0\} \to \tilde{P}$. Recall: $\forall n \in [n_0..\infty)$, $\tilde{\lambda}_n = \mathcal{N}(\tilde{\mu}^n \mid \tilde{\Omega}_n)$. Then: as $n \to \infty$, $\tilde{\lambda}_n \{\tilde{f} \in \tilde{\Omega}_n \mid \tilde{f}_n = \tilde{\varepsilon}_0\} \to \tilde{P}$. Then: as $n \to \infty$, $\tilde{\lambda}_n \{\tilde{f} \in \Omega_n \mid \tilde{f}_n = \varepsilon_{\sigma_0}\} \to \tilde{P}$. So, by Claim 7, as $n \to \infty$, $k \cdot (\lambda_n \{f \in \Omega_n \mid f_n = \sigma_0\}) \to \tilde{P}$. Then: as $n \to \infty$, $\lambda_n \{f \in \Omega_n \mid f_n = \sigma_0\} \to \tilde{P}/k$. So, by Claim 2, as $n \to \infty$, $\nu_{\Omega_n} \{f \in \Omega_n \mid f_n = \sigma_0\} \to \tilde{P}/k$. Recall: $\mu = B_{\beta}^{\varepsilon}$. Then, since $\tilde{P}/k = P = \mu \{\sigma_0\} = B_{\beta}^{\varepsilon} \{\sigma_0\}$, we get: as $n \to \infty$, $\nu_{\Omega_n} \{f \in \Omega_n \mid f_n = \sigma_0\} \to B_{\beta}^{\varepsilon} \{\sigma_0\}$.

The possibility of degeneracy at $\tilde{\varepsilon}_0$ (*i.e.*, the possibility that $k \neq 1$) causes a number of complications in the preceding proof. Here is another approach to proving Theorem 22.1:

By density of the set of injective functions $\Sigma \to \mathbb{R}$ in the topological space of all functions $\Sigma \to \mathbb{R}$, we reduce to the case where ε is injective.

Then the proof can follow the proof of Theorem 17.1, avoiding the degeneracy complications in the preceding proof.

Recall (§2): $\forall t \in \mathbb{R}, [t]$ is the floor of t. Next, we record the $t_n = \lfloor n\alpha \rfloor$ version of the preceding theorem:

THEOREM 22.2. Let Σ be a finite set.

Let $\varepsilon : \Sigma \to \mathbb{Z}$. Assume \mathbb{I}_{ε} is residue-unconstrained.

Let $\alpha \in (\min \mathbb{I}_{\varepsilon}; \max \mathbb{I}_{\varepsilon})$. For all $n \in \mathbb{N}$, let $\Omega_n := \{f \in \Sigma^n \mid (\varepsilon(f_1)) + \dots + (\varepsilon(f_n)) = \lfloor n\alpha \rfloor\}$. Let $\sigma_0 \in \Sigma$. Then: as $n \to \infty$, $\nu_{\Omega_n} \{f \in \Omega_n \mid f_n = \sigma_0\} \to B^{\varepsilon}_{\beta} \{\sigma_0\}$.

We record the $\alpha \in \mathbb{Z}$ special case of the preceding theorem:

THEOREM 22.3. Let Σ be a finite set. Let $\varepsilon : \Sigma \to \mathbb{Z}$. Assume \mathbb{I}_{ε} is residue-unconstrained. Let $\alpha \in (\min \mathbb{I}_{\varepsilon}; \max \mathbb{I}_{\varepsilon})$. Assume $\alpha \in \mathbb{Z}$. Let $\beta := BP_{\alpha}^{\varepsilon}$. For all $n \in \mathbb{N}$, let $\Omega_n := \{f \in \Sigma^n \mid (\varepsilon(f_1)) + \cdots + (\varepsilon(f_n)) = n\alpha\}$. Let $\sigma_0 \in \Sigma$. Then: as $n \to \infty$, $\nu_{\Omega_n} \{f \in \Omega_n \mid f_n = \sigma_0\} \to B_{\beta}^{\varepsilon} \{\sigma_0\}$. *Example:* Suppose $\Sigma = \{0, 1, 10\}$ and $\alpha = 1$. Suppose, also, $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma$. Then Ω_N represents

the set of all GFA dispensations to the N professors. Since ν_{Ω_N} gives equal probability to each dispensation,

$$\begin{split} \nu_{\Omega_N} \text{ represents the GFA's first system for awarding grants.} \\ \text{Since } \beta = \mathrm{BP}_{\alpha}^{\varepsilon} = \mathrm{BP}_{1}^{\varepsilon}, & \text{we calculate: } \beta = (\ln 9)/10. \\ \text{More calculation gives: } (B_{\beta}^{\varepsilon}\{0\}, B_{\beta}^{\varepsilon}\{1\}, B_{\beta}^{\varepsilon}\{10\}) = \frac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}. \\ \text{Since } N \text{ is large, } \text{ by Theorem 22.3, we get:} \\ \nu_{\Omega_N} \{f \in \Omega_N \mid f_N = \sigma_0\} \approx B_{\beta}^{\varepsilon}\{\sigma_0\}. \\ \text{So if } I \text{ are the } N \text{the professore theorem under the first system} \end{split}$$

So, if I am the Nth professor, then, under the first system, my probability of receiving σ_0 dollars

is approximately equal to $B^{\varepsilon}_{\beta}\{\sigma_0\}.$

Thus Theorem 22.3 reproduces the result of $\S13$.

Example: Suppose $\Sigma = ([0..4] \times [0..4]) \setminus \{(4,4)\}.$ Suppose, also, $\alpha = 1$ and $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma_1 + \sigma_2$. Then Ω_N represents

the set of all state-distributions at the BUA. (See §20.) Since $\beta = BP_{\alpha}^{\varepsilon} = BP_{1}^{\varepsilon}$, we calculate:

$$\begin{split} \beta &\approx 1.0670, \quad \text{accurate to four decimal places.} \\ \mathbf{Let} \ M \in \mathbb{R}^{5 \times 5} \ \text{be the matrix defined by:} \quad M_{55} = 0 \quad \text{and} \\ \forall (i,j) \in (\begin{bmatrix} 1..5 \end{bmatrix} \times \begin{bmatrix} 1..5 \end{bmatrix}) \setminus \{ (5,5) \}, \quad M_{ij} = B^{\varepsilon}_{\beta}\{(i-1,j-1)\}. \\ 0.4345 \quad 0.1495 \quad 0.0514 \quad 0.0177 \quad 0.0061 \\ 0.1495 \quad 0.0514 \quad 0.0177 \quad 0.0061 \quad 0.0021 \\ 0.0514 \quad 0.0177 \quad 0.0061 \quad 0.0021 \quad 0.0007 \\ 0.0177 \quad 0.0061 \quad 0.0021 \quad 0.0007 \quad 0.0002 \\ 0.0061 \quad 0.0021 \quad 0.0007 \quad 0.0002 \quad 0 \\ \end{split}$$

all accurate to four decimal places.

(Thanks to C. Prouty for these calculations. See §28.) According to Theorem 22.3, this answers

the "precise mathematical problem" formulated near the end of §20. Since $B_{\beta}^{\varepsilon}\{(0,0)\} = M_{11} \approx 0.4345$, it is possible (cf. §15) to prove:

If N is sufficiently large, then, more than 99% of the time,

over 43% of the BUA professors have \$0 wealth.

23. ∞ -properness and $(-\infty)$ -properness

Recall (§2): the notations \mathbb{I}_f and f^*A .

DEFINITION 23.1. Let Σ be a set. Let $\varepsilon : \Sigma \to \mathbb{R}$.

By ε is **\neg \text{proper}**, we mean: $\forall t \in \mathbb{R}, \# \{ \sigma \in \Sigma \mid \varepsilon(\sigma) \leq t \} < \infty$.

That is, $\forall t \in \mathbb{R}, \ \#(-\infty; t]) < \infty.$

Note that, for any finite set Σ , for any $\varepsilon : \Sigma \to \mathbb{R}$, we have: ε is ∞ -proper.

THEOREM 23.2. Let Σ be a set. If $\exists \varepsilon : \Sigma \to \mathbb{R}$ s.t. ε is ∞ -proper, then Σ is countable.

The next result asserts that, for a nonempty set Σ , if $\varepsilon : \Sigma \to \mathbb{R}$ is ∞ -proper, then its image, \mathbb{I}_{ε} , has a minimal element, *i.e.*, min \mathbb{I}_{ε} exists.

THEOREM 23.3. Let Σ be a set. Let $\varepsilon : \Sigma \to \mathbb{R}$ be ∞ -proper. Assume: $\Sigma \neq \emptyset$. Then: $\exists t_0 \in \mathbb{I}_{\varepsilon}$ s.t., $\forall t \in \mathbb{I}_{\varepsilon}$, $t \ge t_0$.

THEOREM 23.4. Let Σ be a set. Let $\varepsilon : \Sigma \to \mathbb{R}$ be ∞ -proper. Then: \mathbb{I}_{ε} is bounded below and $\forall t \in \mathbb{I}_{\varepsilon}, \ \varepsilon^*\{t\}$ is finite.

The preceding three theorems are basic; we omit proofs. When ε is \mathbb{Z} -valued, the converse of Theorem 23.4 is also true:

THEOREM 23.5. Let Σ be a set. Let $\varepsilon : \Sigma \to \mathbb{Z}$. Then: $[\varepsilon \text{ is } \infty\text{-proper }]$ $\Leftrightarrow [(\mathbb{I}_{\varepsilon} \text{ is bounded below}) \& (\forall t \in \mathbb{I}_{\varepsilon}, \varepsilon^* \{t\} \text{ is finite })].$

The preceding is basic; we omit proof. The following two results are corollaries of Theorem 23.5:

THEOREM 23.6. Let Σ be a set. Let $\varepsilon : \Sigma \to \mathbb{Z}$ be injective. Then: $[\varepsilon \infty$ -proper] $\Leftrightarrow [\mathbb{I}_{\varepsilon} \text{ is bounded below }].$

THEOREM 23.7. Let $\Sigma \subseteq \mathbb{Z}$.

Define $\varepsilon: \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma$. Then: $[\varepsilon \infty$ -proper] $\Leftrightarrow [\Sigma \text{ is bounded below}].$

DEFINITION 23.8. Let Σ be a set. Let $\varepsilon : \Sigma \to \mathbb{R}$. By ε is $(-\infty)$ -proper, we mean: $\forall t \in \mathbb{R}, \ \#\{\sigma \in \Sigma \mid \varepsilon(\sigma) \ge t\} < \infty$. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Then: $(\varepsilon \text{ is } (-\infty)\text{-proper}) \Leftrightarrow (-\varepsilon \text{ is } \infty\text{-proper}).$

THEOREM 23.9. Let Σ be a finite set. Then: $\forall \varepsilon : \Sigma \to \mathbb{R}$, ε is both ∞ -proper and $(-\infty)$ -proper.

THEOREM 23.10. Let Σ be a set.

Assume: $\exists \varepsilon : \Sigma \to \mathbb{R}$ s.t. ε is both ∞ -proper and $(-\infty)$ -proper. Then: Σ is finite.

The preceding two theorems are basic; we omit proofs.

24. Boltzmann distributions on countable sets

In the next few sections,

we generalize our earlier work on Boltzmann distributions (§21) to allow for a countably infinite set of states.

DEFINITION 24.1. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. $\overline{\Delta_{\beta}^{\varepsilon}} := \sum_{\sigma \in \Sigma} \left[e^{-\beta \cdot (\varepsilon(\sigma))} \right] \in [0; \infty].$ Then: $\forall \text{nonempty set } \Sigma, \ \forall \varepsilon : \Sigma \to \mathbb{R}, \ \forall \beta \in \mathbb{R}, \ \Delta_{\beta}^{\varepsilon} > 0.$ We have: **DEFINITION 24.2.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Then the Delta-finite-set of ε is: $\left| \mathrm{DF}_{\varepsilon} \right| := \{ \beta \in \mathbb{R} \mid \Delta_{\beta}^{\varepsilon} < \infty \}.$ $\begin{array}{ll} \forall \text{finite set } \Sigma, & \forall \varepsilon : \Sigma \to \mathbb{R}, \\ \forall \text{finite set } \Sigma, & \forall \varepsilon : \Sigma \to \mathbb{R}, \end{array} \qquad \begin{array}{ll} \forall \beta \in \mathbb{R}, & \Delta_{\beta}^{\varepsilon} < \infty. \\ \text{DF}_{\varepsilon} = \mathbb{R}. \end{array}$ We have: Then: Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. $\forall \beta \in \mathbb{R}, \quad \Delta_{-\beta}^{-\varepsilon} = \Delta_{\beta}^{\varepsilon},$ we get: $DF_{-\varepsilon} = -DF_{\varepsilon}$. Since Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}, \xi \in \mathbb{R}$. $\forall \beta \in \mathbb{R}, \quad \Delta_{\beta}^{\varepsilon + \xi} = e^{-\beta \cdot \xi} \cdot \Delta_{\beta}^{\varepsilon}, \quad \text{we get:} \quad \mathrm{DF}_{\varepsilon + \xi} = \mathrm{DF}_{\varepsilon}.$ Since Recall (§9) the notations: \mathcal{M}_{Θ} , $\mathcal{F}\mathcal{M}_{\Theta}^{\times}$, \mathcal{P}_{Θ} , $\mathcal{N}(\mu)$. **DEFINITION 24.3.** Let Σ be a countable set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. Then $\left| \hat{B}_{\beta}^{\varepsilon} \right| \in \mathcal{M}_{\Sigma}$ is defined by: $\forall \sigma \in \Sigma, \quad \hat{B}_{\beta}^{\varepsilon} \{\sigma\} = e^{-\beta \cdot (\varepsilon(\sigma))}.$ Let Σ be a countable set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. we get: $\Delta_{\beta}^{\varepsilon} = \hat{B}_{\beta}^{\varepsilon}(\Sigma).$ Since $\Delta_{\beta}^{\varepsilon} = \sum_{\sigma \in \Sigma} [\hat{B}_{\beta}^{\varepsilon} \{\sigma\}],$

For any countable set Σ , for any $\varepsilon : \Sigma \to \mathbb{R}$, for any $\beta \in \mathbb{R}$,

$$(\Sigma \neq \emptyset \text{ and } \beta \in \mathrm{DF}_{\varepsilon}) \Leftrightarrow (0 < \Delta_{\beta}^{\varepsilon} < \infty) \Leftrightarrow (0 < \hat{B}_{\beta}^{\varepsilon}(\Sigma) < \infty) \Leftrightarrow (\hat{B}_{\beta}^{\varepsilon} \in \mathcal{FM}_{\Sigma}^{\times}).$$

DEFINITION 24.4. Let Σ be a countable set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. Assume: $0 < \Delta_{\beta}^{\varepsilon} < \infty$. Then: $B_{\beta}^{\varepsilon} := \mathcal{N}(\hat{B}_{\beta}^{\varepsilon}) \in \mathcal{P}_{\Sigma}$.

Let Σ be a countable set, $\varepsilon: \Sigma \to \mathbb{R}$.

If $DF_{\varepsilon} = \emptyset$, then, $\forall \beta \in \mathbb{R}$, since $\hat{B}_{\beta}^{\varepsilon}(\Sigma) = \Delta_{\beta}^{\varepsilon} = \infty$,

we see that $\hat{B}^{\varepsilon}_{\beta}$ cannot be normalized, *i.e.*, there is no B^{ε}_{β} . So, if $DF_{\varepsilon} = \emptyset$, then we have no Boltzmann distributions to study. So, going forward, we often focus on cases where $DF_{\varepsilon} \neq \emptyset$. If $\Sigma = \emptyset$, ε is the empty function, and there is nothing to say.

If Σ is nonempty and finite,

Then:

we already developed a satisfactory Boltzmann theory, in $\S{21}$. So, going forward, we often focus on cases where Σ is infinite.

Recall (§2): the notations \mathbb{I}_f and f^*A .

Let Σ be an infinite set, $\varepsilon : \Sigma \to \mathbb{R}$. Then: $\varepsilon^* \mathbb{R}$ = Σ. $(-\infty; 0]$ [] [0; ∞) = We have: $\mathbb{R}.$ $(\varepsilon^*(-\infty;0]) \bigcup (\varepsilon^*[0;\infty)) = \varepsilon^*\mathbb{R} = \Sigma,$ Since either $\varepsilon^*(-\infty; 0]$ is infinite or $\varepsilon^*[0; \infty)$ is infinite. we get: Assuming Σ is countable, the Boltzmann theory splits into these two cases; replacing ε with $-\varepsilon$ interchanges the two cases, so the theory in one case parallels the theory in the other. by Theorem 24.7 below, if $DF_{\varepsilon} \neq \emptyset$, Also, then only one of the two cases can happen. **THEOREM 24.5.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*[0;\infty)$ is infinite. Then: $DF_{\varepsilon} \subseteq (0; \infty).$ *Proof.* Given $\beta \in DF_{\varepsilon}$, want: $\beta \in (0; \infty)$. $\mathrm{DF}_{\varepsilon} \subseteq \mathbb{R},$ we get: $\beta \in \mathbb{R}$. Since Want: $\beta > 0$. Assume: $\beta \leq 0$. Want: Contradiction. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma).$ For all $\sigma \in \varepsilon^*[0;\infty)$, since $\varepsilon_{\sigma} = \varepsilon(\sigma) \in [0;\infty)$, we get: $\varepsilon_{\sigma} \ge 0$. So, since $\beta \leq 0$, we get:

So, since $\varepsilon^*[0;\infty)$ is infinite, we get: $\sum_{\sigma\in\varepsilon^*[0;\infty)]} [e^{-\beta\cdot\varepsilon_\sigma}] = \infty$. Since $\Delta_{\beta}^{\varepsilon} = \sum_{\sigma\in\Sigma} [e^{-\beta\cdot\varepsilon_\sigma}] \ge \sum_{\sigma\in\varepsilon^*[0;\infty)]} [e^{-\beta\cdot\varepsilon_\sigma}] = \infty$, we get: $\beta \notin \mathrm{DF}_{\varepsilon}$. Contradiction.

THEOREM 24.6. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*(-\infty; 0]$ is infinite. Then: $DF_{\varepsilon} \subseteq (-\infty; 0)$.

Proof. Since $(-\varepsilon)^*[0;\infty) = \varepsilon^*(-\infty;0]$, we get: $(-\varepsilon)^*[0;\infty)$ is infinite. Then, by Theorem 24.5, we get: $\mathrm{DF}_{-\varepsilon} \subseteq (0;\infty)$. Then: $\mathrm{DF}_{\varepsilon} = -\mathrm{DF}_{-\varepsilon} \subseteq -(0;\infty) = (-\infty;0)$. \Box

THEOREM 24.7. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*(-\infty; 0]$ and $\varepsilon^*[0; \infty)$ are both infinite. Then: $DF_{\varepsilon} = \emptyset$.

Proof. By Theorem 24.5,we get: $DF_{\varepsilon} \subseteq (0; \infty)$.By Theorem 24.6,we get: $DF_{\varepsilon} \subseteq (-\infty; 0)$.Since $DF_{\varepsilon} \subseteq (-\infty; 0) \bigcap (0; \infty) = \emptyset$,we get: $DF_{\varepsilon} = \emptyset$.

THEOREM 24.8. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $DF_{\varepsilon} \cap [0; \infty) \neq \emptyset$. Then: ε is ∞ -proper.

THEOREM 24.9. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $DF_{\varepsilon} \cap (-\infty; 0] \neq \emptyset$. Then: ε is $(-\infty)$ -proper. Proof. Since $-(DF_{\varepsilon} \cap (-\infty; 0]) \neq \emptyset$, we get: $DF_{-\varepsilon} \cap [0; \infty) \neq \emptyset$. Then, by Theorem 24.8, $-\varepsilon$ is ∞ -proper, and so ε is $(-\infty)$ -proper. \Box **THEOREM 24.10.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$.

Assume: $DF_{\varepsilon} \neq \emptyset$. Then: Σ is countable.

Proof. Since $(DF_{\varepsilon} \cap (-\infty; 0]) \cup (DF_{\varepsilon} \cap [0; \infty)) = DF_{\varepsilon} \neq \emptyset$, it follows that: either $\mathrm{DF}_{\varepsilon} \bigcap (-\infty; 0] \neq \emptyset$ or $\mathrm{DF}_{\varepsilon} \bigcap [0; \infty) \neq \emptyset$. Then, by Theorem 24.9 or Theorem 24.8, we get: either ε is $(-\infty)$ -proper ε is ∞ -proper. or either $-\varepsilon$ is ∞ -proper Then: ε is ∞ -proper. or In either case, by Theorem 23.2, we get: Σ is countable. **THEOREM 24.11.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $DF_{\varepsilon} \cap (-\infty; 0] \neq \emptyset \neq DF_{\varepsilon} \cap [0; \infty).$ Then: Σ is finite. *Proof.* By Theorem 24.8, we get: ε is ∞ -proper. By Theorem 24.9, we get: ε is $(-\infty)$ -proper. Then, by Theorem 23.10, we get: Σ is finite. **THEOREM 24.12.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*[0;\infty)$ is infinite and $DF_{\varepsilon} \neq \emptyset$. Then: ε is ∞ -proper. *Proof.* By Theorem 24.5, we have: $\mathrm{DF}_{\varepsilon} \subseteq (0; \infty).$ $DF_{\varepsilon} \subseteq (0; \infty) \subseteq [0; \infty)$, we get: Since $\mathrm{DF}_{\varepsilon} \bigcap [0; \infty) = \mathrm{DF}_{\varepsilon}.$ Since $DF_{\varepsilon} \bigcap [0; \infty) = DF_{\varepsilon} \neq \emptyset$, by Theorem 24.8, we get: ε is ∞ -proper. \Box **THEOREM 24.13.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*(-\infty; 0]$ is infinite and $DF_{\varepsilon} \neq \emptyset$. Then: ε is $(-\infty)$ -proper. *Proof.* Since $(-\varepsilon)^*[0;\infty) = \varepsilon^*(-\infty;0]$, we get: $(-\varepsilon)^*[0;\infty)$ is infinite. Since $DF_{-\varepsilon} = -DF_{\varepsilon}$, we get: $DF_{-\varepsilon} \neq \emptyset$. Then, by Theorem 24.12, $-\varepsilon$ is ∞ -proper, so ε is $(-\infty)$ -proper. **DEFINITION 24.14.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. Then, $\forall \rho \in [0; \infty)$, the $|\rho$ -exponent (β, ε) -absolute-sum is:
$$\begin{split} & \overline{\overline{\mathbf{X}}^{\rho}\mathbf{S}_{\beta}^{\varepsilon}} := \overline{\sum_{\sigma \in \Sigma} \left[\left| \varepsilon_{\sigma} \right|^{\rho} \cdot \left| e^{-\beta \cdot \varepsilon_{\sigma}} \right| \right]} \in [0; \infty]. \\ & \forall \rho \in [0..\infty), \quad if \quad \overline{\mathbf{X}}^{\rho}\mathbf{S}_{\beta}^{\varepsilon} < \infty, \end{split}$$
Also. then the ρ -exponent (β, ε) -sum is: $X^{\rho}S^{\varepsilon}_{\beta} := \sum_{\sigma \in \Sigma} [\varepsilon^{\rho}_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}}] \in \mathbb{R}.$ Let Σ be a set, $\varepsilon: \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$, $\rho \in [0..\infty)$.

If $\overline{\mathbf{X}}^{\rho}\mathbf{S}^{\varepsilon}_{\beta} < \infty$, then, by subadditivity of absolute value,

we get: $|\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\beta}| \leq \overline{\mathbf{X}}^{\rho}\mathbf{S}^{\varepsilon}_{\beta}.$

Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. Recall our convention (§2): $0^0 = 1$. Then: $\overline{X}^0 S^{\varepsilon}_{\beta} = \Delta^{\varepsilon}_{\beta}$. Also, if $\overline{X}^0 S^{\varepsilon}_{\beta} < \infty$, then $X^0 S^{\varepsilon}_{\beta} = \Delta^{\varepsilon}_{\beta}$.

THEOREM 24.15. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $DF_{\varepsilon} \neq \emptyset$ and \mathbb{I}_{ε} is bounded below. Let $\rho \ge 0$ be real. Let $\beta \in DF_{\varepsilon}$ and let $\gamma > \beta$ be real. Then: $\overline{X}^{\rho}S_{\gamma}^{\varepsilon} < \infty$.

We cannot replace " $\gamma > \beta$ " with " $\gamma \ge \beta$ "; see Theorem 24.17 below. *Proof.* Since \mathbb{I}_{ε} is bounded below, **choose** $t_0 \in \mathbb{R}$ s.t., $\forall \sigma \in \Sigma, \varepsilon(\sigma) \ge t_0$. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. Then: $\forall \sigma \in \Sigma, \varepsilon_{\sigma} \ge t_0$. Let $\delta := \gamma - \beta$. Then $\delta > 0$, so, as $t \to \infty$, $|t|^{\rho} \cdot e^{-\delta \cdot t} \to 0$.

So, since $t \mapsto |t|^{\rho} \cdot e^{-\delta \cdot t} : [t_0; \infty) \to \mathbb{R}$ is continuous,

by the Extreme Value Theorem, **choose** $M \in \mathbb{R}$ s.t.,

 $\forall \text{real } t \geq t_0, \qquad |t|^{\rho} \cdot e^{-\delta \cdot t} \leq M.$

THEOREM 24.16. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta, \rho \in \mathbb{R}$. Assume: $\rho \ge 0$, ε is ∞ -proper, $\overline{X}^{\rho}S^{\varepsilon}_{\beta} < \infty$. Then: $\beta \in DF_{\varepsilon}$.

The proof below shows that we can weaken the hypothesis

" ε is ∞ -proper" to "{ $\sigma \in \Sigma \mid \varepsilon(\sigma) \leq 1$ } is finite".

However, it cannot be dropped altogether; see Theorem 24.18 below.

$$\begin{array}{rcl} & \text{we get:} & \varepsilon_{\sigma} &= |\varepsilon_{\sigma}|.\\ \text{Since} & \forall \sigma \in \Sigma \backslash F, & 1 < \varepsilon_{\sigma} &= |\varepsilon_{\sigma}|,\\ \text{we get:} & \forall \sigma \in \Sigma \backslash F, & 1^{\rho} &\leqslant |\varepsilon_{\sigma}|^{\rho}.\\ \text{Then:} & \forall \sigma \in \Sigma \backslash F, & 1^{\rho} \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \leqslant |\varepsilon_{\sigma}|^{\rho} \cdot e^{-\beta \cdot \varepsilon_{\sigma}}.\\ \text{Then:} & \sum_{\sigma \in \Sigma \backslash F} \left[e^{-\beta \cdot \varepsilon_{\sigma}} \right] = \sum_{\sigma \in \Sigma \backslash F} \left[1^{\rho} \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \right] \leqslant \sum_{\sigma \in \Sigma \backslash F} \left[|\varepsilon_{\sigma}|^{\rho} \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \right] \\ & \leqslant \sum_{\sigma \in \Sigma} \left[|\varepsilon_{\sigma}|^{\rho} \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \right] = \overline{X}^{\rho} \mathbf{S}_{\beta}^{\varepsilon} < \infty. \end{array}$$

THEOREM 24.17. Let $\Sigma := [3..\infty)$. **Define** $\varepsilon: \Sigma \to \mathbb{R}$ by: $\forall k \in \Sigma, \ \varepsilon(k) = (\ln k) + 2 \cdot (\ln(\ln k)).$ Let $\beta := 1$, $\gamma := 1$, $\rho := 1$. Then: $\beta \in DF_{\varepsilon}$ and $\overline{X}^{\rho}S_{\gamma}^{\varepsilon} = \infty$. *Proof.* For all $k \in \Sigma$, let $\varepsilon_k := \varepsilon(k)$. Then: $\forall k \in [3..\infty), \qquad \varepsilon_k = (\ln k) + 2 \cdot (\ln(\ln k)).$ Since $\Delta_{\beta}^{\varepsilon} = \Delta_1^{\varepsilon} = \sum_{k \in \Sigma} [e^{-\varepsilon_k}] = \sum_{k=3}^{\infty} [e^{-\varepsilon_k}]$ $= \sum_{k=3}^{\infty} \left[\frac{1}{e^{\varepsilon_k}}\right] = \sum_{k=3}^{\infty} \left[\frac{1}{e^{(\ln k) + 2(\ln(\ln k))}}\right] = \sum_{k=3}^{\infty} \left[\frac{1}{k \cdot (\ln k)^2}\right] < \infty,$ we get: $\beta \in DF_{\varepsilon}$. It remains only to show: $\overline{X}^{\rho}S_{\gamma}^{\varepsilon} = \infty$.

We have: $\forall k \in [3..\infty), k > e$, so $\ln k > 1$, so $\ln(\ln k) > 0$. For all $k \in [3..\infty)$, since $\varepsilon_k = (\ln k) + 2 \cdot (\ln(\ln k)) > 1 + 2 \cdot 0 = 1 > 0$, we get: $|\varepsilon_k| = \varepsilon_k.$

Sind

$$\begin{aligned} \overline{\mathbf{X}}^{\rho} \mathbf{S}_{\gamma}^{\varepsilon} &= \overline{\mathbf{X}}^{1} \mathbf{S}_{1}^{\varepsilon} = \sum_{k \in \Sigma} \left[|\varepsilon_{k}| \cdot e^{-\varepsilon_{k}} \right] \\ &= \sum_{k=3}^{\infty} \left[|\varepsilon_{k}| \cdot e^{-\varepsilon_{k}} \right] \\ &= \sum_{k=3}^{\infty} \left[\varepsilon_{k} \cdot e^{-\varepsilon_{k}} \right] \\ &= \sum_{k=3}^{\infty} \left[\frac{\varepsilon_{k}}{e^{\varepsilon_{k}}} \right] = \sum_{k=3}^{\infty} \left[\frac{(\ln k) + 2 \cdot (\ln(\ln k))}{e^{(\ln k) + 2(\ln(\ln k))}} \right] \\ &= \sum_{k=3}^{\infty} \left[\frac{(\ln k) + 2 \cdot (\ln(\ln k))}{k \cdot (\ln k)^{2}} \right] \\ &\geq \sum_{k=3}^{\infty} \left[\frac{\ln k}{k \cdot (\ln k)^{2}} \right] \\ &= \sum_{k=3}^{\infty} \left[\frac{1}{k \cdot (\ln k)} \right] = \infty, \end{aligned}$$
we get: $\overline{\mathbf{X}}^{\rho} \mathbf{S}_{\gamma}^{\varepsilon} = \infty$

THEOREM 24.18. Let $\Sigma := \mathbb{N}$. **Define** $\varepsilon: \Sigma \to \mathbb{R}$ by: $\forall k \in \Sigma$, $\varepsilon(k) = 1/k^2$. Let $\beta := 1$, $\rho := 1$. Then: $\overline{\mathbf{X}}^{\rho} \mathbf{S}_{\beta}^{\varepsilon} < \infty$ and $\beta \notin \mathrm{DF}_{\varepsilon}$.

Proof. For all $k \in \Sigma$, let $\varepsilon_k := \varepsilon(k)$. Then: $\forall k \in \Sigma$, $\varepsilon_k = 1/k^2$. We have: $\forall k \in \mathbb{N}$, both $|\varepsilon_k| = 1/k^2$ and $-\varepsilon_k = -1/k^2$.

Since $\overline{\mathbf{X}}^{\rho}\mathbf{S}_{\beta}^{\varepsilon} = \overline{\mathbf{X}}^{1}\mathbf{S}_{1}^{\varepsilon} = \sum_{k\in\Sigma} \begin{bmatrix} |\varepsilon_{k}| \cdot e^{-\varepsilon_{k}} \end{bmatrix}$ $= \sum_{k=1}^{\infty} \begin{bmatrix} |\varepsilon_{k}| \cdot e^{-\varepsilon_{k}} \end{bmatrix}$ $= \sum_{k=1}^{\infty} \begin{bmatrix} (1/k^{2}) \cdot e^{-1/k^{2}} \end{bmatrix}$ $\leqslant \sum_{k=1}^{\infty} \begin{bmatrix} (1/k^{2}) \cdot & 1 \end{bmatrix}$ $= \sum_{k=1}^{\infty} \begin{bmatrix} 1/k^{2} \end{bmatrix} < \infty,$

it remains only to show: $\beta \notin DF_{\varepsilon}$ Want: $\Delta_{\beta}^{\varepsilon} = \infty$. We have: as $k \to \infty$, $e^{-1/k^2} \to 1$. Then: $\sum_{k=1}^{\infty} [e^{-1/k^2}] = \infty$. Then: $\Delta_{\beta}^{\varepsilon} = \Delta_1^{\varepsilon} = \sum_{k \in \Sigma} [e^{-\varepsilon_k}] = \sum_{k=1}^{\infty} [e^{-\varepsilon_k}] = \sum_{k=1}^{\infty} [e^{-1/k^2}] = \infty$. \Box

THEOREM 24.19. Let
$$\Sigma$$
 be a set, $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $DF_{\varepsilon} \neq \emptyset$ and \mathbb{I}_{ε} is bounded below. Let $\rho \ge 0$ be real.
Let $\beta_0 := \inf DF_{\varepsilon}$ and let $\gamma \in (\beta_0; \infty)$. Then: $\overline{X}^{\rho} S_{\gamma}^{\varepsilon} < \infty$.

Proof. Since $\gamma > \beta_0 = \inf DF_{\varepsilon}$, **choose** $\beta \in DF_{\varepsilon}$ s.t. $\gamma > \beta$. Then, by Theorem 24.15, we get: $\overline{X}^{\rho}S_{\gamma}^{\varepsilon} < \infty$. \Box

THEOREM 24.20. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $DF_{\varepsilon} \neq \emptyset$ and \mathbb{I}_{ε} is bounded below. Let $\beta_0 := \inf DF_{\varepsilon}$ and let $\gamma \in (\beta_0; \infty)$. Then: $\gamma \in DF_{\varepsilon}$.

Proof. By Theorem 24.19, we have: $\overline{X}^0 S_{\gamma}^{\varepsilon} < \infty$. Since $\Delta_{\gamma}^{\varepsilon} = \overline{X}^0 S_{\gamma}^{\varepsilon} < \infty$, we get: $\gamma \in DF_{\varepsilon}$. \Box

THEOREM 24.21. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*[0;\infty)$ is infinite and $\mathrm{DF}_{\varepsilon} \neq \emptyset$. Let $\beta_0 := \inf \mathrm{DF}_{\varepsilon}$. Then: $0 \leq \beta_0 < \infty$ and $(\beta_0;\infty) \subseteq \mathrm{DF}_{\varepsilon}$.

Proof. By Theorem 24.5, $DF_{\varepsilon} \subseteq (0; \infty)$. Then: $\inf DF_{\varepsilon} \ge \inf(0; \infty)$. Since $DF_{\varepsilon} \ne \emptyset$, we get: $\inf DF_{\varepsilon} < \infty$.

Since $\beta_0 = \inf DF_{\varepsilon} \ge \inf(0; \infty) = 0$ and since $\beta_0 = \inf DF_{\varepsilon} < \infty$, we get: $0 \le \beta_0 < \infty$.

It remains to show: $(\beta_0; \infty) \subseteq DF_{\varepsilon}$. Given $\gamma \in (\beta_0; \infty)$, want: $\gamma \in DF_{\varepsilon}$.

By Theorem 24.12, we have: ε is ∞ -proper. Then, by Theorem 23.4, we have: \mathbb{I}_{ε} is bounded below. Then, by Theorem 24.20, we have: $\gamma \in DF_{\varepsilon}$.

THEOREM 24.22. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*[0; \infty)$ is infinite and $\mathrm{DF}_{\varepsilon} \neq \emptyset$. Let $\beta_0 := \inf \mathrm{DF}_{\varepsilon}$. Then either $(\mathrm{DF}_{\varepsilon} = [\beta_0; \infty) \text{ and } 0 < \beta_0 < \infty)$ or $(\mathrm{DF}_{\varepsilon} = (\beta_0; \infty) \text{ and } 0 \leq \beta_0 < \infty)$.

 $\begin{array}{lll} \textit{Proof. By Theorem 24.21, we get:} & 0 \leqslant \beta_0 < \infty \text{ and } (\beta_0; \infty) \subseteq \mathrm{DF}_{\varepsilon}.\\ \text{Since } \beta_0 = \inf \mathrm{DF}_{\varepsilon}, & \text{we get:} & \mathrm{DF}_{\varepsilon} \subseteq [\beta_0; \infty).\\ \text{By Theorem 24.5, we get:} & \mathrm{DF}_{\varepsilon} \subseteq (0; \infty). \end{array}$

 $\beta_0 \in \mathrm{DF}_{\varepsilon}.$ **Want:** $DF_{\varepsilon} = [\beta_0; \infty)$ and $0 < \beta_0 < \infty$. Case 1: $(\beta_0; \infty) \subseteq \mathrm{DF}_{\varepsilon}$ and $\mathrm{DF}_{\varepsilon} \subseteq [\beta_0; \infty)$ and $\mathrm{DF}_{\varepsilon} \subseteq (0; \infty)$. Recall: $\beta_0 \in \mathrm{DF}_{\varepsilon}$ and $(\beta_0; \infty) \subseteq \mathrm{DF}_{\varepsilon}$, Since we get: $\{\beta_0\} \mid \mathsf{J}(\beta_0; \infty) \subseteq \mathrm{DF}_{\varepsilon}.$ $[\beta_0; \infty) = \{\beta_0\} \bigcup (\beta_0; \infty) \subseteq \mathrm{DF}_{\varepsilon} \text{ and since } \mathrm{DF}_{\varepsilon} \subseteq [\beta_0; \infty),$ Since we get: $DF_{\varepsilon} = [\beta_0; \infty).$ $0 < \beta_0 < \infty$. It remains only to show: $\beta_0 < \infty$. Recall: $0 \leq \beta_0 < \infty$. Then: It remains only to show: $0 < \beta_0$. $0 < \beta_0$. Since $\beta_0 \in [\beta_0; \infty) = \mathrm{DF}_{\varepsilon} \subseteq (0; \infty),$ we get: End of Case 1.

Case 2: $\beta_0 \notin DF_{\varepsilon}$. Want: $DF_{\varepsilon} = (\beta_0; \infty)$ and $0 \leq \beta_0 < \infty$. $0 \leq \beta_0 < \infty$. Recall: $\mathrm{DF}_{\varepsilon} = (\beta_0; \infty).$ It remains only to show: Recall: $\mathrm{DF}_{\varepsilon} \subseteq [\beta_0; \infty),$ $\beta_0 \notin \mathrm{DF}_{\varepsilon}$ and $\mathrm{DF}_{\varepsilon} \subseteq [\beta_0; \infty)$, Since $\mathrm{DF}_{\varepsilon} \subseteq [\beta_0; \infty) \setminus \{\beta_0\}.$ Recall: $(\beta_0; \infty) \subseteq \mathrm{DF}_{\varepsilon}$. we get: $\mathrm{DF}_{\varepsilon} \subseteq [\beta_0; \infty) \setminus \{\beta_0\} = (\beta_0; \infty) \text{ and } (\beta_0; \infty) \subseteq \mathrm{DF}_{\varepsilon},$ Since $DF_{\varepsilon} = (\beta_0; \infty).$ we get:

End of Case 2.

THEOREM 24.23. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*(-\infty; 0]$ is infinite and $DF_{\varepsilon} \neq \emptyset$. Let $\beta_0 := -\sup DF_{\varepsilon}$. Then one of the following holds: Either $(DF_{\varepsilon} = (-\infty; -\beta_0] \text{ and } 0 < \beta_0 < \infty)$ or $(DF_{\varepsilon} = (-\infty; -\beta_0) \text{ and } 0 \leq \beta_0 < \infty)$. Proof. Since $(-\varepsilon)^*[0; \infty)$ is infinite and $DF_{-\varepsilon} \neq \emptyset$ and $\beta_0 = \inf DF_{-\varepsilon}$, by Theorem 24.22, we get: either $(DF_{-\varepsilon} = [\beta_0; \infty) \text{ and } 0 < \beta_0 < \infty)$ or $(DF_{-\varepsilon} = (\beta_0; \infty) \text{ and } 0 \leq \beta_0 < \infty)$. Then: either $(DF_{\varepsilon} = (-\infty; -\beta_0] \text{ and } 0 < \beta_0 < \infty)$.

THEOREM 24.24. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $DF_{\varepsilon} \neq \emptyset$. Then one of the following holds:

(i) $DF_{\varepsilon} = \mathbb{R}$ (*ii*) $\exists real \ \beta_0 \ge 0$ s.t. $DF_{\varepsilon} = (\beta_0; \infty)$ or(*ii*') $\exists real \ \beta_0 > 0$ s.t. $DF_{\varepsilon} = [\beta_0; \infty)$ or(iii) $\exists real \ \beta_0 \ge 0$ s.t. $DF_{\varepsilon} = (-\infty; -\beta_0)$ or(*iii*') $\exists real \ \beta_0 > 0$ s.t. $DF_{\varepsilon} = (-\infty; -\beta_0].$ orBelow, in each of Theorem 24.28, Theorem 24.30, Theorem 24.32, we give examples of ∞ -proper $\varepsilon : \Sigma \to \mathbb{Z}$ such that $DF_{\varepsilon} = \emptyset, DF_{\varepsilon} = (\beta_0; \infty), DF_{\varepsilon} = [\beta_0; \infty), respectively;$ $-\varepsilon$ is $(-\infty)$ -proper it follows that and that $DF_{-\varepsilon} = \emptyset$, $DF_{-\varepsilon} = (-\infty; -\beta_0)$, $DF_{-\varepsilon} = (-\infty; -\beta_0]$, respectively. *Proof.* Since $\varepsilon : \Sigma \to \mathbb{R}$, we get: $\varepsilon^* \mathbb{R} = \Sigma$. Since $(-\infty; 0] \bigcup [0; \infty) = \mathbb{R}$, we get: $\varepsilon^*(-\infty; 0] \bigcup \varepsilon^*[0; \infty) = \varepsilon^* \mathbb{R}$. In case $\#\Sigma < \infty$, we get: (i) holds. We therefore assume $\#\Sigma = \infty$. (ii) or (ii') or (iii) or (iii') holds. Want: $\varepsilon^*(-\infty;0] \bigcup \varepsilon^*[0;\infty) = \varepsilon^* \mathbb{R} = \Sigma,$ Because and because Σ is infinite, we get: either $\varepsilon^*(-\infty; 0]$ is infinite or $\varepsilon^*[0;\infty)$ is infinite. Then, by Theorem 24.23 or Theorem 24.22, we get: either (iii) or (iii') holds (ii) or (ii') holds. or Then: (ii) or (ii') or (iii) or (iii') holds. **THEOREM 24.25.** Let $n_1, n_2, ... \in [0..\infty)$. Let $\Sigma := \{(k, j) \in \mathbb{N} \times \mathbb{N} \mid j \leq n_k\}.$ **Define** $\varepsilon : \Sigma \to [0..\infty)$ by: $\forall (k,j) \in \Sigma, \quad \varepsilon(k,j) = k-1.$ $#(\varepsilon^*[k-1;k)) = n_k.$ Then: $\forall k \in \mathbb{N}.$ *Proof.* Given $k \in \mathbb{N}$, want: $\#(\varepsilon^*[k-1;k)) = n_k$. Since $\varepsilon^*[k-1;k) = \{(\ell,j) \in \Sigma \mid \varepsilon(\ell,j) \in [k-1;k)\}$ $= \{(\ell, j) \in \Sigma \mid \ell - 1 \in [k - 1; k)\}$ $= \{ (\ell, j) \in \Sigma \mid \ell - 1 = k - 1 \}$ $= \{ (\ell, j) \in \Sigma \mid \ell = k \}$ $= \{ (\ell, j) \in \mathbb{N} \times \mathbb{N} \mid \ell = k , \ j \leq n_{\ell} \}$ $= \{(\ell, j) \in \mathbb{N} \times \mathbb{N} \mid \ell = k, \ j \leq n_k\}$ $= \{ (k, 1), \ldots, (k, n_k) \},\$

we get: $\#(\varepsilon^*[k-1;k)) = n_k.$

THEOREM 24.26. Let Σ be a set, $\varepsilon : \Sigma \to [0; \infty)$.

For all $k \in \mathbb{N}$, let $n_k := \#(\varepsilon^*[k-1;k))$. Let $\beta \in [0;\infty)$. Then: $(\beta \in \mathrm{DF}_{\varepsilon}) \Leftrightarrow (\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty)$.

Proof. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$.

Assume: $\beta \in DF_{\varepsilon}$. Want: $\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty$. Proof of \Rightarrow : Since $\beta \in DF_{\varepsilon}$, we get: $\Delta_{\beta}^{\varepsilon} < \infty$. Because Σ is the disjoint union, over k = 1 to ∞ , of $\varepsilon^*[k-1;k)$, we get: $\sum_{\sigma \in \Sigma} [e^{-\beta \cdot \varepsilon_{\sigma}}] = \sum_{k=1}^{\infty} \sum_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot \varepsilon_{\sigma}}].$ For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1;k)$, since $\varepsilon_{\sigma} = \varepsilon(\sigma) \in [k-1;k)$, we have: k > ε_{σ} . Since $\beta \in [0; \infty)$, we get: $-\beta \leq 0$. For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1;k)$, we have: $-\beta \cdot k \leq -\beta \cdot \varepsilon_{\sigma}$. For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1;k)$, we have: $e^{-\beta \cdot \varepsilon_\sigma}$.
$$\begin{split} \sum_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot k}] &\leq \sum_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot \varepsilon_\sigma}].\\ \sum_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot k}] &= n_k e^{-\beta \cdot k}.\\ n_k e^{-\beta \cdot k} &\leq \sum_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot \varepsilon_\sigma}].\\ \sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] &\leq \sum_{k=1}^{\infty} \sum_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot \varepsilon_\sigma}]\\ &= \sum_{\sigma \in \Sigma} [e^{-\beta \cdot \varepsilon_\sigma}] = \Delta_{\beta}^{\varepsilon} < \infty. \end{split}$$
Then: $\forall k \in \mathbb{N}$, Also, $\forall k \in \mathbb{N}$, Then: $\forall k \in \mathbb{N}$, Then:

End of proof of \Rightarrow .

$$= e^{\beta} \cdot \sum_{\sigma \in \Sigma} \begin{bmatrix} e^{-\beta \cdot (\varepsilon_{\sigma} + 1)} \end{bmatrix}$$

$$= e^{\beta} \cdot \sum_{k=1}^{\infty} \sum_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot (\varepsilon_{\sigma} + 1)}]$$

$$\leqslant e^{\beta} \cdot \sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty, \qquad \text{we get:} \quad \beta \in \mathrm{DF}_{\varepsilon}.$$

End of proof of \Leftarrow .

End of proof of

THEOREM 24.27. Let Σ be a set, $\varepsilon : \Sigma \to [0; \infty)$. For all $k \in \mathbb{N}$, let $n_k := \#(\varepsilon^*[k-1;k))$. $\forall k \in \mathbb{N}, \ n_k \ge e^{k^2}.$ Then: $\mathrm{DF}_{\varepsilon} = \emptyset.$ Assume: Proof. Since $\forall k \in \mathbb{N}$, $n_k \ge e^{k^2} > 1$, we get: $\sum_{k=1}^{\infty} n_k = \infty$. Since $\#(\varepsilon^*[0;\infty)) = \sum_{k=1}^{\infty} [\#(\varepsilon^*[k-1;k))] = \sum_{k=1}^{\infty} n_k = \infty$, it follows, from Theorem 24.5, that: $DF_{\varepsilon} \subseteq (0; \infty).$ It therefore suffices to show: $\forall \beta \in (0; \infty)$, $\beta \notin \mathrm{DF}_{\varepsilon}.$ **Given** $\beta \in (0; \infty)$, want: $\beta \notin DF_{\varepsilon}$. Since, as $k \to \infty$, $e^{k^2 - \beta \cdot k} \to \infty$, we get: $\sum_{k=1}^{\infty} [e^{k^2 - \beta \cdot k}] = \infty$. Since $\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] \ge \sum_{k=1}^{\infty} [e^{k^2} e^{-\beta \cdot k}] = \sum_{k=1}^{\infty} [e^{k^2 - \beta \cdot k}] = \infty$, and since $\beta \in (0; \infty) \subseteq [0; \infty)$, by Theorem 24.26, we get: $\beta \notin DF_{\varepsilon}$. Recall $(\S 2)$: $\forall t \in \mathbb{R}, |t|$ denotes the floor of t. **THEOREM 24.28.** For all $k \in \mathbb{N}$, let $n_k := |e^{k^2} + 1|$. Let $\Sigma := \{(k, j) \in \mathbb{N} \times \mathbb{N} \mid j \leq n_k\}.$ **Define** $\varepsilon : \Sigma \to [0..\infty)$ by: $\forall (k,j) \in \Sigma, \quad \varepsilon(k,j) = k-1.$ Then: $DF_{\varepsilon} = \emptyset$. We have: $\forall k \in \mathbb{N}, \ n_k \ge e^{k^2}.$ Proof. By Theorem 24.25, we get: $\forall k \in \mathbb{N}, \ \#(\varepsilon^*[k-1;k)) = n_k.$ Then, by Theorem 24.27, we get: $DF_{\varepsilon} = \emptyset$. **THEOREM 24.29.** Let Σ be a set, $\varepsilon : \Sigma \to [0; \infty)$. For all $k \in \mathbb{N}$, let $n_k := \#(\varepsilon^*[k-1;k))$. Let $\beta_0 \in [0;\infty)$. as $k \to \infty$, $n_k e^{-\beta_0 \cdot k} \to 1$. Then: $\mathrm{DF}_{\varepsilon} = (\beta_0; \infty)$. Assume: *Proof.* Since as $k \to \infty$, $n_k e^{-\beta_0 \cdot k} \to 1$, we get: $#\{k \in \mathbb{N} \mid n_k e^{-\beta_0 \cdot k} = 0\} < \infty.$ $#\{k \in \mathbb{N} \mid n_k = 0\} < \infty.$ Then: $\begin{aligned} &\#\{k \in \mathbb{N} \mid n_k \ge 1\} = \infty, \quad \text{and so} \quad \sum_{k=1}^{\infty} n_k = \infty. \\ &\#(\varepsilon^*[0;\infty)) = \sum_{k=1}^{\infty} \left[\#(\varepsilon^*[k-1;k)) \right] = \sum_{k=1}^{\infty} n_k = \infty, \end{aligned}$ Then Since it follows, from Theorem 24.5, that: $DF_{\varepsilon} \subseteq (0; \infty)$. Since $DF_{\varepsilon} \subseteq (0; \infty) \subseteq [0; \infty)$, we get: $DF_{\varepsilon} \bigcap [0; \infty) = DF_{\varepsilon}$. Since $\beta_0 \in [0; \infty)$, we get: $(\beta_0; \infty) \subseteq (0; \infty)$.
Since $(\beta_0; \infty) \subseteq (0; \infty) \subseteq [0; \infty)$, we get: $(\beta_0; \infty) \bigcap [0; \infty) = (\beta_0; \infty)$. We have: $\forall \beta \in \mathbb{R}, \forall k \in \mathbb{N}, [n_k e^{-\beta \cdot k}] / [e^{-(\beta - \beta_0) \cdot k}] = n_k e^{-\beta_0 \cdot k}$ $n_k e^{-\beta_0 \cdot k} \to 1.$ as $k \to \infty$, By hypothesis, as $k \to \infty$, $n_{k} \to \infty$ as $k \to \infty$, $[n_{k}e^{-\beta \cdot k}] / [e^{-(\beta - \beta_{0}) \cdot k}] \to 1$. $(\sum_{k=1}^{\infty} [n_{k}e^{-\beta \cdot k}] < \infty) \Leftrightarrow (\sum_{k=1}^{\infty} [e^{-(\beta - \beta_{0}) \cdot k}] < \infty).$ $(\beta \in (\beta_{0}; \infty)) \Leftrightarrow (\sum_{k=1}^{\infty} [e^{-(\beta - \beta_{0}) \cdot k}] < \infty).$ Then: $\forall \beta \in \mathbb{R}$, Then: $\forall \beta \in \mathbb{R}$, Also, $\forall \beta \in \mathbb{R}$, $\left(\sum_{k=1}^{\infty} \left[n_k e^{-\beta \cdot k}\right] < \infty\right) \Leftrightarrow \left(\beta \in (\beta_0; \infty)\right).$ Then: $\forall \beta \in \mathbb{R}$, Then, by Theorem 24.26, $(\beta \in \mathrm{DF}_{\varepsilon}) \Leftrightarrow (\beta \in (\beta_0; \infty)).$ $\forall \beta \in [0; \infty),$ $DF_{\varepsilon} \bigcap [0; \infty) = (\beta_0; \infty) \bigcap [0; \infty).$ Then: $\mathrm{DF}_{\varepsilon} = \mathrm{DF}_{\varepsilon} \bigcap [0; \infty) = (\beta_0; \infty) \bigcap [0; \infty) = (\beta_0; \infty).$ Then: **THEOREM 24.30.** Let $\beta_0 \in [0; \infty)$. For all $k \in \mathbb{N}$, let $n_k := |e^{\beta_0 \cdot k}|$. Let $\Sigma := \{(k, j) \in \mathbb{N} \times \mathbb{N} \mid j \leq n_k\}.$ **Define** $\varepsilon : \Sigma \to [0..\infty)$ by: $\forall (k, j) \in \Sigma, \quad \varepsilon(k, j) = k - 1.$ Then: $DF_{\varepsilon} = (\beta_0; \infty).$ *Proof.* We have: as $k \to \infty$, $n_k e^{-\beta_0 \cdot k} \to 1$. By Theorem 24.25, we get: $\forall k \in \mathbb{N}, \ \#(\varepsilon^*[k-1;k)) = n_k.$ Then, by Theorem 24.29, we get: $DF_{\varepsilon} = (\beta_0; \infty)$. **THEOREM 24.31.** Let Σ be a set, $\varepsilon: \Sigma \to [0; \infty)$. For all $k \in \mathbb{N}$, let $n_k := \#(\varepsilon^*[k-1;k))$. Let $p \in (1; \infty)$, $\beta_0 \in (0; \infty)$. as $k \to \infty$, $k^p n_k e^{-\beta_0 \cdot k} \to 1$. Then: $DF_{\varepsilon} = [\beta_0; \infty)$. Assume: Proof. Since, as $k \to \infty$, $k^p n_k e^{-\beta_0 \cdot k} \to 1$, we get: $#\{k \in \mathbb{N} \mid k^p n_k e^{-\beta_0 \cdot k} = 0\} < \infty.$ $#\{k \in \mathbb{N} \mid n_k = 0\} < \infty.$ Then $#\{k \in \mathbb{N} \mid n_k \ge 1\} = \infty, \text{ and so } \sum_{k=1}^{\infty} n_k = \infty.$ $#(\varepsilon^*[0;\infty)) = \sum_{k=1}^{\infty} [\#(\varepsilon^*[k-1;k))] = \sum_{k=1}^{\infty} n_k = \infty,$ Then Since it follows, from Theorem 24.5, that: $DF_{\varepsilon} \subseteq (0; \infty)$. Since $DF_{\varepsilon} \subseteq (0; \infty) \subseteq [0; \infty)$, we get: $DF_{\varepsilon} \bigcap [0; \infty) = DF_{\varepsilon}$. Since $\beta_0 \in (0; \infty)$, we get: $[\beta_0; \infty) \subseteq (0; \infty)$. Since $[\beta_0; \infty) \subseteq (0; \infty) \subseteq [0; \infty)$, we get: $[\beta_0; \infty) \bigcap [0; \infty) = [\beta_0; \infty)$. We have: $\forall \beta \in \mathbb{R}, \forall k \in \mathbb{N}, [n_k e^{-\beta \cdot k}]/[k^{-p} e^{-(\beta - \beta_0) \cdot k}] = k^p n_k e^{-\beta_0 \cdot k}$. $k^p n_k e^{-\beta_0 \cdot k} \to 1.$ By hypothesis, as $k \to \infty$, Then: $\forall \beta \in \mathbb{R}$, as $k \to \infty$, $[n_k e^{-\beta \cdot k}]/[k^{-p} e^{-(\beta - \beta_0) \cdot k}] \to 1$. Then: $\forall \beta \in \mathbb{R}$, $(\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty) \Leftrightarrow (\sum_{k=1}^{\infty} [k^{-p} e^{-(\beta - \beta_0) \cdot k}] < \infty).$ Also, since $p \in (1; \infty)$, we get:

 $(\beta \in [\beta_0; \infty)) \Leftrightarrow (\sum_{k=1}^{\infty} [k^{-p} e^{-(\beta - \beta_0) \cdot k}] < \infty).$ $(\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty) \Leftrightarrow (\beta \in [\beta_0; \infty)).$ $\forall \beta \in \mathbb{R},$ Then: $\forall \beta \in \mathbb{R}$, Then, by Theorem 24.26, $(\beta \in \mathrm{DF}_{\varepsilon}) \Leftrightarrow (\beta \in [\beta_0; \infty)).$ $\forall \beta \in [0; \infty),$ $DF_{\varepsilon} \bigcap [0; \infty) = [\beta_0; \infty) \bigcap [0; \infty).$ Then: Then: $DF_{\varepsilon} = DF_{\varepsilon} \bigcap [0; \infty) = [\beta_0; \infty) \bigcap [0; \infty) = [\beta_0; \infty).$ THEOREM 24.32. Let $\beta_0 \in (0; \infty)$. For all $k \in \mathbb{N}$, let $n_k := \lfloor k^{-2} e^{\beta_0 \cdot k} \rfloor$. Let $\Sigma := \{(k, j) \in \mathbb{N} \times \mathbb{N} \mid j \leq n_k\}.$ **Define** $\varepsilon : \Sigma \to [0..\infty)$ by: $\forall (k,j) \in \Sigma, \quad \varepsilon(k,j) = k-1.$ Then: $DF_{\varepsilon} = [\beta_0; \infty).$ *Proof.* We have: as $k \to \infty$, $k^2 n_k e^{-\beta_0 \cdot k} \to 1$. By Theorem 24.25, we get: $\forall k \in \mathbb{N}, \ \#(\varepsilon^*[k-1;k)) = n_k.$ Then, by Theorem 24.31, we get: $DF_{\varepsilon} = [\beta_0; \infty)$. Let Σ be an infinite set. Let $\varepsilon: \Sigma \to [0; \infty)$. For all $k \in \mathbb{N}$, let $n_k := \#(\varepsilon^*[k-1;k))$. In many applications, the sequence n_1, n_2, \ldots is subexponential. By the next theorem, whenever that happens, we get: $DF_{\varepsilon} = (0; \infty)$. **THEOREM 24.33.** Let Σ be an infinite set, $\varepsilon: \Sigma \to [0; \infty)$. For all $k \in \mathbb{N}$, let $n_k := \#(\varepsilon^*[k-1;k))$. Assume: $\forall \beta \in (0; \infty), as k \to \infty, n_k e^{-\beta \cdot k} \to 0.$ Then: $DF_{\varepsilon} = (0; \infty)$. *Proof.* Since $\varepsilon : \Sigma \to [0; \infty)$, we get: $\varepsilon^*[0;\infty) = \Sigma.$ So, since Σ is infinite, we get: $\varepsilon^*[0;\infty)$ is infinite. It follows, from Theorem 24.5, that: $DF_{\varepsilon} \subseteq (0; \infty)$. Want: $(0;\infty)$ \subseteq DF_e. **Given** $\beta \in (0; \infty)$, want: $\beta \in DF_{\varepsilon}$. Since $\beta \in (0, \infty) \subseteq [0, \infty)$, by Theorem 24.26, it suffices to show: $\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty$. Since $\beta \in (0; \infty)$, we get: $\beta' \in (0; \infty)$. Let $\beta' := \beta/2$. as $k \to \infty$, $n_k e^{-\beta' \cdot k} \to 0$. Then, by hypothesis, we have: $\{n_k e^{-\beta' \cdot k} \mid k \in \mathbb{N}\}$ is bounded. It follows that: s.t., $\forall k \in \mathbb{N}, \ n_k e^{-\beta' \cdot k} \leq M.$ Choose $M \in \mathbb{R}$ Since $\beta' \in (0; \infty)$, it follows that $1 - e^{-\beta'} > 0$ $e^{-\beta'} + e^{-2\beta'} + e^{-3\beta'} + \cdots = e^{-\beta'}/(1 - e^{-\beta'}).$ and that $e^{-\beta'} + e^{-2\beta'} + e^{-3\beta'} + \cdots < \infty.$ Then: $M \cdot (e^{-\beta'} + e^{-2\beta'} + e^{-3\beta'} + \cdots) < \infty.$ Then:

Then:
$$\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] = \sum_{k=1}^{\infty} [n_k e^{-2\beta' \cdot k}]$$
$$= \sum_{k=1}^{\infty} [(n_k e^{-\beta' \cdot k}) \cdot e^{-\beta' \cdot k}] \leq \sum_{k=1}^{\infty} [M e^{-\beta' \cdot k}] = M \cdot \sum_{k=1}^{\infty} [e^{-\beta' \cdot k}]$$
$$= M \cdot (e^{-\beta'} + e^{-2\beta'} + e^{-3\beta'} + \cdots) < \infty.$$

The next theorem is a corollary of Theorem 24.33.

THEOREM 24.34. Let Σ be an infinite set, $\varepsilon : \Sigma \to [0..\infty)$. Assume: ε is injective. Then: $DF_{\varepsilon} = (0; \infty)$.

Example: Let $\Sigma := [0..\infty)$. Define $\varepsilon : \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma, \varepsilon(\sigma) = \sigma$. Then, $\forall k \in \mathbb{N}, \quad \varepsilon^*[k-1;k] = \{k-1\}, \text{ and so } \#(\varepsilon^*[k-1;k]) = 1$. Then, by Theorem 24.34, we get: $\mathrm{DF}_{\varepsilon} = (0;\infty)$.

DEFINITION 24.35. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. *Then:* IDF_{ε} *denotes the interior in* \mathbb{R} *of* DF_{ε} .

THEOREM 24.36. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $DF_{\varepsilon} \neq \emptyset$. Then one of the following holds:

(i) $IDF_{\varepsilon} = \mathbb{R}$ or (ii) $\exists \beta_0 \in [0; \infty)$ s.t. $IDF_{\varepsilon} = (\beta_0; \infty)$ or (iii) $\exists \beta_0 \in [0; \infty)$ s.t. $IDF_{\varepsilon} = (-\infty; -\beta_0)$.

The preceding theorem is a corollary of Theorem 24.24.

THEOREM 24.37. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Then: $(DF_{\varepsilon} = \emptyset) \iff (IDF_{\varepsilon} = \emptyset).$ *Proof.* Since $IDF_{\varepsilon} \subseteq DF_{\varepsilon}$, we get: $(DF_{\varepsilon} = \emptyset)$ \Rightarrow (IDF $_{\varepsilon} = \emptyset$). Want: $(DF_{\varepsilon} \neq \emptyset)$ \Rightarrow (IDF_{\varepsilon} $\neq \emptyset$). Assume $DF_{\varepsilon} \neq \emptyset$. Want: $IDF_{\varepsilon} \neq \emptyset$. By Theorem 24.36, one of the following is true: (i) $IDF_{\varepsilon} = \mathbb{R}$ (ii) $\exists \beta_0 \in [0; \infty)$ s.t. $\text{IDF}_{\varepsilon} = (\beta_0; \infty)$ or (iii) $\exists \beta_0 \in [0; \infty)$ s.t. $IDF_{\varepsilon} = (-\infty; -\beta_0).$ or $IDF_{\varepsilon} \neq \emptyset$. Then:

THEOREM 24.38. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Then: $(IDF_{\varepsilon} = \mathbb{R}) \Leftrightarrow (DF_{\varepsilon} = \mathbb{R}) \Leftrightarrow (\Sigma \text{ is finite }).$

Proof. Since
$$IDF_{\varepsilon} \subseteq DF_{\varepsilon} \subseteq \mathbb{R}$$
, we get:
($IDF_{\varepsilon} = \mathbb{R}$) \Rightarrow ($DF_{\varepsilon} = \mathbb{R}$).
By Theorem 24.11, ($DF_{\varepsilon} = \mathbb{R}$) \Rightarrow (Σ is finite).

It remains to show: (Σ is finite) $(IDF_{\varepsilon} = \mathbb{R}).$ \Rightarrow Want: $IDF_{\varepsilon} = \mathbb{R}$. Assume: Σ is finite. Since the interior in \mathbb{R} of \mathbb{R} is equal to \mathbb{R} , it suffices to show: $DF_{\varepsilon} = \mathbb{R}$. Since $DF_{\varepsilon} \subseteq \mathbb{R}$, want: $\mathbb{R} \subseteq DF_{\varepsilon}$. Given $\beta \in \mathbb{R}$, want: $\beta \in DF_{\varepsilon}$. Since Σ is finite, we get: $\sum_{\sigma \in \Sigma} [e^{-\beta \cdot (\varepsilon(\sigma))}] < \infty$. Since $\Delta_{\beta}^{\varepsilon} = \sum_{\sigma \in \Sigma} [e^{-\beta \cdot (\varepsilon(\sigma))}] < \infty$, we get: $\beta \in \mathrm{DF}_{\varepsilon}$. **THEOREM 24.39.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\exists \beta_0 \in [0; \infty) \ s.t. \ \text{IDF}_{\varepsilon} = (\beta_0; \infty).$ Then: ε is ∞ -proper. *Proof.* We have: $IDF_{\varepsilon} \bigcap [0; \infty) \neq \emptyset$. since $DF_{\varepsilon} \supseteq IDF_{\varepsilon}$, we get: $DF_{\varepsilon} \bigcap [0; \infty) \neq \emptyset$. So, Then, by Theorem 24.8, we get: ε is ∞ -proper. **THEOREM 24.40.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Then: $(\exists \beta_0 \in [0; \infty) \ s.t. \ \text{IDF}_{\varepsilon} = (-\infty; -\beta_0)) \Rightarrow (\varepsilon \ is \ (-\infty) \text{-proper}).$ *Proof.* We have: $IDF_{\varepsilon} \cap (-\infty; 0] \neq \emptyset$. since $DF_{\varepsilon} \supseteq IDF_{\varepsilon}$, we get: $DF_{\varepsilon} \cap (-\infty; 0] \neq \emptyset$. So, by Theorem 24.9, we get: ε is $(-\infty)$ -proper. Then, **THEOREM 24.41.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $DF_{\varepsilon} \neq \emptyset$. Then: $(\varepsilon is \infty$ -proper) or $(\varepsilon is (-\infty)$ -proper). *Proof.* MORE LATER. **THEOREM 24.42.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathrm{IDF}_{\varepsilon},$ $\overline{\mathbf{X}}^{\rho}\mathbf{S}^{\varepsilon}_{\beta} < \infty.$ $\rho \in [0, \infty)$. Assume ε is proper. Then: We can remove the properness hypothesis from Theorem 24.42: **THEOREM 24.43.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in IDF_{\varepsilon}$, $\rho \in [0; \infty). \text{ Then:} \quad \overline{\mathbf{X}}^{\rho} \mathbf{S}^{\varepsilon}_{\beta} < \infty.$ *Proof.* By Theorem 24.41, we have: (ε is ∞ -proper) or (ε is $(-\infty)$ -proper). USE Theo-

25. Convergence, complex-differentiation, C^{ω} results

Recall (§2): the notations \mathbb{I}_f and f^*A . Recall (§8): for $f: S \to \mathbb{C}$, the notation $\sum_{x \in S} [f(x)]$.

rem 24.19. MORE LATER.

THEOREM 25.1. Let Σ be an infinite set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\emptyset \neq \mathrm{DF}_{\varepsilon} \subseteq (0; \infty)$. Let $\gamma \in \mathrm{IDF}_{\varepsilon}$. For all $n \in \mathbb{N}$, let $\Sigma_n := \varepsilon^*(-\infty; n]$ and let $\varepsilon_n := \varepsilon |\Sigma_n$. $(\overline{\mathbf{X}}^{\rho}\mathbf{S}^{\varepsilon}_{\gamma} < \infty)$ and $(as n \to \infty, \mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\gamma} \to \mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\gamma}).$ Then: *Proof.* Since $DF_{\varepsilon} \subseteq (0; \infty) \subseteq [0; \infty)$, we get: $DF_{\varepsilon} \bigcap [0; \infty) = DF_{\varepsilon}$. Since $DF_{\varepsilon} \cap [0; \infty) = DF_{\varepsilon} \neq \emptyset$, by Theorem 24.8, we get: ε is ∞ -proper. Let $\beta_0 := \inf DF_{\varepsilon}$. By Theorem 24.22, we get: $IDF_{\varepsilon} = (\beta_0; \infty)$. Since $\gamma \in (\beta_0; \infty)$, by Theorem 24.19, we get: $\overline{X}^{\rho} S_{\gamma}^{\varepsilon} < \infty$. as $n \to \infty$, $X^{\rho} S_{\gamma}^{\varepsilon_n} \to X^{\rho} S_{\gamma}^{\varepsilon}$. It remains to show: For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. **Define** $f: \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma, f(\sigma) = \varepsilon_{\sigma}^{\rho} \cdot e^{-\gamma \cdot \varepsilon_{\sigma}}.$ By Theorem 8.4, as $n \to \infty$, $\sum_{\sigma \in \Sigma_n} [f(\sigma)] \to \sum_{\sigma \in \Sigma} [f(\sigma)]$. So, since $\forall n \in \mathbb{N}, \sum_{\sigma \in \Sigma_n} [f(\sigma)] = X^{\rho} S_{\gamma}^{\varepsilon_n}$ and since $\sum_{\sigma \in \Sigma} [f(\sigma)] = X^{\rho} S_{\gamma}^{\varepsilon}$, we get: as $n \to \infty$, $X^{\rho} S_{\gamma}^{\varepsilon_n} \to X^{\rho} S_{\gamma}^{\varepsilon}$.

Recall (§2): the notations $\Re(z)$ and $\Im(z)$. Note: $\forall z \in \mathbb{C}, |e^z| = e^{\Re(z)}$. Also, $\forall S \subseteq \mathbb{R}, \ \Re^*S = \{x + y\sqrt{-1} | x \in S\}.$

THEOREM 25.2. Let U be an open subset of \mathbb{C} , $g, h: U \to \mathbb{C}$. Let $f_1, f_2, \ldots : U \to \mathbb{C}$ all be complex-differentiable on U. Assume, as $n \to \infty$, we have:

both $f_n \to g$ pointwise on U and $f'_n \to h$ uniformly on U. Then: g is complex-differentiable on U and g' = h on U.

Theorem 25.2 is a standard result about commuting of limit and differentiation. We omit proof.

It will be helpful to extend Definition 24.14 to \mathbb{C} :

DEFINITION 25.3. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{C}$, $z \in \mathbb{C}$. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. Then, $\forall \rho \in [0; \infty)$, the ρ -exponent (z, ε) -absolute-sum is: $\overline{X}^{\rho} S_{z}^{\varepsilon} := \sum_{\sigma \in \Sigma} [|\varepsilon_{\sigma}|^{\rho} \cdot |e^{-z \cdot \varepsilon_{\sigma}}|] \in [0; \infty]$. Also, $\forall \rho \in [0..\infty)$, if $\overline{X}^{\rho} S_{z}^{\varepsilon} < \infty$, then the ρ -exponent (z, ε) -sum is: $\overline{X}^{\rho} S_{z}^{\varepsilon} := \sum_{\sigma \in \Sigma} [\varepsilon_{\sigma}^{\rho} \cdot e^{-z \cdot \varepsilon_{\sigma}}] \in \mathbb{C}$. We continue to focus on the case where $\mathbb{I}_{\varepsilon} \subseteq \mathbb{R}$.

 $\forall \text{set } \Sigma, \quad \forall \varepsilon : \Sigma \to \mathbb{R}, \quad \forall z \in \mathbb{C}, \qquad \quad \forall \rho \in [0; \infty),$ We have: $\overline{\mathbf{X}}^{\rho} \mathbf{S}_{z}^{\varepsilon} = \overline{\mathbf{X}}^{\rho} \mathbf{S}_{\Re(z)}^{\varepsilon}.$ Recall: Theorem 24.43. $\forall \text{set } \Sigma, \quad \forall \varepsilon : \Sigma \to \mathbb{R}, \quad \forall z \in \Re^* \text{IDF}_{\varepsilon}, \quad \forall \rho \in [0; \infty),$ Then: $\overline{\mathbf{X}}^{\rho}\mathbf{S}_{z}^{\varepsilon} < \infty$, and so $\mathbf{X}^{\rho}\mathbf{S}_{z}^{\varepsilon}$ is defined. **DEFINITION 25.4.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\rho \in [0..\infty)$. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. $\begin{array}{rcl} Then \ \overline{\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}} & : & \mathrm{IDF}_{\varepsilon} \to \mathbb{R} \ is \ defined \ by: \\ & \forall \beta \in & \mathrm{IDF}_{\varepsilon}, & (\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet})(\beta) = \mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\beta}. \\ Also, \ \overline{\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}}} & : \ \Re^{*}\mathrm{IDF}_{\varepsilon} \to \mathbb{C} \ is \ defined \ by: \end{array}$ $\overline{\forall z} \in \Re^* \mathrm{IDF}_{\varepsilon}, \qquad (\mathrm{X}^{\rho} \mathrm{S}^{\varepsilon}_{\bullet})(z) = \mathrm{X}^{\rho} \mathrm{S}^{\varepsilon}_{\bullet}.$ **THEOREM 25.5.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\emptyset \neq \mathrm{DF}_{\varepsilon} \subseteq (0; \infty)$. Let $\gamma \in \mathrm{IDF}_{\varepsilon}$. For all $n \in \mathbb{N}$, let $\Sigma_n := \varepsilon^*(-\infty; n]$ and let $\varepsilon_n := \varepsilon |\Sigma_n$. Then: as $n \to \infty$, $X^{\rho} S^{\varepsilon_n}_{\bullet \mathbb{C}} \to X^{\rho} S^{\varepsilon}_{\bullet \mathbb{C}}$ uniformly on $\Re^*(\gamma; \infty)$. *Proof.* Given $\delta > 0$, want: $\exists n_0 \in \mathbb{N} \text{ s.t.}, \forall n \in [n_0..\infty)$, $|\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}} - \mathbf{X}^{\rho}\mathbf{S}^{\varepsilon_n}_{\bullet\mathbb{C}}| < \delta \quad \text{ on } \Re^*(\gamma; \infty).$ Let $\beta_0 := \inf DF_{\varepsilon}$. Then $IDF_{\varepsilon} = (\beta_0; \infty)$. On $\Re^*(\gamma; \infty)$, $|X^{\rho} S^{\varepsilon}_{\bullet \mathbb{C}} - X^{\rho} S^{\varepsilon_n}_{\bullet \mathbb{C}}| \leq X^{\rho} S^{\varepsilon}_{\gamma} - X^{\rho} S^{\varepsilon_n}_{\gamma}$. MORE LATER **THEOREM 25.6.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: Σ is finite. Let $\rho \in [0..\infty)$. $X^{\rho}S_{\bullet\mathbb{C}}^{\varepsilon}$ is complex-differentiable on \mathbb{C} . Then: $(\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}})' = -\mathbf{X}^{\rho+1}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}} \quad on \ \mathbb{C}.$ and *Proof.* For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. $(\mathbf{X}^{\rho}\mathbf{S}_{\bullet\mathbb{C}}^{\varepsilon})(z) = \sum_{\sigma\in\Sigma} \left[\varepsilon_{\sigma}^{\rho} \cdot e^{-z\cdot\varepsilon_{\sigma}}\right]$ We have: $\forall z \in \mathbb{C}$, Since Σ is finite, we may differentiate term-by-term, yielding: $(\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}})'(z) = \sum_{\sigma\in\Sigma} \left[\varepsilon^{\rho}_{\sigma} \cdot e^{-z\cdot\varepsilon_{\sigma}} \cdot (-\varepsilon_{\sigma})\right]$ $\forall z \in \mathbb{C},$ Thus $X^{\rho}S^{\varepsilon}_{\bullet\mathbb{C}}$ is complex-differentiable on \mathbb{C} . It remains to show: $(X^{\rho}S_{\bullet\mathbb{C}}^{\varepsilon})' = -X^{\rho+1}S_{\bullet\mathbb{C}}^{\varepsilon}$ on \mathbb{C} . Since $\forall z \in \mathbb{C}$, we have: $(X^{\rho}S^{\varepsilon}_{\bullet\mathbb{C}})'(z) = \sum_{\sigma \in \Sigma} \left[\varepsilon^{\rho}_{\sigma} \cdot e^{-z \cdot \varepsilon_{\sigma}} \cdot (-\varepsilon_{\sigma})\right]$ we conclude: $(X^{\rho}S_{\bullet\mathbb{C}}^{\varepsilon})' = -X^{\rho+1}S_{\bullet\mathbb{C}}^{\varepsilon}$ on \mathbb{C} . \Box In Theorem 25.6, we assumed Σ was finite.

We next investigate what happens without that assumption:

THEOREM 25.7. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Let $\rho \in [0..\infty)$. $\emptyset \neq \mathrm{DF}_{\varepsilon} \subseteq (0;\infty).$ Assume: $X^{\rho}S_{\bullet\mathbb{C}}^{\varepsilon} \text{ is complex-differentiable } on \ \Re^*IDF_{\varepsilon}.$ Then: $(\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}})' = -\mathbf{X}^{\rho+1}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}} \quad on \ \Re^*\mathrm{IDF}_{\varepsilon}.$ and *Proof.* Let $\beta_0 := \inf DF_{\varepsilon}$. Then $IDF_{\varepsilon} = (\beta_0; \infty)$. For all $n \in \mathbb{N}$, let $\Sigma_n := \varepsilon^*(-\infty; n]$ and let $\varepsilon_n := \varepsilon |\Sigma_n$. **Given** $z \in \Re^*(\beta_0; \infty)$, want: $X^{\rho} S^{\varepsilon}_{\bullet \mathbb{C}}$ is complex-differentiable at z and $(\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}})'(z) = -(\mathbf{X}^{\rho+1}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}})(z).$ Let $\beta := \Re(z)$. Let $\gamma := (\beta_0 + \beta)/2$. Then $\beta_0 < \gamma < \beta$. It suffices to show: $X^{\rho}S_{\bullet\mathbb{C}}^{\varepsilon}$ is complex-differentiable on $\Re^*(\gamma; \infty)$ and $(X^{\rho}S_{\bullet\mathbb{C}}^{\varepsilon})' = -X^{\rho+1}S_{\bullet\mathbb{C}}^{\varepsilon}$ on $\Re^*(\gamma;\infty)$. By Theorem 25.5, as $n \to \infty$, we have both $X^{\rho}S^{\varepsilon_n}_{\bullet\mathbb{C}} \to X^{\rho}S^{\varepsilon}_{\bullet\mathbb{C}}$ uniformly on $\Re^*(\gamma;\infty)$ $X^{\rho+1}S^{\varepsilon_n}_{\bullet\mathbb{C}} \to X^{\rho+1}S^{\varepsilon}_{\bullet\mathbb{C}}$ uniformly on $\Re^*(\gamma; \infty)$. and For all $n \in \mathbb{N}$, since Σ_n is finite, by Theorem 25.6, we see that $\mathcal{X}^{\rho} \mathcal{S}^{\varepsilon_n}_{\bullet \mathbb{C}}$ is complex-differentiable at zand $(X^{\rho}S_{\bullet\mathbb{C}}^{\varepsilon_n})' = -X^{\rho+1}S_{\bullet\mathbb{C}}^{\varepsilon_n}$ on $\Re^*(\beta_0;\infty)$. Then, as $n \to \infty$, we have both $X^{\rho}S^{\varepsilon_n}_{\bullet\mathbb{C}} \to X^{\rho}S^{\varepsilon}_{\bullet\mathbb{C}}$ pointwise on $\Re^*(\gamma;\infty)$ $(\mathbf{X}^{\rho}\mathbf{S}_{\bullet\mathbb{C}}^{\bullet_{n}})' \to -\mathbf{X}^{\rho+1}\mathbf{S}_{\bullet\mathbb{C}}^{\varepsilon} \text{ uniformly on } \Re^{*}(\gamma;\infty).$ and Then, by Theorem 25.2, we get: $X^{\rho}S^{\varepsilon}_{\bullet\mathbb{C}}$ is complex-differentiable on $\Re^{*}(\gamma;\infty)$ $(\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}})' = -\mathbf{X}^{\rho+1}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}} \text{ on } \Re^{*}(\gamma;\infty).$ and " $\emptyset \neq \mathrm{DF}_{\varepsilon} \subseteq (0;\infty)$ " We can remove the hypothesis from the preceding theorem: **THEOREM 25.8.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Let $\rho \in [0..\infty)$. $X^{\rho}S^{\varepsilon}_{\bullet\mathbb{C}}$ is complex-differentiable on $\Re^{*}IDF_{\varepsilon}$. Then: $(\mathbf{X}^{\rho}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}})' = -\mathbf{X}^{\rho+1}\mathbf{S}^{\varepsilon}_{\bullet\mathbb{C}} \quad on \ \Re^*\mathrm{IDF}_{\varepsilon}.$ and " C^{ω} " means "real-analytic". Recall $(\S 2)$: **THEOREM 25.9.** Let V be an open subset of \mathbb{C} , $U := V \cap \mathbb{R}$. Let $g: V \to \mathbb{C}$. Assume $g_*U \subseteq \mathbb{R}$. Let f := g|U. Assume: $g: V \to \mathbb{C}$ is complex-differentiable on V. $f: U \to \mathbb{R}$ is C^{ω} Then: on U. $((g')_*U \subseteq \mathbb{R})$ and (f' = g'|U).Also,

THEOREM 25.10. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\rho \in [0..\infty)$. Then: $X^{\rho}S^{\varepsilon}_{\bullet}$ is C^{ω} on IDF_{ε} and $(X^{\rho}S^{\varepsilon}_{\bullet})' = -X^{\rho+1}S^{\varepsilon}_{\bullet}$ on IDF_{ε} .

Proof. Let $\beta_0 := \inf DF_{\varepsilon}$. Then $IDF_{\varepsilon} = (\beta_0; \infty)$. Use Theorem 25.9. MORE LATER

26. Boltzmann averages on countable sets

DEFINITION 26.1. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. Assume: $\overline{X}^1 S_{\beta}^{\varepsilon} < \infty$. Then: $\Gamma_{\beta}^{\varepsilon} := X^1 S_{\beta}^{\varepsilon}$. Let Σ be a countable set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. If $\overline{X}^1 S_{\beta}^{\varepsilon} < \infty$, then $\Gamma_{\beta}^{\varepsilon} = \sum_{\sigma \in \Sigma} [\varepsilon_{\sigma} \cdot (\widehat{B}_{\beta}^{\varepsilon} \{\sigma\})]$, and so $\Gamma_{\beta}^{\varepsilon}$ is the integral of ε wrt $\widehat{B}_{\beta}^{\varepsilon}$.

In the next definition, in order that $\Gamma^{\varepsilon}_{\beta}/\Delta^{\varepsilon}_{\beta}$ is defined,

we need: both $\Gamma^{\varepsilon}_{\beta}$ is defined and $0 < \Delta^{\varepsilon}_{\beta} < \infty$. We therefore assume $\overline{X}^1 S_{\beta}^{\varepsilon} < \infty$, to ensure that $\Gamma_{\beta}^{\varepsilon}$ is defined. assume Σ is nonempty, to ensure that $\Delta_{\beta}^{\varepsilon} > 0$. We also assume $\beta \in DF_{\varepsilon}$, Finally, we to ensure that $\Delta_{\beta}^{\varepsilon} < \infty$. **DEFINITION 26.2.** Let Σ be a nonempty set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$. Assume: $\overline{\mathbf{X}}^1 \mathbf{S}^{\varepsilon}_{\beta} < \infty$ and $\beta \in \mathrm{DF}_{\varepsilon}$. Then: $\overline{A^{\varepsilon}_{\beta}} := \Gamma^{\varepsilon}_{\beta} / \Delta^{\varepsilon}_{\beta}$. Note that, by Theorem 24.16, if ε is ∞ -proper, then $\left(\,\overline{\mathbf{X}}^{1}\mathbf{S}_{\beta}^{\varepsilon} < \infty\,\right) \;\Rightarrow\; \left(\,\beta \in \mathrm{DF}_{\varepsilon}\,\right).$ Without ∞ -properness, this fails, by Theorem 24.18, $(\overline{\mathbf{X}}^{1}\mathbf{S}_{1}^{\varepsilon} < \infty) \Rightarrow (1 \in \mathrm{DF}_{\varepsilon}).$ By Theorem 24.17, even with ∞ -properness, $(1 \in \mathrm{DF}_{\varepsilon}) \Rightarrow (\overline{\mathrm{X}}^1 \mathrm{S}_1^{\varepsilon} < \infty).$ **THEOREM 26.3.** Let Σ be a nonempty countable set, $\varepsilon: \Sigma \to \mathbb{R}$. Let $\beta \in DF_{\varepsilon}$. Assume $\overline{X}^1 S_{\beta}^{\varepsilon} < \infty$. Then $|\varepsilon_* B_{\beta}^{\varepsilon}|_1 < \infty$ and $A_{\beta}^{\varepsilon} = M_{\varepsilon_* B_{\beta}^{\varepsilon}}$.

 $\begin{array}{l} \textit{Proof. BY Theorem 8.5,} \\ \text{SHOW: } \sum_{t \in \mathbb{I}_{\varepsilon}} \left[|t| \cdot \left((\varepsilon_* B_{\beta}^{\varepsilon}) \{t\} \right) \right] = \sum_{\sigma \in \Sigma} \left[|\varepsilon_{\sigma}| \cdot \left(B_{\beta}^{\varepsilon} \{\sigma\} \right) \right]. \\ \text{IDEA: BY Theorem 8.5,} \\ \sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^* \{t\}} \left[|t| \cdot \left(B_{\beta}^{\varepsilon} \{\sigma\} \right) \right] = \sum_{\sigma \in \Sigma} \left[|\varepsilon_{\sigma}| \cdot \left(B_{\beta}^{\varepsilon} \{\sigma\} \right) \right]. \end{array}$

THEN: $|\varepsilon_* B^{\varepsilon}_{\beta}|_1 = (\overline{X}^1 S^{\varepsilon}_{\beta}) / \Delta^{\varepsilon}_{\beta}$. THEN: $|\varepsilon_* B^{\varepsilon}_{\beta}|_1 < \infty$. THEN: $M_{\varepsilon_* B_{\beta}^{\varepsilon}} = \sum_{t \in \mathbb{I}_{\varepsilon}} [t \cdot ((\varepsilon_* B_{\beta}^{\varepsilon}) \{t\})].$ Since $\Sigma \neq \emptyset$, we get: $\Delta_{\beta}^{\varepsilon} > 0$ Since $\beta \in DF_{\varepsilon}$, we get: $\Delta_{\beta}^{\varepsilon} < \infty$. Then $0 < \Delta_{\beta}^{\varepsilon} < \infty$, so, since $\Delta_{\beta}^{\varepsilon} = \widehat{B}_{\beta}^{\varepsilon}(\Sigma)$, we get: $0 < B_{\beta}^{\varepsilon}(\Sigma) < \infty$. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. $\sum_{\sigma \in \Sigma} |\varepsilon_{\sigma} \cdot (\hat{B}^{\varepsilon}_{\beta} \{\sigma\})| = \sum_{\sigma \in \Sigma} [|\varepsilon_{\sigma}| \cdot e^{-\beta \cdot \varepsilon_{\sigma}}] = \overline{X}^{1} S^{\varepsilon}_{\beta} < \infty,$ Since dividing by $\hat{B}^{\varepsilon}_{\beta}(\Sigma)$, we get: $\sum_{\sigma \in \Sigma} |\varepsilon_{\sigma} \cdot (B^{\varepsilon}_{\beta}\{\sigma\})| < \infty$. Then, by Theorem 8.6, $\begin{array}{ll} \forall t \in \mathbb{I}_{\varepsilon}, & \sum_{\sigma \in \varepsilon^*\{t\}} |\varepsilon_{\sigma} \cdot (B_{\beta}\{\sigma\}) | < \infty & \text{and} \\ & \sum_{t \in \mathbb{I}_{\varepsilon}} & |\sum_{\sigma \in \varepsilon^*\{t\}} [\varepsilon_{\sigma} \cdot (B_{\beta}\{\sigma\})] | < \infty & \text{and} \\ & \sum_{t \in \mathbb{I}_{\varepsilon}} & \sum_{\sigma \in \varepsilon^*\{t\}} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})] = \sum_{\sigma \in \Sigma} [\varepsilon_{\sigma} \cdot (B_{\beta}^{\varepsilon}\{\sigma\})]. \end{array}$ we get: $\sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^*\{t\}} [\varepsilon^{\sigma} (D_{\beta}(\varepsilon))] = \sum_{\sigma \in \Sigma} [\varepsilon^{\sigma} (D_{\beta}(\varepsilon))],$ so, $A_{\beta}^{\varepsilon} = \sum_{\sigma \in \Sigma} [\varepsilon^{\sigma} (D_{\beta}^{\varepsilon}(\varepsilon))],$ en: $\sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^*\{t\}} [\varepsilon_{\sigma} (B_{\beta}^{\varepsilon}\{\sigma\})] = A_{\beta}^{\varepsilon},$ we want: $\sum_{t \in \mathbb{I}_{\varepsilon}} [t \cdot ((\varepsilon_* B_{\beta}^{\varepsilon})\{t\})] = \sum_{t \in \mathbb{I}_{\varepsilon}} \sum_{\sigma \in \varepsilon^*\{t\}} [\varepsilon_{\sigma} (B_{\beta}^{\varepsilon}\{\sigma\})],$ ant: $\forall t \in \mathbb{I}_{\varepsilon}, \quad t \cdot ((\varepsilon_* B_{\beta}^{\varepsilon})\{t\}) = \sum_{\sigma \in \varepsilon^*\{t\}} [\varepsilon_{\sigma} (B_{\beta}^{\varepsilon}\{\sigma\})],$ ven $t \in \mathbb{I}_{\varepsilon},$ want: $t \cdot ((\varepsilon_* B_{\beta}^{\varepsilon})\{t\}) = \sum_{\sigma \in \varepsilon^*\{t\}} [\varepsilon_{\sigma} (B_{\beta}^{\varepsilon}\{\sigma\})],$ Also, Then: So, since Want: $\forall t \in \mathbb{I}_{\varepsilon}$, **Given** $t \in \mathbb{I}_{\varepsilon}$, want: $t \cdot ((\varepsilon_* B_{\beta}^{\varepsilon}) \{t\}) =$ For all $\sigma \in \varepsilon^* \{t\}$, since $\varepsilon_\sigma = \varepsilon(\sigma) \in \{t\}$, we get: $t = \varepsilon_\sigma$. Want: $t \cdot ((\varepsilon_* B^{\varepsilon}_{\beta}) \{t\}) = \sum_{\sigma \in \varepsilon^* \{t\}} [t \cdot (B^{\varepsilon}_{\beta} \{\sigma\})].$ $\varepsilon^*{t}$ is the disjoint union, over $\sigma \in \varepsilon^*{t}$, of $\{\sigma\}$, Because $B_{\beta}^{\varepsilon}(\varepsilon^{*}\{t\}) = \sum_{\sigma \in \varepsilon^{*}\{t\}} \begin{bmatrix} B_{\beta}^{\varepsilon}\{\sigma\} \end{bmatrix}$ we get: $(\varepsilon_* B_\beta^\varepsilon)\{t\}) =$ $B^{\varepsilon}_{\beta}(\varepsilon^*\{t\}).$ Also, Then: $t \cdot ((\varepsilon_* B^{\varepsilon}_{\beta}) \{t\}) = t \cdot (B^{\varepsilon}_{\beta}(\varepsilon^* \{t\})) = \sum_{\sigma \in \varepsilon^* \{t\}} [t \cdot (B^{\varepsilon}_{\beta} \{\sigma\})].$ **THEOREM 26.4.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*[0;\infty)$ is infinite and $DF_{\varepsilon} \neq \emptyset$. Let $\beta_0 := \inf DF_{\varepsilon}$. Then: $\forall real \ \gamma > \beta_0, \quad \forall real \ \rho > 0, \quad \overline{\mathbf{X}}^{\rho} \mathbf{S}^{\varepsilon}_{\gamma} < \infty.$ *Proof.* Given a real $\gamma > \beta_0$ and a real $\rho > 0$, want: $\overline{X}^{\rho} S_{\gamma}^{\varepsilon} < \infty$. By Theorem 24.12, ε is ∞ -proper. Then, by Theorem 23.4, \mathbb{I}_{ε} is bounded below. By Theorem 24.19, we have: $\overline{\mathbf{X}}^{\rho} \mathbf{S}^{\varepsilon}_{\gamma} < \infty$.

DEFINITION 26.5. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Then $\overline{A_{\bullet}^{\varepsilon}} : \mathrm{IDF}_{\varepsilon} \to \mathbb{R}$ is defined by: $\forall \beta \in \mathrm{IDF}_{\varepsilon}, A_{\bullet}^{\varepsilon}(\beta) = A_{\beta}^{\varepsilon}$.

THEOREM 26.6. Let Σ be a set.

Let $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\#\mathbb{I}_{\varepsilon} \ge 2$. Then: $A_{\bullet}^{\varepsilon}$ is a strictly-decreasing C^{ω} -diffeomorphism from $\mathrm{IDF}_{\varepsilon}$ onto $(\inf \mathbb{I}_{A_{\varepsilon}}; \sup \mathbb{I}_{A_{\varepsilon}})$.

Proof. For all $\sigma \in \Sigma$, let $\varepsilon_{\sigma} := \varepsilon(\sigma)$. We have: $\forall \beta \in \text{IDF}_{\varepsilon}, A^{\varepsilon}_{\bullet}(\beta) = \frac{\sum_{\sigma \in \Sigma} [\varepsilon_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}}]}{\sum_{\tau \in \Sigma} [e^{-\beta \cdot \varepsilon_{\tau}}]}$. We have: $\forall \beta \in \text{IDF}_{\varepsilon}, A^{\varepsilon}_{\bullet}(\beta) = \frac{\Gamma^{\varepsilon}_{\bullet}(\beta)}{\Delta^{\varepsilon}_{\bullet}(\beta)}$. We have: $\forall \beta \in \mathrm{IDF}_{\varepsilon}, A^{\varepsilon}_{\bullet}(\beta) = \frac{\overline{(X^{1}S^{\varepsilon}_{\bullet})(\beta)}}{(X^{0}S^{\varepsilon}_{\bullet})(\beta)}.$ By Theorem 25.7, $X^1S^{\varepsilon}_{\bullet}$ and $X^0S^{\varepsilon}_{\bullet}$ are both C^{ω} . So, since $X^0 S^{\varepsilon} \neq 0$ on IDF_{ε} , we conclude: $A^{\varepsilon}_{\bullet}$ is C^{ω} . By Theorem 25.10, we have: $\forall \beta \in \mathrm{IDF}_{\varepsilon}, (\mathrm{X}^{1}\mathrm{S}_{\bullet}^{\varepsilon})'(\beta) = -(\mathrm{X}^{2}\mathrm{S}_{\bullet}^{\varepsilon})(\beta),$ $\forall \beta \in \mathrm{IDF}_{\varepsilon}, \, (\mathrm{X}^{0} \mathrm{S}_{\bullet}^{\varepsilon})'(\beta) = -(\mathrm{X}^{1} \mathrm{S}_{\bullet}^{\varepsilon})(\beta).$ and Then: $\forall \beta \in \mathrm{IDF}_{\varepsilon}, (X^{1}S_{\bullet}^{\varepsilon})'(\beta) = \sum_{\sigma \in \Sigma} [(-\varepsilon_{\sigma}^{2}) \cdot e^{-\beta \cdot \varepsilon_{\sigma}}],$ and $\forall \beta \in \mathrm{IDF}_{\varepsilon}, (X^{0}S_{\bullet}^{\varepsilon})'(\beta) = \sum_{\tau \in \Sigma} [(-\varepsilon_{\tau}) \cdot e^{-\beta \cdot \varepsilon_{\tau}}].$ So, by the C^{ω} -Inverse Function Theorem and the Mean Value Theorem, it suffices to show: $(A^{\varepsilon})' < 0$ on IDF_{ε} . Given $\beta \in IDF_{\varepsilon}$, want: $(A^{\varepsilon})'(\beta) < 0$. $P := \sum_{\sigma \in \Sigma} \left[\varepsilon_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \right], \qquad P' := \sum_{\sigma \in \Sigma} \left[\left(-\varepsilon_{\sigma}^2 \right) \cdot e^{-\beta \cdot \varepsilon_{\sigma}} \right].$ $Q := \sum_{\tau \in \Sigma} \left[e^{-\beta \cdot \varepsilon_{\tau}} \right], \qquad Q' := \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\tau} \right) \cdot e^{-\beta \cdot \varepsilon_{\tau}} \right].$ Let Let Then Q > 0. Also, by the Quotient Rule, $(A^{\varepsilon}_{\bullet})'(\beta) = [QP' - PQ']/Q^2$. Want: QP' - PQ' < 0.CITE NEW THEOREM THAT SAYS

$$\begin{split} & \left(\sum_{x} \left[f(x)\right]\right) \cdot \left(\sum_{y} \left[g(y)\right]\right) = \sum_{x} \sum_{y} \left[\left(f(x)\right) \cdot \left(g(y)\right)\right].\\ & \text{We have: } QP' \qquad = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\sigma}^{2}\right) \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})}\right].\\ & \text{We have: } PQ' \qquad = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\sigma} \varepsilon_{\tau}\right) \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})}\right].\\ & \text{Then: } QP' - PQ' = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\sigma}^{2} + \varepsilon_{\sigma} \varepsilon_{\tau}\right) \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})}\right].\\ & \text{Interchanging } \sigma \text{ and } \tau, \text{ we get: } \end{split}$$

$$QP' - PQ' = \sum_{\tau \in \Sigma} \sum_{\sigma \in \Sigma} \left[\left(-\varepsilon_{\tau}^2 + \varepsilon_{\tau} \varepsilon_{\sigma} \right) \cdot e^{-\beta \cdot (\varepsilon_{\tau} + \varepsilon_{\sigma})} \right].$$

By commutativity of addition and multiplication,

adding the last two equations gives:

$$2 \cdot (QP' - PQ') = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[\left(-\varepsilon_{\sigma}^2 - \varepsilon_{\tau}^2 + 2\varepsilon_{\sigma}\varepsilon_{\tau} \right) \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})} \right].$$

Then:
$$2 \cdot (QP' - PQ') = \sum_{\sigma \in \Sigma} \sum_{\tau \in \Sigma} \left[-(\varepsilon_{\sigma} - \varepsilon_{\tau})^2 \cdot e^{-\beta \cdot (\varepsilon_{\sigma} + \varepsilon_{\tau})} \right].$$

Then:
$$2 \cdot (QP' - PQ') < 0.$$

Then:
$$QP' - PQ' < 0.$$

Recall (Theorem 23.3):

If ε is ∞ -proper, then \mathbb{I}_{ε} has a minimum element, *i.e.*, min \mathbb{I}_{ε} exists.

THEOREM 26.7. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$. Assume: $\varepsilon^*[0;\infty)$ is infinite and $DF_{\varepsilon} \neq \emptyset$. $\varepsilon \text{ is } \infty\text{-proper} \quad and \quad as \ \beta \to \infty, \ A_{\beta}^{\varepsilon} \to \min \mathbb{I}_{\varepsilon}.$ Then: *Proof.* By Theorem 24.12, ε is ∞ -proper. It remains to show: as $\beta \to \infty$, $A_{\beta}^{\varepsilon} \to \min \mathbb{I}_{\varepsilon}$. Let $t_0 := \min \mathbb{I}_{\varepsilon}$. Want: $A_{\beta}^{\varepsilon} \to t_0$. Let $\Sigma' := \Sigma \setminus (\varepsilon^* \{t_0\}).$ Let $n_0 := \#(\varepsilon^* \{t_0\}).$ Since $\{t_0\} \subseteq (-\infty; t_0]$, we get $\varepsilon^* \{t_0\} \subseteq \varepsilon^* (-\infty; t_0]$. Since ε is ∞ -proper, we get: $\varepsilon^*(-\infty; t_0]$ is finite. Then $\varepsilon^* \{t_0\}$ is finite. That is, $n_0 < \infty$. Since $t_0 \in \mathbb{I}_{\varepsilon}$, we get $\varepsilon^* \{t_0\} \neq \emptyset$, and so $n_0 > 0$. Then $0 < n_0 < \infty$. For all $\beta \in (\beta_0; \infty)$, we have: $A_{\beta}^{\varepsilon} = \frac{n_{0} \cdot t_{0} \cdot e^{-\beta \cdot t_{0}} + \sum_{\sigma \in \Sigma'} [\varepsilon_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}}]}{n_{0} \cdot e^{-\beta \cdot t_{0}} + \sum_{\sigma \in \Sigma'} [\varepsilon_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}}]}$ $= \frac{n_{0} \cdot t_{0} \cdot e^{-\beta \cdot t_{0}} + \sum_{\sigma \in \Sigma'} [\varepsilon_{\sigma} \cdot e^{-\beta \cdot \varepsilon_{\sigma}}]}{n_{0} \cdot e^{-\beta \cdot t_{0}} + \sum_{\sigma \in \Sigma'} [e^{-\beta \cdot \varepsilon_{\sigma}}]} \cdot \frac{e^{\beta \cdot t_{0}}}{e^{\beta \cdot t_{0}}}$ $= \frac{n_{0} \cdot t_{0} + \sum_{\sigma \in \Sigma'} [\varepsilon_{\sigma} \cdot e^{-\beta \cdot (\varepsilon_{\sigma} - t_{0})}]}{n_{0} + \sum_{\sigma \in \Sigma'} [e^{-\beta \cdot (\varepsilon_{\sigma} - t_{0})}]}.$ Let $\beta_1 := \beta_0 + 1$. Then, for all $\beta \in [\beta_1; \infty)$, for all $\sigma \in \Sigma$, we have $\begin{aligned} |\varepsilon_{\sigma} \cdot e^{-\beta \cdot (\varepsilon_{\sigma} - t_0)}| &\leq |\varepsilon_{\sigma}| \cdot e^{-\beta_1 \cdot (\varepsilon_{\sigma} - t_0)} \\ |e^{-\beta \cdot (\varepsilon_{\sigma} - t_0)}| &\leq e^{-\beta_1 \cdot (\varepsilon_{\sigma} - t_0)} \\ \sum_{\sigma \in \Sigma} [|\varepsilon_{\sigma}| \cdot e^{-\beta_1 \cdot (\varepsilon_{\sigma} - t_0)}] = \overline{X}^1 S_{\beta_1}^{\varepsilon}. \end{aligned}$ and We have: $\sum_{\sigma \in \Sigma} \left[e^{-\beta_1 \cdot (\varepsilon_\sigma - t_0)} \right] = \overline{\mathbf{X}}^0 \mathbf{S}_{\beta_1}^{\varepsilon}.$ Also, By Theorem 26.4, we have: $\overline{X}^1 S_{\beta_1}^{\varepsilon} < \infty$ and $\overline{X}^0 S_{\beta_1}^{\varepsilon} < \infty$. So, by the Dominated Convergence Theorem, as $\beta \to \infty$, $\sum_{\sigma \in \Sigma'} \left[\varepsilon_{\sigma} \cdot e^{-\beta \cdot (\varepsilon_{\sigma} - t_0)} \right] \to 0$

$$\frac{\sum_{\sigma \in \Sigma'} [\varepsilon_{\sigma} \cdot e^{-\beta \cdot (\varepsilon_{\sigma} - t_0)}]}{\sum_{\sigma \in \Sigma'} [e^{-\beta \cdot (\varepsilon_{\sigma} - t_0)}]}$$

Then:

and
$$\sum_{\sigma \in \Sigma'} \left[e^{-\beta \cdot (\varepsilon_{\sigma} - t_0)} \right] \to 0.$$

Then:
as $\beta \to \infty$, $A_{\beta}^{\varepsilon} \to \frac{n_0 \cdot t_0 + 0}{n_0 + 0}.$
Then:
as $\beta \to \infty$, $A_{\beta}^{\varepsilon} \to t_0.$

 $\rightarrow 0.$

Let Σ be a set and let $\varepsilon : \Sigma \to [0; \infty)$ be ∞ -proper. Assume: $\varepsilon^*[0;\infty)$ is infinite and $\sup \mathbb{I}_{\varepsilon} = \infty$ and $DF_{\varepsilon} \neq \emptyset$. Let $\beta_0 := \inf DF_{\varepsilon}$. By Theorem 24.21, $(\beta_0; \infty) \subseteq DF_{\varepsilon}$. Even though $\sup \mathbb{I}_{\varepsilon} = \infty$,

it does NOT necessarily follow that: as $\beta \to (\beta_0)^+$, $A_{\beta}^{\varepsilon} \to \infty$. Here is an example:

THEOREM 26.8. For all $k \in \mathbb{N}$, let $n_k := \lfloor e^k / k^3 \rfloor$. Let $\Sigma := \{(k, j) \in \mathbb{N} \times \mathbb{N} \mid k \in \mathbb{N}, j \in [1..n_k]\}.$ **Define** $\varepsilon : \Sigma \to [0..\infty)$ by: $\forall k \in \mathbb{N}, \forall j \in [1..n_k], \quad \varepsilon(k,j) = k-1.$ Then $\mathbb{I}_{A_{\bullet}^{\varepsilon}}$ is bounded. *Proof.* We have $DF_{\varepsilon} = [1; \infty)$, so $\inf DF_{\varepsilon} = 1$. Also, $\Gamma_1^{\varepsilon} < \infty$ and $0 < \Delta_1^{\varepsilon} < \infty$, so $A_1^{\varepsilon} < \infty$. Also, by the Dominated Convergence Theorem, we have: as $\beta \to 1^+$, both $\Gamma_{\beta}^{\varepsilon} \to \Gamma_1^{\varepsilon}$ and $\Delta_{\beta}^{\varepsilon} \to \Delta_1^{\infty}$. Then, as $\beta \to 1^+$, $A_{\beta}^{\varepsilon} \to A_1^{\varepsilon} < \infty$. Then $\mathbb{I}_{A_{\bullet}^{\varepsilon}}$ is bounded. Theorem 26.8 leads to an open problem, as follows: For all $k \in \mathbb{N}$, let $n_k := |e^k/k^3|$. Let $\Sigma := \{(k, j) \in \mathbb{N} \times \mathbb{N} \mid k \in \mathbb{N}, j \in [1..n_k]\}.$ **Define** $\varepsilon : \Sigma \to \mathbb{N}$ by: $\forall k \in \mathbb{N}, \forall j \in [1..n_k], \quad \varepsilon(k, j) = k.$ By Theorem 26.6, $A^{\varepsilon}_{\bullet}$ is strictly-decreasing, and so and since as $\beta \to 1^+$, $A_{\beta}^{\varepsilon} \to A_1^{\varepsilon}$, we get: $\mathbb{I}_{A^{\varepsilon}_{\bullet}}$ is bounded above by A^{ε}_{1} . Let $\alpha \in \mathbb{N}$. Assume: $\alpha > A_1^{\varepsilon}$. Then: $\alpha \notin \mathbb{I}_{A_{\varepsilon}^{\varepsilon}}$. Suppose N professors, numbered 1 to N, have states in Σ . Suppose each state $\sigma \in \Sigma$ has wealth $\varepsilon(\sigma)$. Suppose the total wealth of all professors is $N\alpha$. Give equal probability to every dispensation of states. For each $\sigma_0 \in \Sigma$, we seek a method to approximate the probability that Professor #N is in state σ_0 . For all $n \in \mathbb{N}$, More precisely: let $\Omega_n := \{ \omega : [1..n] \to \Sigma \mid \sum_{\ell=1}^n [\varepsilon(\omega(\ell))] = n\alpha \}.$ Then Ω_N represents the set of all state-dispensations. **Open Problem:** For each $\sigma_0 \in \Sigma$, determine whether the limit, as $n \to \infty$, of $\nu_{\Omega_n} \{ \omega \in \Omega_n \, | \, \omega(n) = \sigma_0 \}$ exists, and, if it does, compute it. This is a well-defined mathematical problem. However, since $\alpha \notin \mathbb{I}_{A_{\bullet}^{\varepsilon}}$, we cannot solve $A_{\beta}^{\varepsilon} = \alpha$ for β ,

so our earlier techniques do not immediately apply.

THEOREM 26.9. Let $\beta_0 \in \mathbb{R}$, $I := (\beta_0; \infty)$, $g : I \to \mathbb{R}$. q is differentiable on I and q' is semi-decreasing on I. Assume: as $\beta \to (\beta_0)^+$, $g(\beta) \to -\infty$. Assume: as $\beta \to (\beta_0)^+$, $g'(\beta) \to \infty$. Then: *Proof.* Since $q: I \to \mathbb{R}$ and q is differentiable on I, we get: $q': I \to \mathbb{R}$. Since $I = (\beta_0; \infty) \neq \emptyset$ and $g' : I \to \mathbb{R}$, we get: $\mathbb{I}_{q'} \neq \emptyset$. Since $\emptyset \neq \mathbb{I}_{q'} \subseteq \mathbb{R}$, we get: $\sup \mathbb{I}_{q'} \neq -\infty$. Let $M := \sup \mathbb{I}_{q'} \in (-\infty; \infty].$ Since g' is strictly-decreasing, we get: as $\beta \to (\beta_0)^+, g'(\beta) \to M$. Want: $M = \infty$. Assume $M < \infty$. Want: Contradiction. Let $\beta_1 := \beta_0 + 1$. Since, as $\beta \to (\beta_0)^+$, $g(\beta) \to -\infty$, choose $\gamma \in (\beta_0; \beta_1)$ s.t. $g(\gamma) < (g(\beta_1)) - M$. By the Mean Value Theorem, choose $\xi \in (\gamma; \beta_0 + 1)$ s.t. $\frac{(g(\beta_1) - (g(\gamma)))}{\beta_1 - \gamma} = g'(\xi).$ Since $M = \sup \mathbb{I}_{g'}$, we get: $g'(\xi)$ $\leq M$. $\beta_1 - \gamma > 0.$ Since $\gamma \in (\beta_0; \beta_1)$, we get: $(g'(\xi)) \cdot (\beta_1 - \gamma) \leq M \cdot (\beta_1 - \gamma).$ Then $(g(\beta_1) - (g(\gamma)) = (g'(\xi)) \cdot (\beta_1 - \gamma) \leq M \cdot (\beta_1 - \gamma),$ Since $g(\gamma) \geqslant (g(\beta_1)) - M \cdot (\beta_1 - \gamma).$ we get: By the choice of γ , we get $\gamma \in (\beta_0; \beta_1)$, and so $\gamma - \beta_0 > 0$. By the choice of γ , we get: $(g(\beta_1)) - M > g(\gamma).$ Since $(g(\beta_1)) - M > g(\gamma) \ge (g(\beta_1)) - M \cdot (\beta_1 - \gamma),$ we get: M < $M \cdot (\beta_1 - \gamma).$ $M \cdot (\gamma + 1 - \beta_1) < 0.$ Then: Since $\beta_1 = \beta_0 + 1$, we get: $1 - \beta_1 = -\beta_0$. $M \cdot (\gamma - \beta_0) < 0.$ Then: So, since $\gamma - \beta_0 > 0$, we get: M < 0. Recall: $I = (\beta_0; \infty)$ and g is differentiable on I and $\sup \mathbb{I}_{q'} = M$. So, since M < 0, we get: g' < 0 on I. Then, by the Mean Value Theorem, g is strictly-decreasing on I. We conclude: $\forall \beta \in (\beta_0; \beta_1), \quad q(\beta) > q(\beta_1).$ This contradicts the hypothesis that, as $\beta \to (\beta_0)^+$, $g(\beta) \to -\infty$. \Box

Next, we prove that the pathology observed in Theorem 26.8

does not happen when DF_{ε} is open in \mathbb{R} and bounded below.

By Theorem 24.33, in typical Boltzmann applications,

 $DF_{\varepsilon} = (0; \infty)$, and so DF_{ε} is open in \mathbb{R} and bounded below.

THEOREM 26.10. Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta_0 \in \mathbb{R}$. Assume: $DF_{\varepsilon} = (\beta_0; \infty)$. Then: as $\beta \to (\beta_0)^+$, $\Delta_{\beta}^{\varepsilon} \to \infty$. *Proof.* By Theorem 24.39, ε is ∞ -proper. Then, by Theorem 23.4, \mathbb{I}_{ε} is bounded below. **Choose** $\xi \in \mathbb{R}$ s.t. $\xi + \mathbb{I}_{\varepsilon} \subseteq (0; \infty)$. Let $\widetilde{\varepsilon} := \varepsilon + \xi$. Then $\Delta_{\beta}^{\varepsilon} = e^{\beta \cdot \xi} \cdot \Delta_{\beta}^{\widetilde{\varepsilon}}$. Want: as $\beta \to (\beta_0)^+$, $\Delta_{\beta}^{\widetilde{\varepsilon}} \to \infty$. Otherwise, since $\beta \mapsto \Delta_{\beta}^{\tilde{\varepsilon}}$ is strictly-decreasing, we get $\{\Delta_{\beta}^{\tilde{\varepsilon}} \mid \beta \in DF_{\tilde{\varepsilon}}\}$ is bounded above. Let M be an upper bound. Since $\beta_0 \notin (\beta_0; \infty) = \mathrm{DF}_{\tilde{\varepsilon}}$, we get: $\Delta_{\beta}^{\tilde{\varepsilon}} = \infty$. That is, $\sum_{\sigma \in \Sigma} \left[e^{-\beta \cdot \tilde{\varepsilon}_{\sigma}} \right] = \infty$. Choose a finite subsum that is > M. Perturb β_0 to a slightly larger β . If the perturbation is small enough, then the finite subsum stays > M. Then $\Delta_{\beta}^{\tilde{\varepsilon}} \geq$ the perturbed finite subsum > M, contradicting that M is an upper bound. **THEOREM 26.11.** Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta_0 \in \mathbb{R}$. Then: as $\beta \to (\beta_0)^+$, $A_\beta^\varepsilon \to \infty$. Assume: $DF_{\varepsilon} = (\beta_0; \infty).$ *Proof.* Let $I := (\beta_0; \infty)$. Define $f : I \to \mathbb{R}$ by: $\forall \beta \in I, f(\beta) = \Delta_{\beta}^{\varepsilon}$. Then $f = X^0 S_{\bullet}^{\varepsilon}$, so, by Theorem 25.7, we get: $f' = -X^1 S_{\bullet}^{\varepsilon}$. $\forall \beta \in I, \ \mathbf{X}^1 \mathbf{S}^{\varepsilon}_{\beta} = \Gamma^{\varepsilon}_{\beta}.$ We have: $\forall \beta \in I, \ f'(\beta) = -\Gamma_{\beta}^{\varepsilon}.$ Then: **Define** $g: I \to \mathbb{R}$ by: $\forall \beta \in I, g(\beta) = -(\ln(f(\beta))).$ q is differentiable on I and, Then: by the Chain Rule, $\forall \beta \in I, \ g'(\beta) = -(f'(\beta))/(f(\beta)).$ $\forall \beta \in I, \quad g'(\beta) = \Gamma^{\varepsilon}_{\beta} / \Delta^{\varepsilon}_{\beta}.$ Then: $\forall \beta \in I, \ g'(\beta) = A_{\beta}^{\varepsilon}.$ Then: Want: as $\beta \to (\beta_0)^+$, $g'(\beta) \to \infty$. By Theorem 26.6, we get: g' is strictly-decreasing on I. By Theorem 26.10, we get: as $\beta \to (\beta_0)^+$, $\Delta_{\beta}^{\varepsilon} \to \infty$. Then: as $\beta \to (\beta_0)^+$, $f(\beta) \to \infty$. Then: as $\beta \to (\beta_0)^+$, $\ln(f(\beta)) \to \infty$.

Then: as $\beta \to (\beta_0)^+$, $g(\beta) \to -\infty$. Then, by Theorem 26.9, we get: as $\beta \to (\beta_0)^+$, $g'(\beta) \to \infty$.

27. Countably infinite sets of states

MORE LATER

Thanks once again to C. Prouty, for writing the Python code to do the Boltzmann computations in this paper:

First code: The GFA and 0, 2, 20 dollar awards, with average 3 dollars.

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
def F(beta):
z = np.zeros(3)
z[0] = 1
z[1] = np.exp(-2 * beta)
z[2] = np.exp(-20 * beta)
return z
def G(beta):
z = np.zeros(3)
z[0] = 0
z[1] = 2 * np.exp(-2 * beta)
z[2] = 20 * np.exp(-20 * beta)
return z
def f(beta):
return np.sum(F(beta))
def g(beta):
return np.sum(G(beta))
def bisection(minval, maxval, y, fn):
mid = (maxval + minval) / 2
while((fn(mid) - y) ** 2 > 0.0000001):
if(fn(mid) < y):
\max val = \min d
else:
minval = mid
mid = (maxval + minval) / 2
return mid
fn = lambda x: g(x) / f(x)
```

```
\begin{array}{l} \mathrm{target} = \mathrm{bisection}(-25,\,25,\,3,\,\mathrm{fn}) \\ \mathrm{b} = 0.07410049 \ \# \ \mathrm{hard}\text{-coded result of bisection} \\ \mathrm{r} = \mathrm{F}(\mathrm{b}) \ / \ \mathrm{f}(\mathrm{b}) \\ \mathrm{df} = \mathrm{pd}.\mathrm{DataFrame}(\mathrm{r}) \\ \mathrm{df.to\_excel}(\ "\mathrm{results2.xlsx", index=False}) \\ \mathrm{betas} = \mathrm{np}.\mathrm{linspace}(-25,25,100000) \\ \mathrm{z} = \mathrm{np}.\mathrm{zeros}(\mathrm{len}(\mathrm{betas})) \\ \mathrm{for \ i \ in \ range}(\mathrm{len}(\mathrm{betas})) \\ \mathrm{for \ i \ in \ range}(\mathrm{len}(\mathrm{betas})) \\ \mathrm{z[i]} = \mathrm{fn}(\mathrm{betas}[\mathrm{i}]) \\ \mathrm{plt.plot}(\mathrm{betas,z}) \\ \mathrm{plt.show}() \end{array}
```

Second code: The BUA and red bags and blue bags

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
def F(beta):
z = np.zeros(25).reshape(5,5)
for i in range(5):
for j in range(5):
z[i,j] = np.exp(-(i+j)*beta)
z[4,4] = 0
return z
def G(beta):
z = np.zeros(25).reshape(5,5)
for i in range(5):
for j in range(5):
z[i,j] = (i+j) * np.exp(-(i+j)*beta)
z[4,4] = 0
return z
def f(beta):
return np.sum(F(beta))
def g(beta):
return np.sum(G(beta))
def bisection(minval, maxval, y, fn):
```

```
mid = (maxval + minval) / 2
while((fn(mid) - y) ** 2 > 0.000001):
if(fn(mid) < y):
\maxval = mid
else:
minval = mid
mid = (maxval + minval) / 2
return mid
fn = lambda x: g(x) / f(x)
target = bisection(-25, 25, 1, fn)
b = 1.06697083 \# hard-coded result of bisection
\mathbf{r} = \mathbf{F}(\mathbf{b}) / \mathbf{f}(\mathbf{b})
df = pd.DataFrame(r)
df.to_excel("results5.xlsx", index=False)
betas = np.linspace(-25,25,100000)
z = np.zeros(len(betas))
for i in range(len(betas)):
z[i] = fn(betas[i])
plt.plot(betas, z)
plt.show()
```