Risk-neutral modelling with exponential Levy processes

For a martingale measure $\mathcal{Q}$

$$e^{-rt}S_t = \hat{S}_t = E^{\mathcal{Q}}[\hat{S}_T|\mathcal{F}_t]$$

An option with terminal payoff H_T has a value at time t given by the discounted expectation

$$\Pi_t(H_T) = e^{-r(T-t)}E^{\mathcal{Q}}[H_T|\mathcal{F}_t]$$

Generalize geometric Brownian motion with exponential Levy process

$$S_t = S_0\text{exp}((r-\sigma^2/2)t+\sigma W_t) \rightarrow S_t = S_0\text{exp}(rt+X_t)$$

with the restrictions on exponential moments

$$\int_{|x|\geq 1} e^x\nu(dx) < \infty$$

and the martingale condition

$$\gamma + \sigma^2/2 + \int(e^x - 1 - x1_{|x|\leq 1})\nu(dx) = \psi(-i) = 0$$

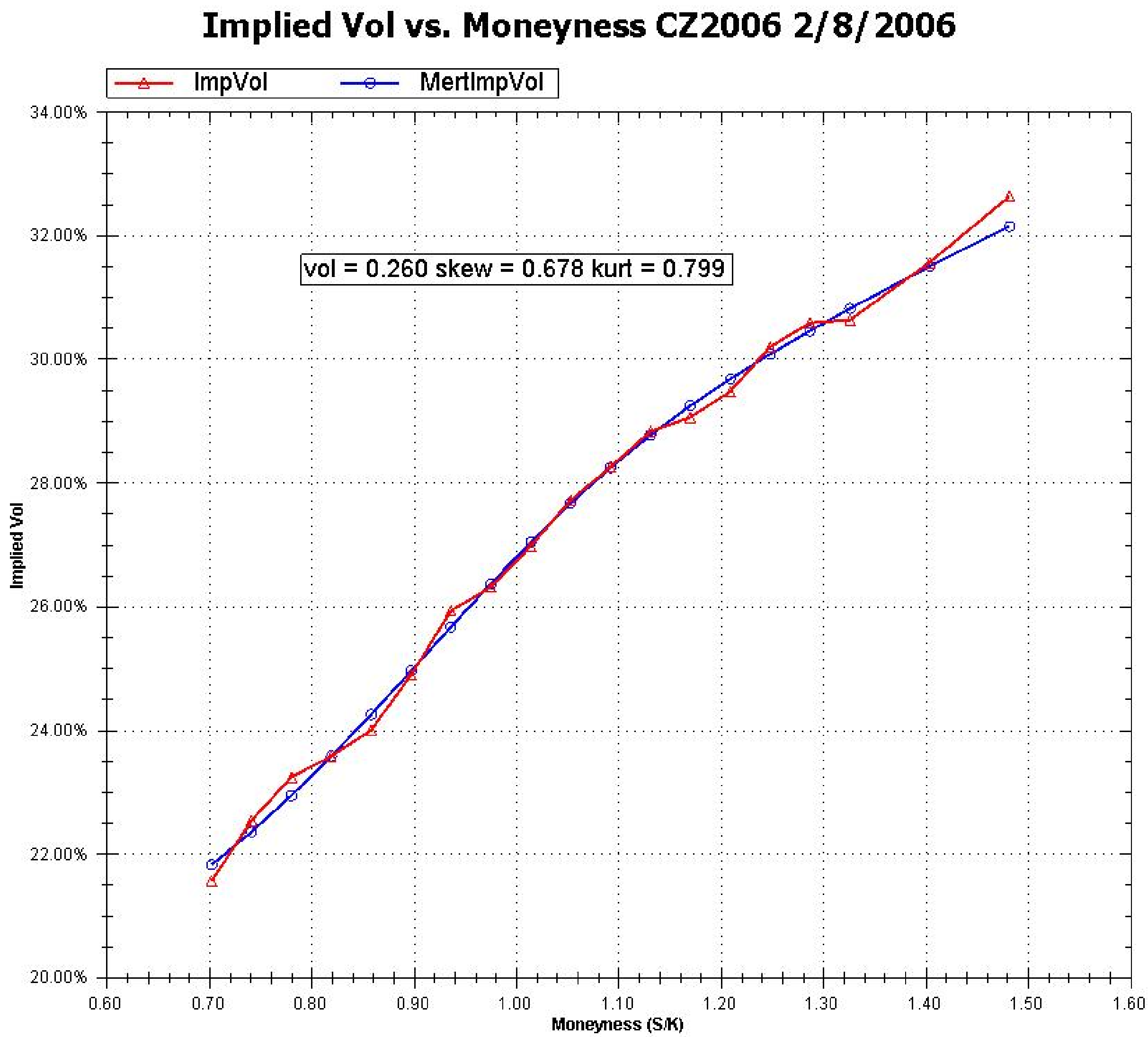
European Call options form a basis for other European payoffs

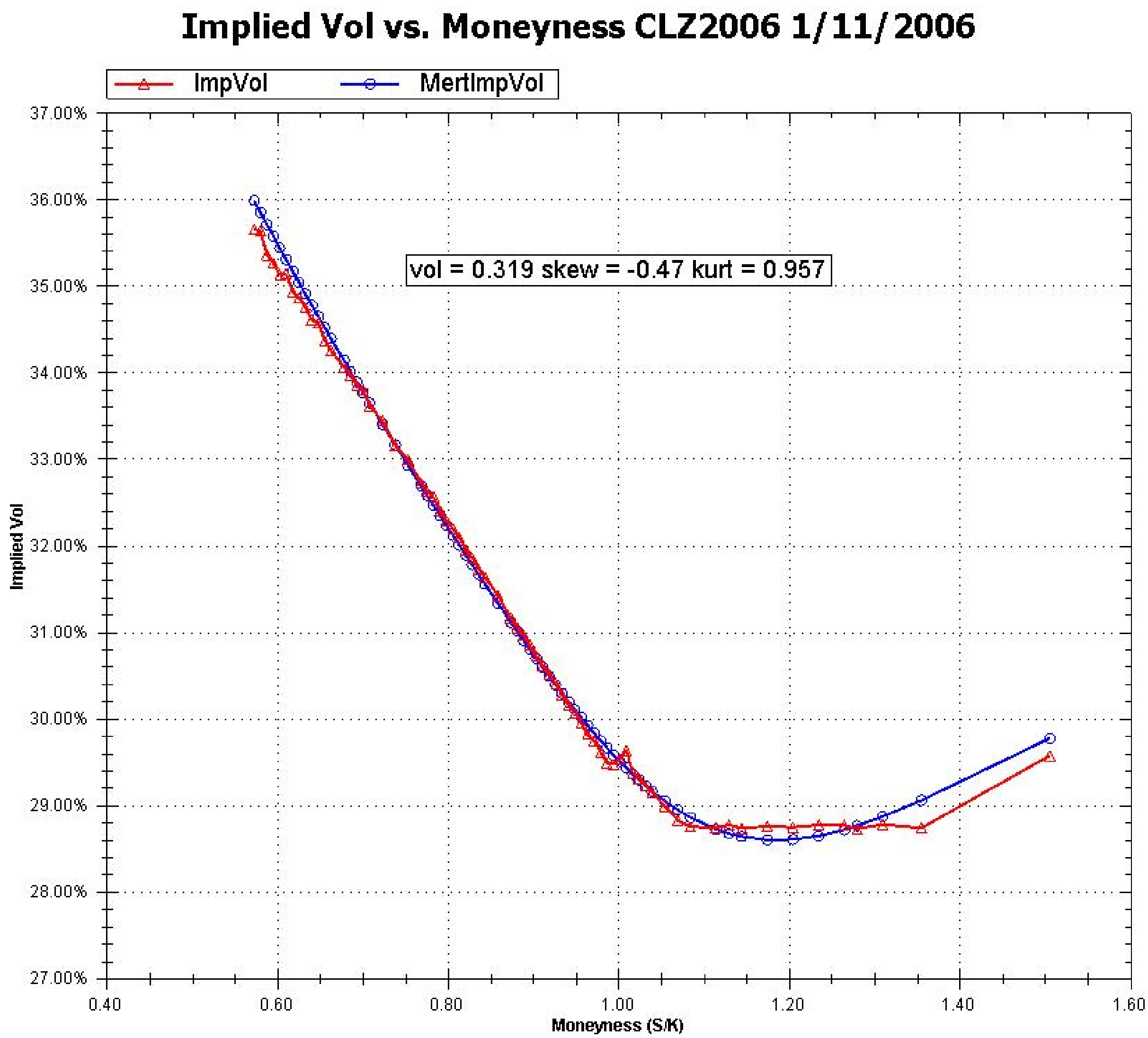
$$C_t(t,S_t;T,K) = e^{-r(T-t)}E^{\mathcal{Q}}[(S_T-K)^+|\mathcal{F}_t] = Se^{-r\tau}\int_k^\infty dx\rho_T(x)(e^{r\tau+x}-e^k)$$

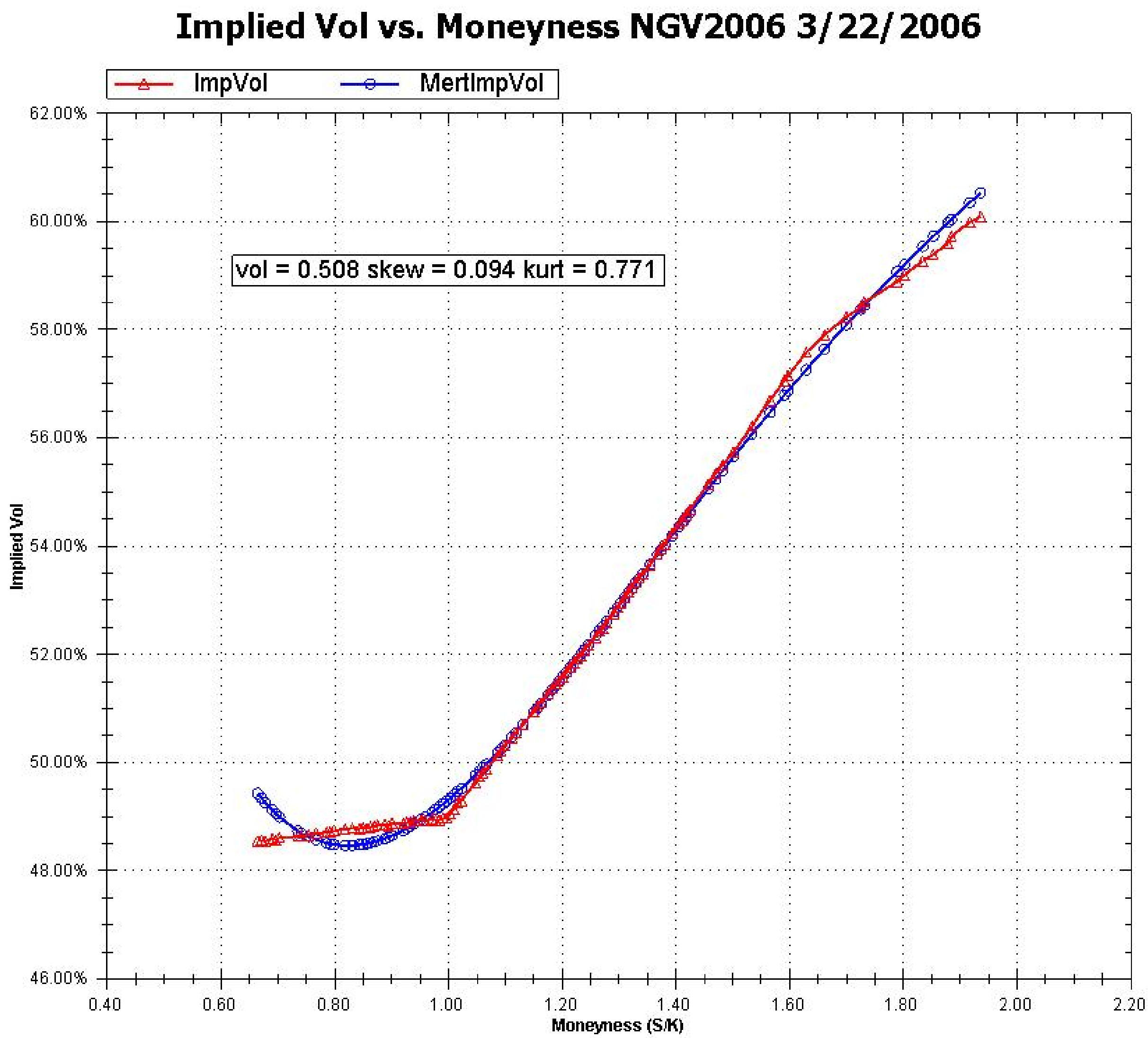
$\tau = T-t$ and $k = \ln(K/S)$

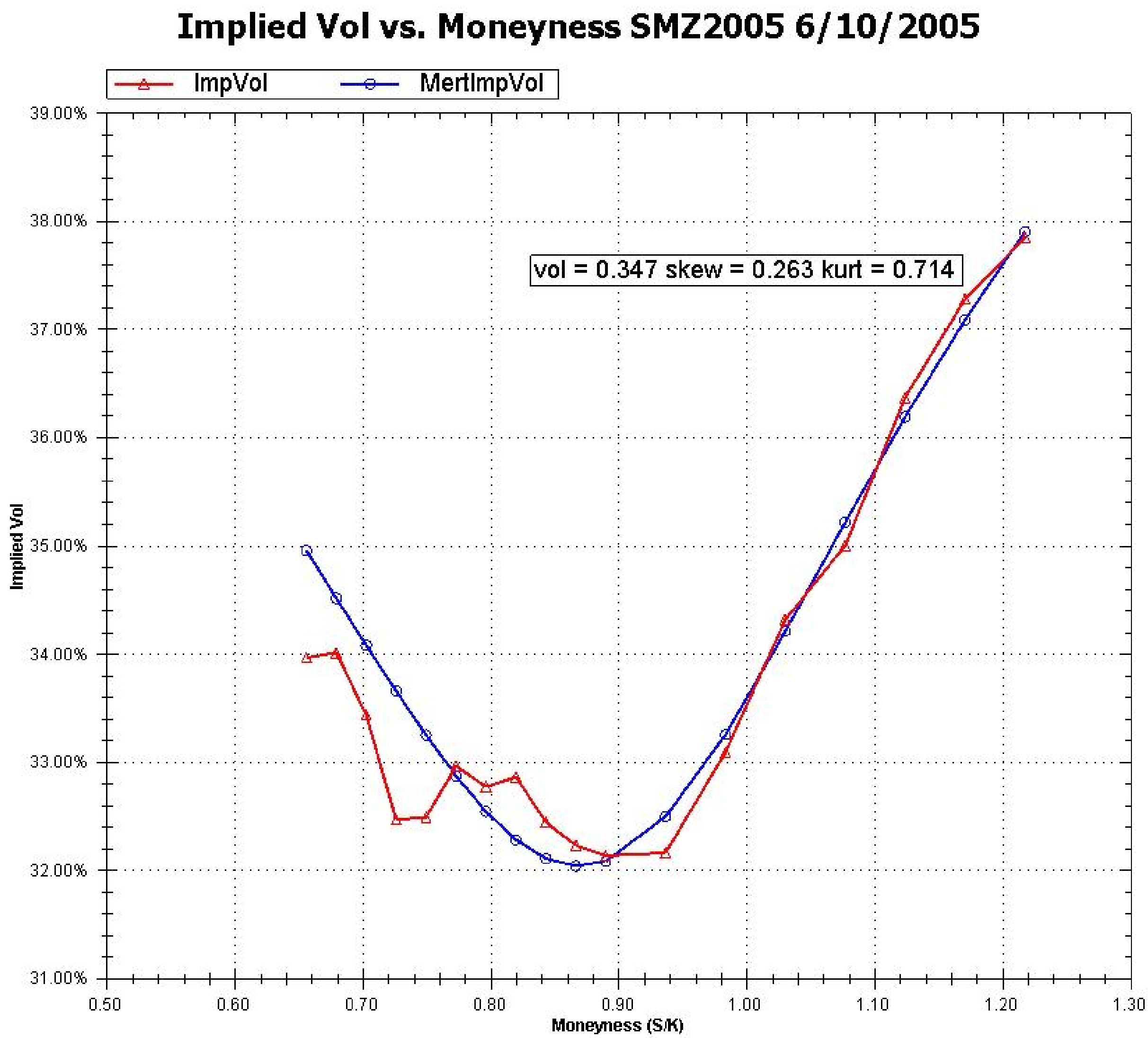
Implied volatility can be computed for each strike and maturity as an "inconsistency check" for Black-Scholes. $C_{BS}(T, K, \sigma^{imp}(T, K)) = C^{market}(T, K)$

Geometric Brownian motion alone constricts $\sigma^{imp}(T, K)$ to a 2-d flat plane. Here are a few cross sections of $\sigma^{imp}(T, K)$ vs. K/S (moneyness) for a given T for a few different commodity futures markets.




Implied Vol vs. Moneyness CLZ2006 1/11/2006
ImpVol
MertImpVol
vol = 0.319 skew = -0.47 kurt = 0.957
Implied Vol
Moneyness (S/K)


Implied Vol vs. Moneyness NGV2006 3/22/2006
ImpVol
MertImpVol
vol = 0.508 skew = 0.094 kurt = 0.771
62.00%
60.00%
58.00%
56.00%
54.00%
52.00%
50.00%
48.00%
46.00%
Implied Vol
0.40
0.60
0.80
1.00
1.20
1.40
1.60
1.80
2.00
2.20
Moneyness (S/K)



General statements about implied volatility surfaces of exponential Levy processes

- Addition of jumps allows *an* explanation of implied volatility surfaces seen in financial market. Skew (slope) and Smile (curvature) of these surfaces can be fit to model parameters with good precision.
- IV surfaces are constant with time to expiration (floating smile) and

independent of S_t (sticky delta or moneyness). The second feature is opposed sticky strike (generally stochastic volatility models. cf Chap 15) which have a correlation between S_t and $\sigma^{imp}(T-t,K)$

- Short term skew is well represented by the jumps of levy processes
- Flattening of the skew with option maturity. This occurs in accord with the central limit theorem $c_3 \propto 1/\sqrt{T}$ and $c_4 \propto 1/T$ as shown in Chap 3. However Additive processes of Chap 14 can modify this moment decay.

So how do we compute $C_T(k) = Se^{-r\tau}\int_k^\infty dx\rho_T(x)(e^{r\tau+x}-e^k)$?

Given the characteristic function $\Phi_T(\nu)$ we can use an assortment of Fourier Transform techniques that all have a similar smell to them.

1. Method of Scott, Chen, Heston, Bates, et. al.: Calculate delta (Π_1) and probability option expires in the money (Π_2) with Fourier variable k =ln(K)

   $$C(k) = S\Pi_1 - e^{-r\tau}K\Pi_2$$
   $$\Pi_1 = \tfrac{1}{2} + \tfrac{1}{2\pi}\int d\nu e^{-i\nu k}e^{i\nu r\tau}\frac{\Phi_\tau(\nu-i)}{i\nu\Phi_\tau(-i)}$$
   $$\Pi_2 = \tfrac{1}{2} + \tfrac{1}{2\pi}\int d\nu e^{-i\nu k}e^{i\nu r\tau}\frac{\Phi_\tau(\nu)}{i\nu}$$

   Although delta (not necessarily hedge ratio) is computed for free, convergence is slow and FFT difficult

2. Method A of Carr and Madan: Since $C(k)$ is not integrable, subtract time value $= (e^{s+r\tau}-e^k)^+$

   $$C(k) = (e^{s+r\tau}-e^k)^+ + \tfrac{1}{2\pi}\int d\nu e^{-i\nu k}e^{i\nu r\tau}\frac{\Phi_\tau(\nu-i)-1}{i\nu(1+i\nu)}$$

For better convergence, one can replace time value with a Black-Scholes option value $C_{BS}^{\sigma^*}(k)$ to yield a smooth function:

$C(k) = C_{BS}^{\sigma^*} + \frac{1}{2\pi}\int d\nu e^{-i\nu k} e^{i\nu r\tau} \frac{\Phi_\tau(\nu - i) - \Phi_{BS}^{\sigma^*}(\nu - i)}{i\nu(1 + i\nu)}$

Still converges slowly but FFT can be utilized.

3. Method B of Carr and Madan: Dampened Call price $e^{\alpha k}C(k)$ is integrable with $\alpha > 0$

   $\Xi(\nu) = \int d\nu e^{i\nu k} e^{\alpha k} C(k) = \frac{e^{-r\tau}\Phi_\tau(\nu - i(\alpha + 1))}{\alpha^2 + \alpha - \nu^2 + i(2\alpha + 1)\nu}$

   $C(k) = \frac{e^{-\alpha k}}{2\pi}\int d\nu e^{-i\nu k}\Xi(\nu)$

   Converges much quicker, but there might be additional conditions on α to insure only imaginary roots of $\Xi(\nu)$ and thus integrability along Re(ν)

4. Method of Lewis using Generalized FT: $\mathbf{F}g(z) = \int e^{izx} g(x) dx$ with z complex and $x = \ln(S_\tau/S_0)$ as Fourier variable.

   $\mathbf{F}C(z) = \frac{\Phi_\tau(-z)e^{(1+iz)(k-r\tau)}}{iz(iz + 1)}$

   $1 < \mathcal{I}z = \mu < 1 + \alpha$

   $C(x) = \frac{e^{\mu x + (1-\mu)(k-r\tau)}}{2\pi}\int_{-\infty}^{\infty} du \frac{e^{iu(k-r\tau-x)}\Phi_\tau(-i\mu - u)}{(iu - \mu)(1 + iu - \mu)}$

   Similar convergence and α choice issues as (3)

From limited experience, (3) seems to work quite well except when calibration takes model parameters into integrand into delinquent territory.