Basic Tools

Characteristic Functions

cadlag, caglad, price processes, trading strategies.

filtrations

Exponential random variables

\[ \lambda e^{-\lambda y} 1_{y \geq 0} \]

We can define stack one after another

\[ \tau_1 + \ldots + \tau_n \]

If we define

\[ N_t = \inf\{ n \geq 1, \sum_{i=1}^{n} \tau_i > t \} \]

then
$P(N_t = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$

So, $N_t$ follows a Poisson distr with parameter $\lambda t$.

$N_t$ is called a Poisson process.

It has indep increments and the distr of $N_t - N_s$ is the same as the one of the $N_{t-s}$.

The characteristic function is:

$E(e^{iuN_t}) = e^{\lambda t(e^{iu} - 1)}$

It is not a martingale (we keep on adding 1s)

The expected value is $\lambda t$.  

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Defining a new process as $\tilde{N}_t = N_t - \lambda t$ (the compensated process) gives us a martingale.

The Poisson process will be used to model jumps, but we have to modify it (all the jumps are of size 1)

Random Measures

Call $T_1, T_2, \ldots$ the jump times of a Poisson process

given a set $A$ on the $\mathbb{R}^+$ we can define

$$M(\omega, A) = \#\{i \geq 1, T_i(\omega) \in A\}$$

It satisfies:

$$E(M(A)) = \lambda |A|$$
$M$ is the derivative of the Poisson process: it is a sum of Dirac deltas.

For any $A$ (measurable) $M(A)$ follows a Poisson distr with parameter $\lambda|A|$.

We can also associate a random measure to the compensated Poisson process.

\[ \tilde{M}(\omega, A) = M(\omega, A) - \lambda|A| \]

We can also define a notion of Poisson random measure (without starting with a Poisson process) following these ideas.