Levy Processes

Essentially a superposition of:

a) A Wiener Process

b) A number of Poisson processes.

Definition: A cadlag process X_t is a Levy process if it satisfies:

1. Independent increments.

2. Stationary increments.

3. Stochastic continuity: for any $\epsilon > 0$

$$lim_{h->0} P(|X_{t+h} - X_t| \geq \epsilon) = 0$$

Condition 3 excludes jumps at fixed times.

Definition: A prob distr F is called infinitely divisible if for any $n \geq 2$ there exists iid rv's $Y_1 ... Y_n$ so that $Y_1 + ... + Y_n = F$.

Then if X is a Levy process then the distr of X_t is inf div.

Proposition 3.1: The reciprocal is also true. IF F is an inf div distr then there exist a Levy process X so that X_1 has distr F.

Now, the characteristic function of X_t is:

$$\Phi_t(z) = E(e^{iz \cdot X_t})$$

By ind and stat increments:

$$\Phi_{t+s}(z) = \Phi_t(z)\Phi_s(z)$$

Also, as a fction of t it is continuous (stoch cont is stronger than distr conv). Then Φ is an exponential fction.

$$\Phi_t(z) = e^{t\Psi(z)}$$

Compound Poisson Processes

Def: A compound Poisson process with intensity λ and jump size distr f is a process X_t such that:

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where jump sizes Y_i are iid with distr f and N_t is a Poisson process with intensity λ indel from the Ys.

Interpretation: If we construct a random walk R with distr f then

$$R(n) = \sum_{i=1}^{n} Y_i$$

Now change the time using a Poisson process N_t.

What is the expected value of the compound Poisson process?

If $E(Y_i) = \mu$ then

$$E(X_t) = E(E(\sum_{i=1}^{N_t} Y_i | N_t)) = E(N_t E(Y_i)) = \lambda t \mu$$

Proposition 3.3: X_t is a compound Poisson process iff it is a Levy process and its sample paths are piecewise constant.

Now, any cadlag fction can be approx by piecewise constant fctions. So, maybe, compound a Poisson processes have something to say about general Levy processes.

Proposition 3.4: The characteristic fction of a compound Poisson process is:

$$E(e^{iu\cdot X_t}) = e^{t\lambda\int(e^{iu\cdot x}-1)f(dx)}$$

(Remember that the char fction of a Poisson process is $E(e^{iu\cdot N_t}) = e^{t\lambda(e^{iu}-1)}$)

We will see in a second that we will combine $\lambda f(dx)$ and we will call that $\nu(dx)$.

Jump Measures of Compound Poisson Processes

$$J_X(B) = \#\{(t, X_t - X_{t-}) \in B\}$$

Prop 3.5: If X_t is a compound Poisson process with intensity λ and jump distr f then J_X is a Poisson random measure with intensity measure $\mu(dxxdt) = \nu(dx)dt = \lambda f(dx)dt$

So, on average, per unit of time, $\lambda f(dx)$

counts how many jumps of a certain size there are.

Then, for a compound Poisson process X_t we can write:

$$X_t = \sum_{s \in [0,t]} \Delta X_s = \int_{[0,t] \mathrm{X} R^d} x J_X(ds \mathrm{x} dx) \tag{1}$$

where J_X is a Poisson random measure with intensity $\nu(dx)dt$.

Remark: Compensated Compound Poisson Process

We know that the expected value of X_t is $\lambda t \mu$, so $X_t - \lambda t \mu$ is a centered version of X_t.

Written in terms of measures this is

$$X_t - \lambda t \mu = \int_{[0,t] \mathrm{x} R^d} x J_X(ds \mathrm{x} dx) - \int_{[0,t] \mathrm{x} R^d} x \lambda f(dx) dt$$

This motivates the definition of of the Levy measure:

Given X_t a Levy process we define

$$\nu(A) = E(\#\{t \in [0,1] : \Delta X \neq 0, \Delta X \in A\})$$

$\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A.

As we said before Levy processes are essentially BMs mixed up with (compound) Poisson processes.

If we define X_t as

$$X_t = bt + W_t + N_t, \ (W_t, N_t \text{ indep})$$

where N_t is compound Poisson with intensity λ and jump size distr q then its characteristic exponent is:

$$\Psi(z) = ib\cdot z + 1/2 z\cdot Az + \int (e^{iz\cdot x} - 1)\lambda q(dx)$$

Is this the most general type of characteristic exponents that we can find for Levy processes?

In the 1920s Italian mathematician B. de Finetti conjectured that the answer was yes.

He was close, however the most general Ψs are found by replacing the finite measure λq with a σ-finite measure ν.

But if we do this we might not be able to integrate $(e^{iz\cdot x} - 1)$ so something needs to be done.

Idea:

We would like to say that

$$X_t = bt + W_t + N_t$$

but by (1) that is

$$X_t = bt + W_t + \sum_{s \in [0,t]} \Delta X_s = \int_{[0,t] \mathrm{x} R^d} x J_X(ds \mathrm{x} dx)$$

saying that the measure ν might not be finite amounts to saying that there might be infinitely many (small since cadlag) jumps so that integral might not converge

so we can truncate

$$\int_{[0,t] \mathrm{x} R^d} x J_X(ds \mathrm{x} dx) =$$

$$\int_{|x|>1} x J_X(ds \mathrm{x} dx) +$$

$$\int_{1/n \leq |x| < 1} x J_X(ds \mathrm{x} dx)$$

and try to take the $lim_{n->\infty}$.

However that limit might not exist.

So, Levy proposed to deal with the second term by considering the compensated process. So, lets define:

$$M_n(t) = \int_{1/n \leq |x| < 1} x(J_X(ds\text{x}dx) - \nu(ds))ds$$

The M_ns are martingales and it can be proved that they converge (in mean square) to a martingale

$$M(t) = \int_{0 < |x| < 1} x(J_X(ds\text{x}dx) - \nu(ds))ds$$

So, the Levy-Ito decomposition says that the piece of the compound Poisson process has to be written as two pieces.

Infinite activity Levy processes

Given a BM with drift indep from X^0 (compound Poisson process) we can define a new Levy process as:

$$X_t = \gamma t + W_t + \int_{[0,t]\mathrm{x}R^d} x J_X(ds\mathrm{x}dx)$$

- Can every Levy process be represented in this form?

- $\nu(A) < \infty$ if A does not contain 0.

Prop 3.7. Levy-Ito decomposition

Let X_t be a Levy process and ν its Levy measure.

- ν is a Radon measure on $R^d - 0$ and

$$\int_{|x|\leq 1} |x|^2 \nu(dx) \leq \infty$$

$$\int_{|x|geq1} \nu(dx) \leq \infty$$

- The jump measure is a Poisson random measure with intensity measure $\nu(dx)dt$.

-There exist a vector γ and a d-dimensional BM B_t with covariance A st

$$X_t = \gamma t + B_t + X_t^l + lim_{\epsilon->0} X_t^\epsilon$$

where

$$X_t^l = \int_{|x|\geq 1, s\leq t} x J_x(ds\text{x}dx)$$

$$X_t^\epsilon = \int_{\epsilon\leq |x|<1, s\leq t} x(J_x(ds\text{x}dx) - \nu(dx))ds$$

Meaning:
X_t^l has a finite number of jumps bigger than 1

so this is a compound Poisson process.

However, for a general Levy process, the measure ν can have as singularity at 0. There can be infinitely small jumps as we approach zero and the sum does not need to converge.

To force convergence we need to center this term, replace it with its compensated version.

Using this we get the representation due to Levy and Khinchin:

If X_t is a Levy process with characteristic triplet (A, ν, γ) then

$$E(e^{iz \cdot X_t} = e^{t\Psi(t)}$$

where

$$\Psi(z) = -1/2z \cdot Az + i\gamma \cdot z +$$

$$\int (e^{iz \cdot x} - 1 - iz \cdot x 1_{|x| \leq 1}) \nu(dx)$$

Piecewise constant paths: if it is Compound Poisson.

Finite Variation paths: If it has no Gaussian part ($A = 0$) and

$$\int_{|x|\leq 1} |x|\nu(dx) < \infty$$

Stable Laws

The BM is selfsimilar

$$\frac{W_{at}}{\sqrt{a}} \sim W_t$$

If it has a drift we need to translate:

$$\frac{B_{at}}{\sqrt{a}} \sim B_t + \sqrt{a}\gamma t$$

where γ is the drift

A Levy process is called selfsimilar if for any a there is a $b(a)$ st:

$$\frac{X_{at}}{b(a)} \sim X_t$$

But the characteristic fction is

$$\Phi_{X_t}(z) = e^{-t\Psi(z)}$$

so

the condition is equivalent to

$$\Phi_{X_t}(z)^a = \Phi_{X_t}(zb(a))$$

Distributions satisfying this are called strictly stable.

They are called stable if for every a there is a $b(a) > 0$ and $c(a)$ st

$$\Phi_{X_t}(z)^a = \Phi_{X_t}(zb(a))e^{ic\cdot z}$$

The name comes from the fact that if X has stable distr and $X^1, ..., X^n$ are indep copies of X then there is a c_n and d so that

$$X^1 + ... + X^n \sim c_n X + d$$

It can be proved that for every stable distr there exists α in $(0, 2]$ such that $b(a) = a^{1/\alpha}$.

In 1-dim X is a α-stable variable with $0 < \alpha < 2$ then its Levy measure is of the form

$$\nu(x) = \frac{A}{x^{\alpha+1}} 1_{x>0} + \frac{B}{|x|^{\alpha+1}} 1_{x<0}$$

for some A, B positive constants.

so α-stable distributions on R never admit a second moment and they only admit a first moment if $\alpha > 1$.

Tail Behaviour

Proposition 3.13: X_t a Levy process with (A, ν, γ).

$E(|X_t|^n) < \infty$iff $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$

The kurtosis of X_t is higher than 3 so they are leptokurtic. (cumulant is positive as long as the Levy measure is not zero)

The kurtosis of the increments decay as $1/h$ (timestep).