Chapter 4

Mathematical Background
Stopping Times

- Stopping time:
  \[ \{ \tau \leq t \} \in \mathcal{F}_t \]

- Indicator process:
  \[ N_{\tau}(t) := 1_{\{\tau \leq t\}} \]

- Predictable stopping time: it has an announcing sequence.

- Totally inaccessible stopping time: No predictable stopping time can give any information.

\[ P(\tau = \tau' < \infty) = 0 \]

for any \( \tau' \) predictable.
Let $\tau$ be a stopping time and $F(T)$ its distribution function. Its hazard rate is defined as:

$$h(t, T) = \frac{f(t, T)}{1 - F(t, T)}$$

where $F(t, T) = P(\tau \leq T|F_t)$

Interpretation:

$$h(t, T) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(\tau \leq T + \Delta t|\tau > t)$$

Or, by looking at:

$$F(t, T) = 1 - e^{-\int_t^T h(t,s) ds}$$

we see that, again, it is like forward rates.
\{ \tau_i, i \in \mathbb{N} \} = \{ \tau_1, \tau_2, \ldots \}

Counting Process

\[ N(t) := \sum_{i} 1_{\{\tau_i \leq t\}} \]

Predictable Compensator Process

\[ M(t) = N(t) - A(t) \text{ is a martingale} \]

If \( A \) is differentiable we define the intensity as:

\[ A(t) = \int_{0}^{t} \lambda(s)ds \]
Assume that $A$ is differentiable.

These type of models are called intensity models (chapter 7).

All the models in chapter 9 don't satisfy this.
Hazard rates and intensity are related, under some conditions:

\[ \lambda(t) = h(t, t) \]

There are two ways of viewing a counting process:

- As a stochastic process (predictable compensator, intensities, etc.)
- By looking at the distribution of the next jump time (using hazard rates)
If we know $P(t, T)$ and it is differentiable wrt $T$ (at $T = t$) then (under conditions of theorem 4.1):

$$\frac{dA(t)}{dt} = -\frac{\partial}{\partial T}|_{T=t} P(t, T) = h(t, t)$$

Converse is not true.

Starting from the intensity does not always give easy access to the survival probability.
A marked point process is a point process in which the jumps are stochastic:

\[ \{(\tau_i, Y_i), i \in \mathbb{N}\} = \{\tau_1, \tau_2, \ldots\} \]

One way to generalize the counting process is:

\[ X(t) := \sum_i Y_i 1_{\{\tau_i \leq t\}} \]

However, sometimes \( Y \) could take values that are not numbers (the name of the defaulting company, jumps in the rating classes etc.)

Because of this we use a different approach: the jump measure.
We first define the concept of random measure:
\( \nu : \Omega \times \mathcal{E} \times \mathcal{B}(\mathbb{R}_+) \rightarrow \mathbb{R}_+ \) is a random measure if for every \( \omega \in \Omega \), \( \nu(\omega, \cdot, \cdot) \) is a measure on \( ((E \times \mathbb{R}_+), \mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+)) \) and \( \nu(\omega, E, 0) = 0 \) identically.

We can use random measures to construct stochastic processes by integrating.

The jump measure of a marked point process is a random measure:
\[
\mu(\omega, E', [0, t]) = \int_0^t \int_{E'} \mu(\omega, de, ds) := \sum_{i=1}^{\infty} 1_{\{\tau_i(\omega) \leq t\}} 1_{\{Y_i(\omega) \in E'\}}
\]

By integrating against the jump measure we can represent functionals of the marked point process.
The idea here is that, given a random measure, there exists a predictable random measure so that for every predictable stochastic function $f(\omega, e, t)$ the process defined by:

$$M(\omega, t) := \int_0^t \int_E f(\omega, e, s) \mu(\omega, de, ds) - \int_0^t \int_E f(\omega, e, s) \nu(\omega, de, ds)$$

is a local martingale.

Many times we can separate the probability that an event occurs from the conditional distribution of the marker given that an event has occurred.

$$\nu(de, dt) = K(t, de)dA(t) \quad \text{with} \quad \int_E K(t, de) = 1$$
The Compensator Measure II

In discrete time:

- Suppose

  \[ X(\omega, t) = \int_{-\infty}^{+t} \int_{E} f(s, e) \mu(de, ds) \]

- In discrete time:

  \[ X(t_n) - X(t_{n-1}) = \int_{E} f(t_n, e) \mu_n(de) \]

- \( f \) has to be adapted (for \( X \) to be).

- We will ask it to be predictable: at time \( t_{n-1} \) we will know what \( f \) will be at time \( t_n \) conditioned on \( Y \).

- Define \( \nu_n(de) = P(Y \in de \text{ and } \tau = t_n|\mathcal{F}_n) \)
So:

\[ E((X(t_n) - X(t_{n-1}))|\mathcal{F}_{n-1}) = \int_{E} f(t_n, e)\nu_n(de) \]

We can now construct the compensator:

\[ A(t_n) - A(t_{n-1}) = \int_{E} f(t_n, e)\nu_n(de) \]

Then \( A \) is predictable and \( X - A \) is a martingale.
Examples

- **Poisson Process** $N(t)$ with intensity $\lambda$ (constant)
  - Compensator measure $\nu(de, dt) = \delta_{Y=1}(de)\lambda dt$
  - Conditional distribution $dA(t) = \lambda dt, K(de) = \delta_{Y=1}(de)$

- **Poisson Process** $N(t)$ with intensity $\lambda(t)$ (stochastic)
  - Compensator measure $\nu(de, dt) = \delta_{Y=1}(de)\lambda(t)dt$
  - Conditional distribution $dA(t) = \lambda(t)dt, K(de) = \delta_{Y=1}(de)$

- **Marked inhomogeneous Poisson Process I**
  - Marker: $Y \sim N(0, 1)$.
  - Compensator measure $\nu(de, dt) = \frac{1}{\sqrt{2\pi}} e^{-1/2e^2} \lambda(t) de dt$
  - Conditional distribution $dA(t) = \lambda(t)dt, K(de) = \frac{1}{\sqrt{2\pi}} e^{-1/2e^2} de$
More Examples

- **Marked Poisson process II**
  - Marker $Y$ is the value of a geometric brownian motion at time $t$ (the time of the jump).
  - Compensator measure $\nu(de, dt) = \delta_{Y=S(t-)}(de)\lambda(t)dt$
  - Conditional distribution
    $$dA(t) = \lambda(t)dt, K(de) = \delta_{Y=S(t-)}(de)$$

- **Lognormal Jump Diffusion**
  - Jump times triggered by a Poisson process with parameter $\lambda$.
  - Marker $Y$ (log of the jump size) is $N(0, 1)$.
  - Compensator measure $\nu(de, dt) = \frac{1}{\sqrt{2\pi}} e^{-1/2e^2} \lambda(t)dedt$
More Examples

- First hitting time process
  - Arrival time is the first time that a geometric brownian motion $S(t)$ hits a barrier.
  - No marker
  - Compensator measure $\nu(dt) = dA(t)$ where
    
    $$dA(x) = \begin{cases} 
    1 & \text{if the barrier is hit } S(t) = \bar{K} \\
    0 & \text{otherwise}
    \end{cases}$$

- Since the default arrival is predictable its compensator is the process itself.
The Compensator Measure VII

More Examples

- A (maybe not so) unusual process
  - Compensator measure \( \nu(de, dt) = \frac{1}{|e|} \, dedt \) for \( 0 \notin de \) where
  - This process has an infinite number of very small jumps and a few larger ones.
  - If \([a, b]\) is an interval away from zero then jumps of a size in \([a, b]\) occur with an intensity of

\[
\lambda_{[a,b]} = \int_a^b \frac{1}{|e|} \, de
\]

- So, the process can be viewed as a collection of Poisson processes, one Poisson process per interval in \( \mathbb{R} \). The intensity converges to infinity the closer we get to zero.
- In the book he assumes that the processes have a finite number of jumps in any finite interval. So, processes like this are excluded.
More Examples

- A very simple process
  - Jumps occur at $\tau_1 = 2, \tau_2 = 4, \tau_3 = 8, \ldots$
  - This is known at the beginning.
  - This is known at the beginning so it is predictable and then its compensator is the jump measure itself:

$$\nu(de, dt) = \delta_{t=\tau_i}(dt)$$
Itô’s Lemma For Jump Processes

- The processes considered have RCLL paths.
- Notation $\Delta X(t) := X(t) - X_-(t)$, $X^d(t) := \sum_{s \leq t} \Delta X(s)$, $X^c(t) := X(t) - X^d(t)$.
- Let $X = (X^1, ..., X^n)$ be an $n$-dimensional semi-martingale with a finite number of jumps and $f$ a twice differentiable function on $\mathbb{R}^d$. Then $f(X)$ is also a semi-martingale and:

$$f(X(t)) - f(X(0)) = \sum_{i=1}^n \int_0^t \frac{\partial f(X_-(s))}{\partial x_i} dX^c,i(s) +$$

$$\frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f(X_-(s))}{\partial x_i \partial x_j} d < X^c,i, X^c,j > (s) +$$

$$\sum_{s \leq t} \Delta f(X(s))$$
The jump times $\tau_i$ and the jump sizes $\Delta X(\tau_i)$ define a marked point process.

This marked point process has a jump measure $\mu_x$ (which puts mass 1 on the jump times and sizes of the jumps), and a compensator measure $\nu_x$.

The process $X$ can be rewritten:

$$dX(t) = dX^c(t) + \int_{\mathbb{R}^k} x \mu_x(dx, dt)$$
Using the jump measure:

\[ f(X(t)) - f(X(0)) = \sum_{i=1}^{n} \int_{0}^{t} \frac{\partial f(X_-(s))}{\partial x_i} dX^{c,i}(s) + \]

\[ \frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{t} \frac{\partial^2 f(X_-(s))}{\partial x_i \partial x_j} d < X^{c,i}, X^{c,j} > (s) + \]

\[ \int_{0}^{t} \int_{\mathbb{R}^n} f(X_-(s) + x) - f(X_-(s)) \mu_x(dx, ds) \]
Applications of Itô’s Lemma

- In a lot of applications $X$ can be written as a jump diffusion process

$$dX^i = \alpha^i dt + \sum_{k=1}^{K} \sigma_{ik} dW_k + \int_{\mathbb{R}^n} h_i(x) \mu_X(dx, dt)$$

- And the compensator measure $\nu$ can be decomposed as

$$\nu_X(dx, dt) = K(t, dx) dA(t)$$
Can do Itô to find \( f(X) \) and its compensator.

In this case the predictable compensator is the sum of the usual drift and:

\[
\int_0^t \left( \int_{\mathbb{R}^n} f(X_-(s) + x)K(s, dx) - f(X_-(s)) \right) dA(s)
\]

which compensates for the influence of the jumps.

\[
\int_{\mathbb{R}^n} f(X_-(s) + x)K(s, dx)
\]

represents the expected value of \( f \) after a jump at time \( s \).
Applications of Itô’s Lemma III

Itô product and quotient rule.

Let $Y$ and $Z$ be

$$
\frac{dY}{Y} = \alpha^y + \sum_{k=1}^{K} \sigma^y_k dW_k(s) + \int_{\mathbb{R}^n} h^y(x) \mu_X(dx, dt)
$$

$$
\frac{dZ}{Z} = \alpha^z + \sum_{k=1}^{K} \sigma^z_k dW_k(s) + \int_{\mathbb{R}^n} h^z(x) \mu_X(dx, dt)
$$

so, the jumps of both processes are driven by the jumps of a third process $X$.

So doing Itô can find the process $g(Y, Z) = YZ$. 

The stochastic exponential

- Let $X$ be a stochastic process with $\Delta X \geq -1$. Then $Y(t)$ is called the stochastic exponential of $X$ iff $Y$ solves:

  $$dY(t) = Y_-(t)dX(t)$$

- If $X$ has finitely many jumps:

  $$Y(t) = e^{X^c(t)-X^c(0)-\frac{1}{2}<X^c>(t)} \prod_{s \leq t}(1 + \Delta X(s))$$
Let $Q$ be a probability measure If for every dividend-free traded asset with price process $p(t)$ the discounted process $\frac{p(t)}{b(t)}$ is a martingale under $Q$ then $Q$ is called a martingale measure.

This is important because its existence is equivalent to absence of arbitrage.
Radon-Nikodym: Given two measures $Q$ and $P$ so that $P \ll Q$ ($Q(A) = 0 \Rightarrow P(A) = 0$) there exists a density $L$ so that $E^P(X) = E^Q(LX)$ for all measurable $X$.

In a dynamic model we define $L(t) = E^Q(L|\mathcal{F}_t)$ then, if $X$ is $\mathcal{F}_T$-measurable:

$$E^P(X|\mathcal{F}_t) = E^Q(LX|\mathcal{F}_t) = E^Q(E^Q(LX|\mathcal{F}_T)|\mathcal{F}_t) =$$

$$= E^Q(E^Q(L|\mathcal{F}_T)X|\mathcal{F}_t) = E^Q(L(T)X|\mathcal{F}_t) =$$

$$L(t)E^Q\left(\frac{L(T)}{L(t)}X\right)|\mathcal{F}_t$$
Girsanov Theorem

- It tells us how probabilistic properties of processes change when we change measures.
- A brownian motion under a measure $Q$ does not need to be a brownian motion under $P$.
- Jump measures don’t change (since path are unchanged) but compensator measures will change (since compensators determine probabilities.)
Assume a probability space with a brownian motion \((W_Q(t))\) and a marked point process \(\mu(de, dt)\) with its compensator \(\nu_Q(de, dt) = K_Q(de)\lambda_Q(t)dt\).

Define a process \(L\) as:

\[
\frac{dL(t)}{L(t-)} = \varphi(t)dW_Q(t) + \int_E (\Phi(e, t) - 1)(\mu(de, dt) - \nu_Q(de, dt))
\]

Then:

\[
dW_P(t) = dW_Q(t) - \varphi(t)dt\] is a \(P\)-brownian motion
Girsanov Theorem III

- The compensator under $P$ is:

\[ \nu_P(de, dt) = \Phi(t, e)\nu_Q(de, dt) \]

- If $\psi(t) = \int_E \Phi(e, t)K_Q(t, de)$ and $L_E(e, t) = \Phi(e, t)/\psi(t)$ for $\psi(t) > 0$ and $L_E(e, t) = 1$ otherwise. Then the intensity under $P$ becomes:

\[ \lambda_P(t) = \psi(t)\lambda_Q(t) \]

- The conditional distribution of the marker is

\[ K_P(t, de) = L_E(e, t)K_Q(de) \]
Let $p(t)$ be a price (under the money market numeraire, so discounted..).

Then:

$$\frac{p(t)}{b(t)} = E^Q\left(\frac{X_T}{b(T)}\bigg|\mathcal{F}_t\right)$$

Consider a different numeraire $A(t)$, and consider also the process $\frac{dP}{dQ}\big|_t = \frac{A(t)}{A(0)b(t)}$

Then $X/A$ is a $P$-martingale iff $\frac{X_t}{A(t)} \frac{A(t)}{A(0)b(t)}$ is a $Q$-martingale.