Building Levy Processes

Jump-diffusion : Brownian + compound Poisson with triplet (σ, ν, γ)

- $X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$
  - Merton 4 parameters $(\sigma, \lambda, \mu, \delta)$ with normal jumps $\nu(x) = \frac{\lambda}{\sqrt{2\pi\delta^2}} \exp[\frac{(x-\mu)^2}{2\delta^2}]$
  - Kou 5 parameters $(\sigma, \lambda\lambda_+\lambda_-, p)$ with asymmetric double exponential jumps $\nu(x) = p\lambda\lambda_+ e^{-\lambda_+ x} 1_{x>0} + (1-p)\lambda\lambda_- e^{-\lambda_-|x|} 1_{x<0}$
- Simulation ease and distribution of jump sizes known
- Fit Implied Volatility surfaces well
- Jumps are rare occurrences: interpretation of normal course of business plus event risk

Infinite activity: moves essentially by jumps with triplet $(0, \nu, \gamma)$. σ is not necessary for non-trivial behavior.

Three typical construction methods

- Brownian subordination
- Specify Levy measure ν
- Specify density $p_t(x)$

Brownian Subordination in $(\Omega, \mathcal{F}, \mathbf{P})$

- Subordinator is an almost surely increasing Levy process, S_t, with $(0, \rho, b)$, $\mathrm{E}[e^{uS_t}] = e^{tl(u)}$, and $\mathrm{E}[S_t] = t$

- Levy Process $X(t)$ with $(\sigma, 0, \mu)$ and $\mathrm{E}[e^{iuX_t}] = e^{t\Psi(u)}$

- $Y(t, \omega) = X(S(t, \omega), \omega)$ for each $\omega \in \Omega$ is a new Levy process with $\mathrm{E}[e^{iuY_t}] = e^{tl(\Psi(u))}$

- $(\sigma^Y)^2 = b\sigma^2$, $\nu^Y = \int_0^\infty \mathrm{N}(\mu, \sigma^2 s)\rho(ds)$, $\gamma^Y = b\gamma$

- identity subordinator $S_t = t$ triplet $(0, 0, 1)$ is deterministic

Common examples of Brownian Subordination

- Tempered Stable Subordinator $\rho(x) = \dfrac{ce^{-\lambda x}}{x^{\alpha+1}} 1_{x>0}$

- Variance Gamma ($\alpha = 0$), $\rho = \dfrac{e^{-x/\kappa}}{\kappa x}$
  $\Psi(u) = -\dfrac{1}{\kappa}\ln(1 + u^2\sigma^2\kappa/2 - i\gamma\kappa u)$
- Normal Inverse Gaussian ($\alpha = 1/2$), $\rho = \dfrac{e^{-x/(2\kappa)}}{\sqrt{2\pi\kappa}x^{3/2}}$
  $\Psi(u) = \dfrac{1}{\kappa} - \dfrac{1}{\kappa}\sqrt{1 + u^2\sigma^2\kappa - 2i\gamma\kappa u}$
- 3 Parameters σ, γ (brownian) and κ (subordinator)

- Additional tractability; densities known in terms of special functions
- Interpretation of subordinator as real time $\rightarrow$ business time

Specifying the Levy measure: Generalized Tempered Stable process

- $\nu(x) = \dfrac{c_+ e^{-\lambda_+ x}}{x^{\alpha_+ + 1}} 1_{x>0} + \dfrac{c_- e^{-\lambda_- |x|}}{| x |^{\alpha_- + 1}} 1_{x<0}$

- With exponential tempering $\alpha_\pm < 2$ yields a Levy measure

- 6 parameters (3 for each side of ν) provides a rich structure

- Certain parameter values and density properties contain a brownian subordinator and compound poisson processes

- Small jump have stable-like behavior if $\alpha_\pm > 0$. Levy density and probability density behave exponentially with decay rates $\lambda_\pm$ for $x \to \pm\infty$

Specifying the Probability Density: Generalized Hyperbolic model

- $p(x; \lambda, \alpha, \beta, \delta, \mu) = C(\delta^2 + (x-\mu)^2)^{\frac{\lambda}{2}-\frac{1}{4}} K_{\lambda-1/2}(\alpha\sqrt{\delta^2 + (x-\mu)^2})e^{\beta(x-\mu)}$

- GH laws are not closed under convolution which is a drawback when working with several maturities

- Though cumbersome, it provides rich structure and a variety of shapes

- Normal, hyperbolic, Student t, Variance Gamma, Normal Inverse Gaussian are all subclasses of GH

All distributions decay exponentially ($\lambda_+ > 1$) which will be necessary for exponential Levy models of Chapter 11