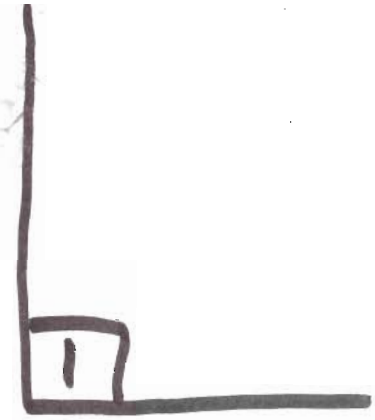


$\Omega = [0,1]$

SRV = piecewise const

$$\Omega \longrightarrow \mathbb{R}$$

$$\boxed{Z^{\perp}} := Z - \underbrace{\mathbb{E}[Z]}_{\int_\Omega Z} \cdot \mathbb{1}$$

orthog. proj. Z into $(\mathbb{R}\mathbb{1})^{\perp}$

$$\langle Y, Z \rangle = \mathbb{E}[Y^{\perp} Z^{\perp}]$$

"covariance"

$\langle Z, Z \rangle$ = variance of Z

≥ 0

$\boxed{2}$

$\sqrt{\langle Z, Z \rangle}$ = std dev of Z

$\langle Z, Z \rangle = 0 \iff Z \in \mathbb{R}\mathbb{1}$

$\boxed{\rho}: \{SRV_s\} \longrightarrow \mathbb{R}^2$

$\rho(Z) = \left(\sqrt{\langle Z, Z \rangle}, \mathbb{E}[Z]\right)$

$\mathbb{R}^2$ = "risk-return space"

ρ = "risk-return map"

$X_1, \dots, X_n$ lin. indep. SRVs

Invest \$1 in asset i

Get \$ $1 + X_i(\omega)$ at T_{end}

Tyche

Portfolio: \$4 in asset 1

& \$3 in asset 2

At T_{end}: \$ $4 + 4X_1(\omega) + 3 + 3X_2(\omega)$

Known: $\forall i,\ \mathbb{E}[X_i]$ &

$\forall i, j,\ \langle X_i, X_j \rangle$

$V := \mathbb{R}X_1 + \cdots + \mathbb{R}X_n$

✓ n-dim'l "portfolio space"

\$1 to invest

$V_1 := \{\alpha_1 X_1 + \cdots + \alpha_n X_n$

✓ $\ni: \alpha_1 + \cdots + \alpha_n = 1\}$

"feasible portfolios"

$$\underbrace{\text{Assume } \mathbb{1} \notin V,\ n \geqslant 3}_{\text{"all portfolios are risky"}}$$

$$Y = \alpha_1 X_1 + \cdots + \alpha_n X_n \Longrightarrow$$

$$\mathbb{E}[Y] = \sum_i \alpha_i \mathbb{E}[X_i]$$

$$\langle Y, Y \rangle = \sum_{i,j} \alpha_i \alpha_j \langle X_i, X_j \rangle$$

$$p|V : V \longrightarrow \mathbb{R}^2$$

known

$$0 \neq Y \in V \Rightarrow \langle Y, Y \rangle > 0$$ $\boxed{6}$

$$\rho(Y) = \left(\sqrt{\langle Y, Y \rangle}, \mathbb{E}[Y]\right)$$

$$\rho \mid V = \left(\sqrt{\text{pos. def.}}, \text{ linear}\right)$$

<u>Goal</u>: Understand

$$\rho(V_1) \subseteq \mathbb{R}^2$$

"risk-return of feasible portfolios"

☑ risk ←, return ↑

$$Q(Y) = \langle Y, Y \rangle$$

$$L(Y) = \mathbb{E}[Y]$$

$$\rho = (\sqrt{Q}, L)$$

Fix $A_0 : \mathbb{R}^{n-1} \longrightarrow V_1$

affine ~~linear~~ bijection

$$Q \circ A_0 : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$$

is a (not necessarily homogeneous) degree two polynomial

<u>e.g.</u> $n = 3$

$$Q \circ A_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(Q \circ A_0)(x,y) =$$

$$ax^2 + bxy + cy^2$$
$$+ dx + ey$$
$$+ f$$

pos. def.

rotate: kill $bxy \longrightarrow 0$

dilate: kill $a,c \longrightarrow 1,1$

translate: kill $d,e \longrightarrow 0,0$

$$(Q \circ A_1)(x,y) = x^2 + y^2 + \Delta^2$$

$$\rho = (\sqrt{Q},\ L)$$

$$(L \circ A_1)(x,y) = gx + hy + t$$

rotate: kill $hy \longrightarrow 0$

$$(Q \circ A_2)(x,y) = x^2 + y^2 + \Delta^2$$

$$(L \circ A_2)(x,y) = ux + t$$

WMA $u \geq 0, \quad \Delta \geq 0$

$$(\rho \circ A_2)(x,y) =$$

$$\left(\sqrt{x^2 + y^2 + \Delta^2},\ ux + t\right)$$

n general, $\geqslant 3$

__Lem.__ $\exists$ aff. bij. $A: \mathbb{R}^{n-1} \to V_1$

$\exists\, s, u \geqslant 0,\ \ \exists\, t \in \mathbb{R}$

$\ni: \forall (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$,

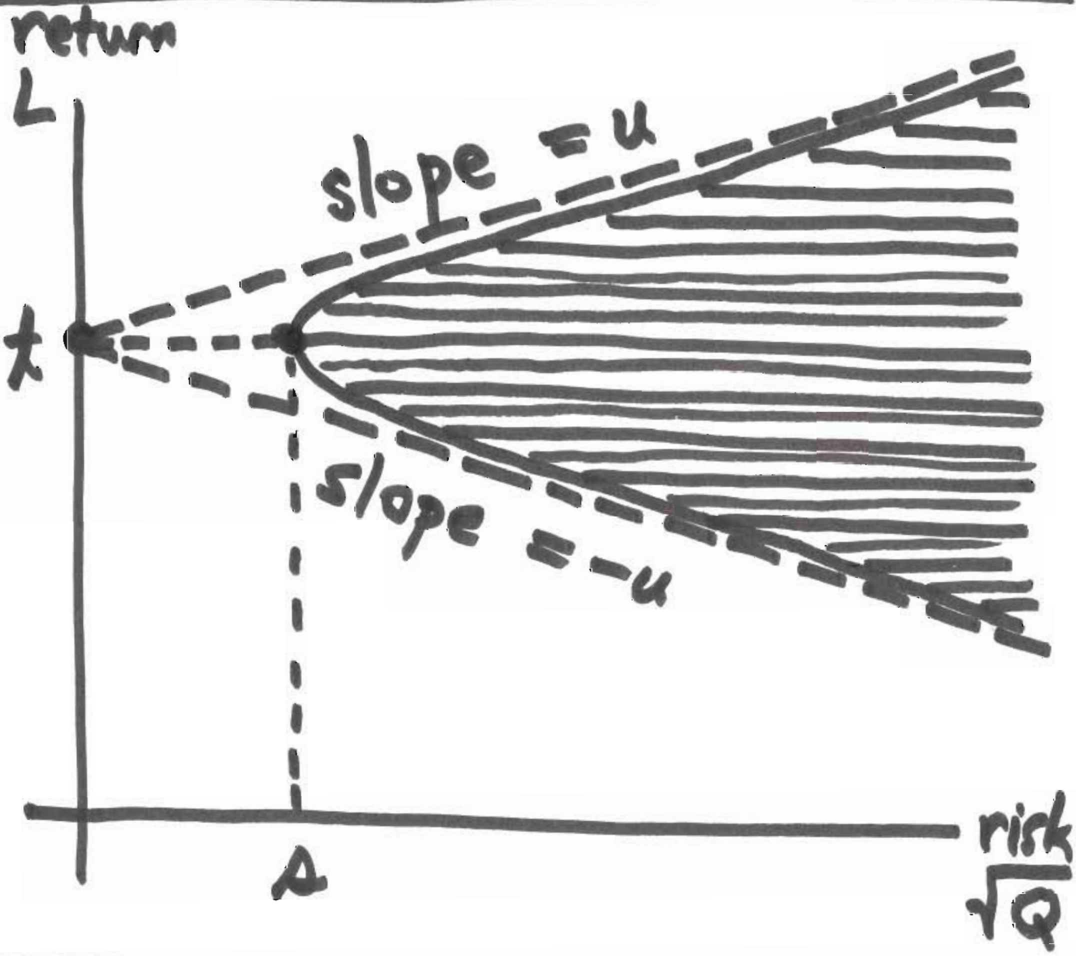
$(\rho \circ A)(x_1, \ldots, x_{n-1}) =$

$\left(\sqrt{x_1^2 + \cdots + x_{n-1}^2 + s^2},\quad u x_1 + t\right)$

✓ $(\rho \circ A)(x) = \left(\sqrt{x \cdot x + s^2},\ u x_1 + t\right)$

$(\rho \circ A)(x) = \left(\sqrt{x \cdot x + \Delta^2},\ u x_1 + t\right)$

<u>Goal</u>: $\rho(V_1) = \mathrm{im}(\rho \circ A)$



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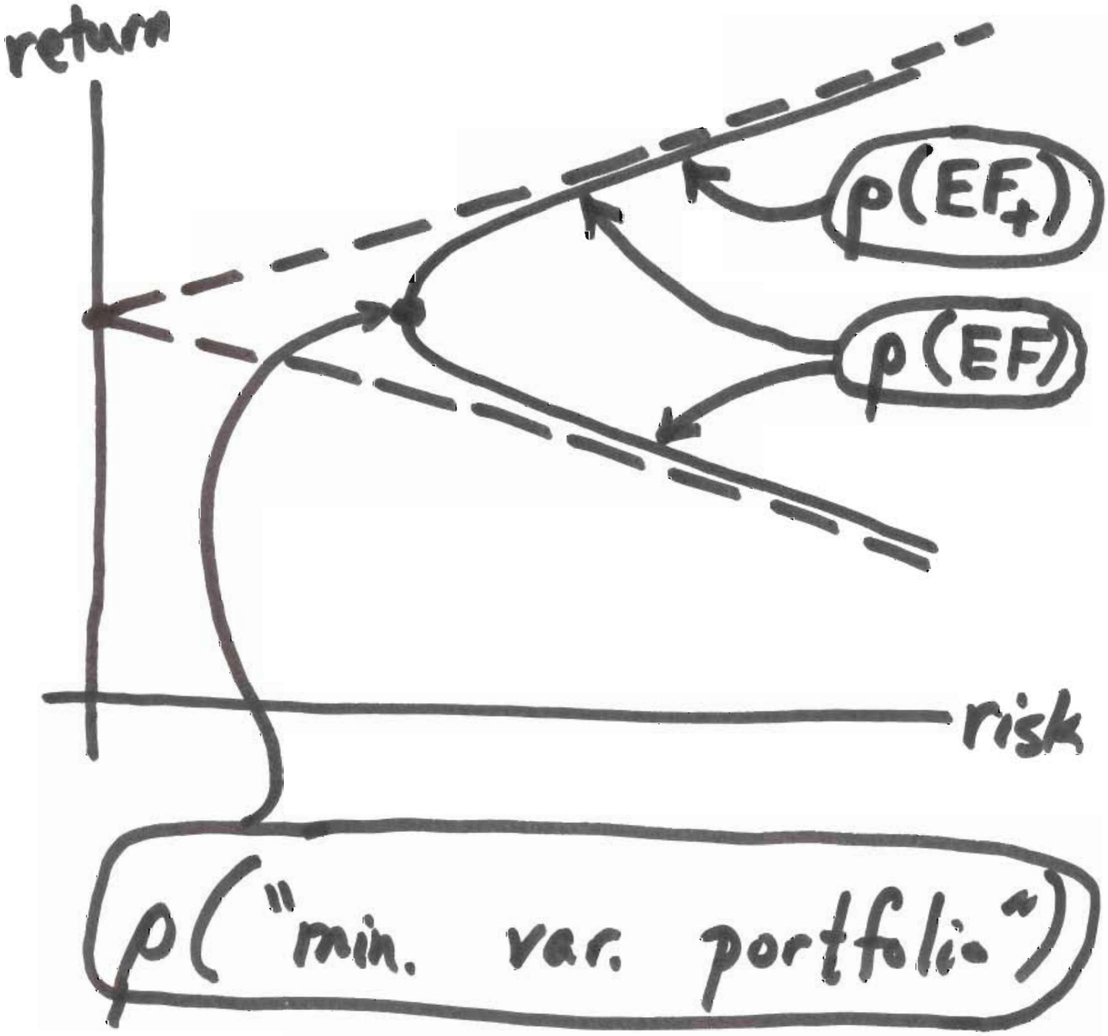
risk ←, return ↑

Assume $\Delta > 0$, $u > 0$

$$(\rho \circ A)(x) = \left(\overleftarrow{\sqrt{x \cdot x + \Delta^2}},\ \overset{\uparrow}{u x_1 + t}\right)$$

$$EF := A(\{(*, 0, \ldots, 0)\})$$

$$EF_+ := A(\{(\geq 0, 0, \ldots, 0)\})$$



$$(\rho \circ A)(x) = \left(\sqrt{x \cdot x + \Delta^2},\ ux_1 + t\right)$$

$$EF = A\left(\{(*, 0, \ldots, 0)\}\right)$$

Each pt on $\rho(EF)$

has a unique preimage in V_1 under ρ

<u>Pf</u>: $x \in \{(*, 0, \ldots, 0)\}$

$y \in \mathbb{R}^{n-1}$

$(\rho \circ A)(x) = (\rho \circ A)(y)$

<u>Want</u>: $x = y$ <u>etc.</u>

Fix $r > 0$. $X_0 := r\mathbb{1}$.

Bank = asset 0

Invest \$1 in Bank

Get $\$\underbrace{1+r}_{\parallel}$ at T_{end}

$$1 + X_0(\omega)$$

$$V' := \mathbb{R}X_0 + V \quad (n+1)\text{-dim'l}$$
$$= \mathbb{R}X_0 + \cdots + \mathbb{R}X_n$$

$$V_1' := \{\alpha_0 X_0 + \cdots + \alpha_n X_n$$
$$\ni : \alpha_0 + \cdots + \alpha_n = 1\}$$

<u>New goal</u>: $\rho(V_1')$

$\forall Y \in V_1, \quad \forall w \in \mathbb{R}$

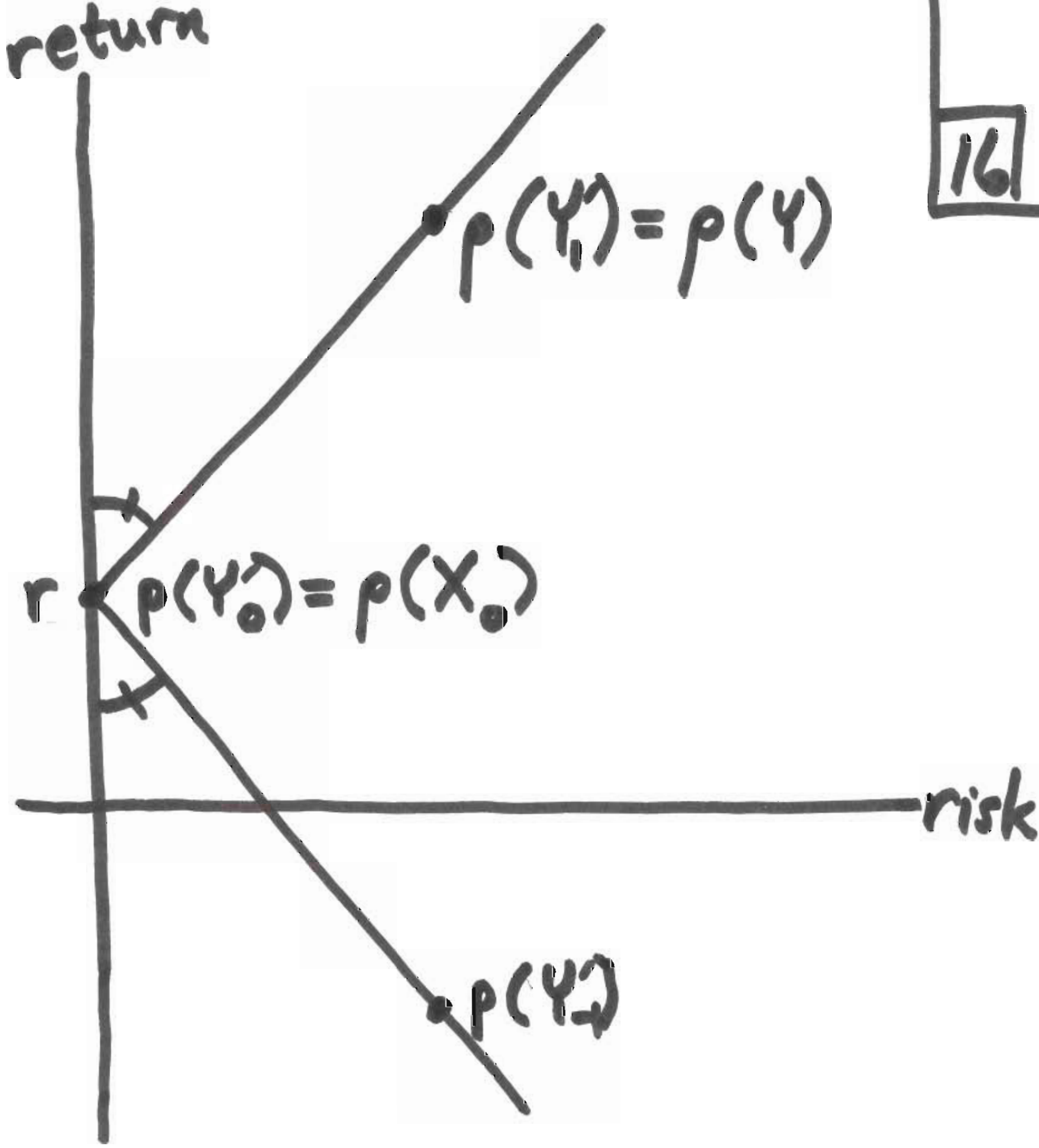
$$Y'_w := (1-w)X_0 + wY$$

$$\in V_1'$$

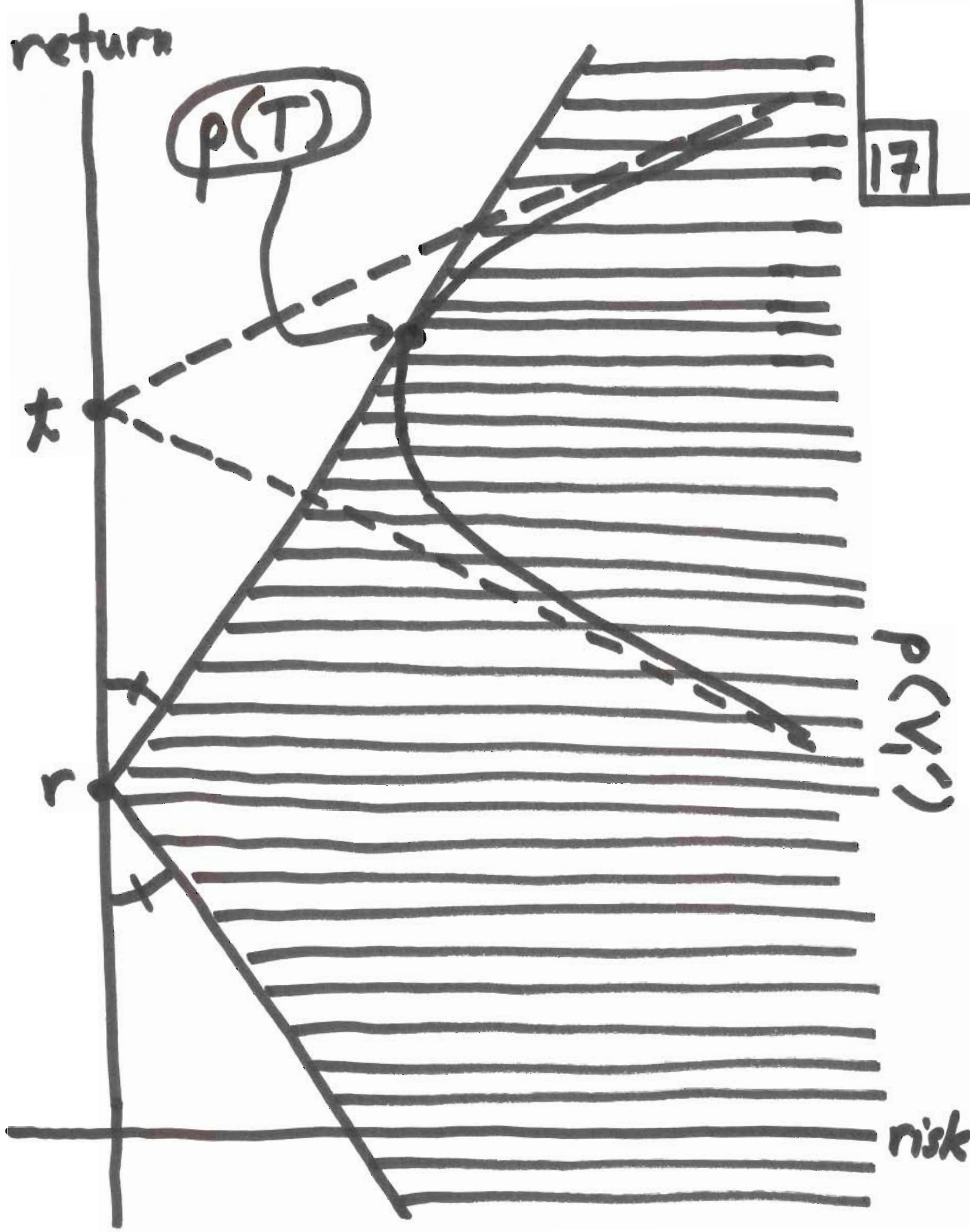
$$V_1' = \left\{ Y'_w \ni: \begin{array}{l} Y \in V_1 \\ w \in \mathbb{R} \end{array} \right\}$$

Fix $Y \in V_1$

$$Y'_0 = X_0 \quad , \qquad Y'_1 = Y$$



Take union of these "reflecting lines" over $Y \in V_1$ to get $\rho(V_1^-)$

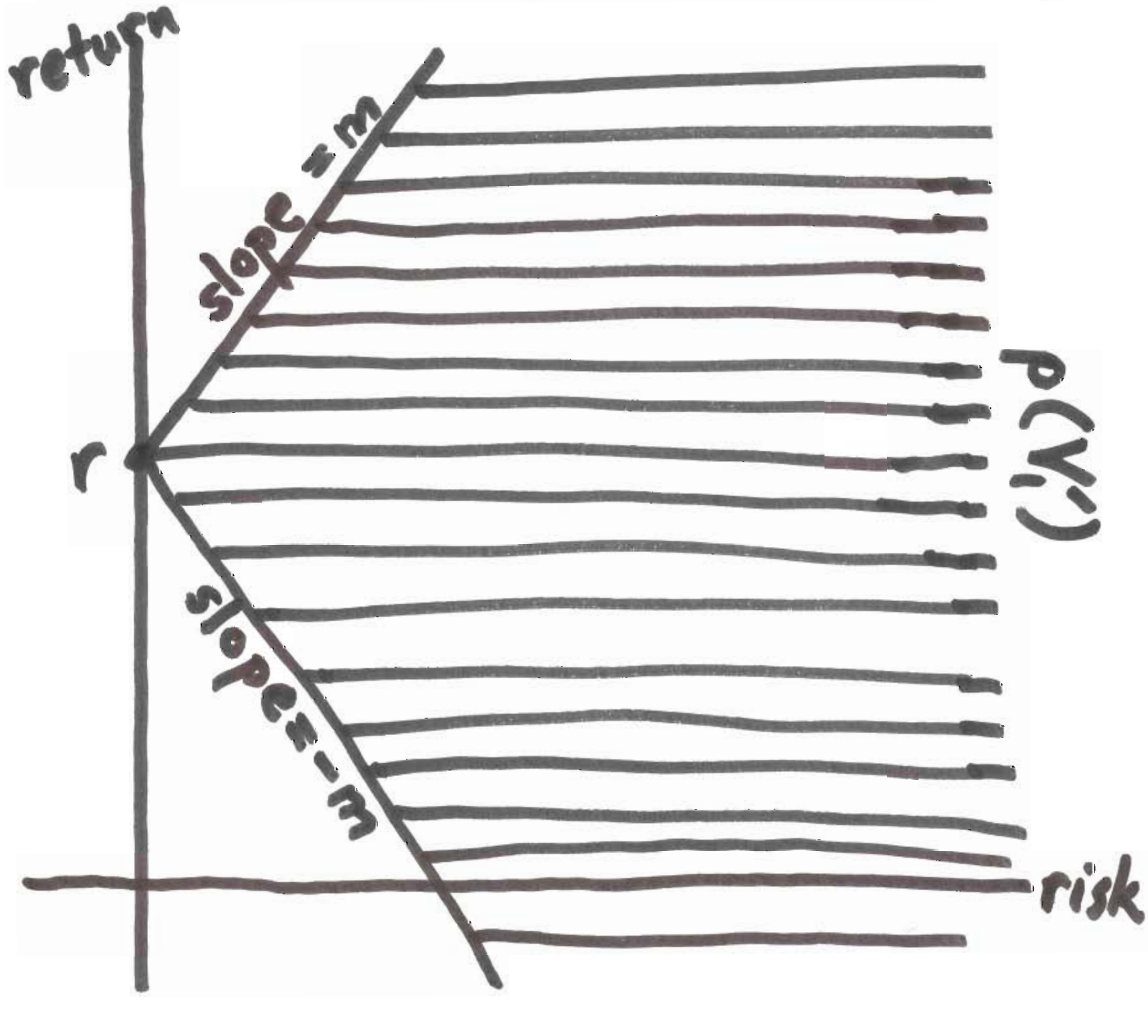


risk ← , return ↑

Lem. $\exists$ aff. bij. $A': \mathbb{R}^n \to V_1'$

$\exists m > 0 \ni: \forall x \in \mathbb{R}^n$

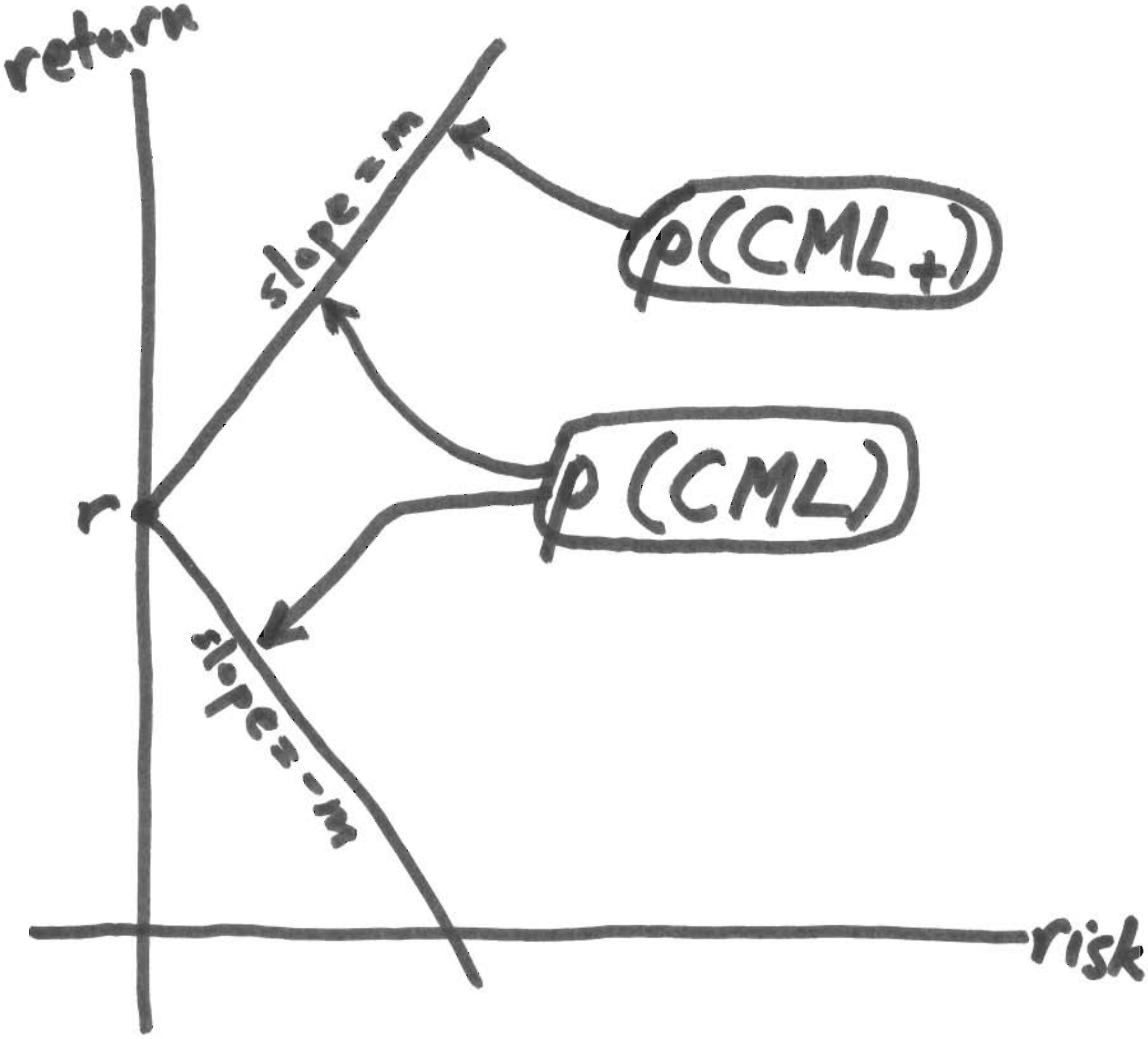
$(\rho \circ A')(x) = (\sqrt{x \cdot x},\ mx_1 + r)$



$$(p \circ A')(x) = (\sqrt{x \cdot x},\ m x_1 + r)$$

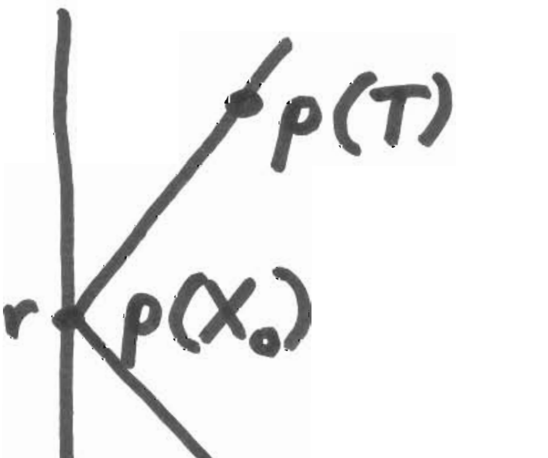
$$CML = A'(\{(*, 0, \ldots, 0)\})$$

$$CML_+ = A'(\{(\geq 0, 0, \ldots, 0)\})$$



Each pt on $\rho(EF)$ has a unique preimage in V_1 under ρ

Each pt on $\rho(CML)$ has a unique preimage in V_1' under ρ



$T, X_0 \in CML$

$CML = \{(1-w)X_0 + wT \ni: w \in \mathbb{R}\}$

Assume N investors

Portfolios $a_1 B_1, \dots, a_N B_N$

$\ni : a_1, \dots, a_N > 0, \quad B_1, \dots, B_N \in V_1$

Rationality $\Longrightarrow B_1, \dots, B_N \in CML_+$

$$M := \frac{a_1 B_1 + \cdots + a_N B_N}{a_1 + \cdots + a_N} \in CML_+$$

Bank balance $\Longrightarrow M \in V_1$

$$\rho(M) \in (\rho(CML_+)) \cap (\rho(V_1))$$
$$= \{\rho(T)\} \quad \therefore M = T$$

$$\rho(Z) = \left(\sqrt{\langle Z, Z\rangle},\ \mathbb{E}[Z]\right)$$

$$(\rho \circ A')(x) = \left(\sqrt{x \cdot x},\ m x_1 + r\right)$$

$Z = A'(x) \Longrightarrow$

$\langle Z, Z\rangle = x \cdot x$ &

$\mathbb{E}[Z] = m x_1 + r$

$\forall x \in \mathbb{R}^n,$

$\langle A'(x), A'(x)\rangle = x \cdot x$

$\mathbb{E}[A'(x)] = m x_1 + r$

$$2\langle A'(x), A'(y)\rangle = \boxed{22}$$

$$\langle A'(x+y), A'(x+y)\rangle -$$

$$\langle A'(x), A'(x)\rangle - \langle A'(y), A'(y)\rangle$$

$$= (x+y)\cdot(x+y) -$$

$$x\cdot x - y\cdot y$$

$$= 2\,x\cdot y$$

$$\langle A'(x), A'(y)\rangle = x\cdot y$$

$\forall Z \in \{SRV_A\}$, $\boxed{\pi_Z} := \mathbb{E}[Z] - r$

$\boxed{\beta_Z} := \langle Z, T \rangle / \langle T, T \rangle$

"risk premium" & "beta"

$\forall Z \in V_1'$, π_Z, β_Z known

$$\mathbb{E}[A'(x)] = mx_1 + r$$

✓ $\pi_{A'(x)} = mx_1$

$$\beta_{A'(x)} = ?$$

$$T \in CML = A(\{(*, 0, \ldots, 0)\})$$

Choose $c \in \mathbb{R} \ni$:

$$T = A(c, 0, \ldots, 0)$$

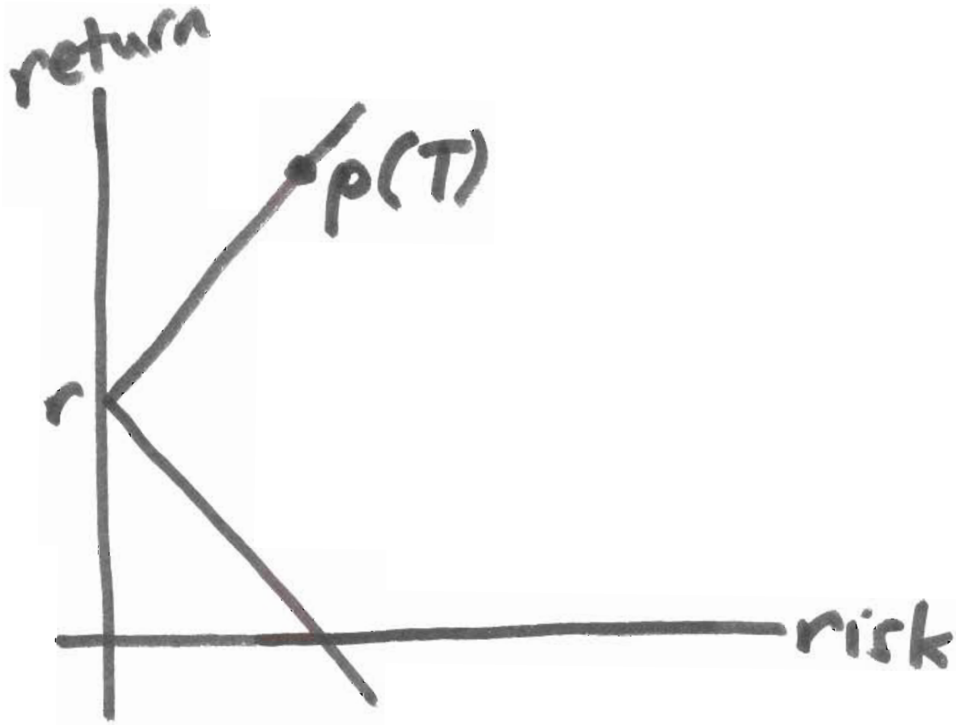
$$\langle A'(x), T \rangle = x_1 c$$

$$0 \neq \langle T, T \rangle = c^2$$

✓ $$\beta_{A'(x)} = \frac{x_1}{c}$$

$$K := mc = \pi_T$$

$$\pi_{A'(x)} = m x_1 = K \frac{x_1}{c} = K \beta_{A'(x)}$$

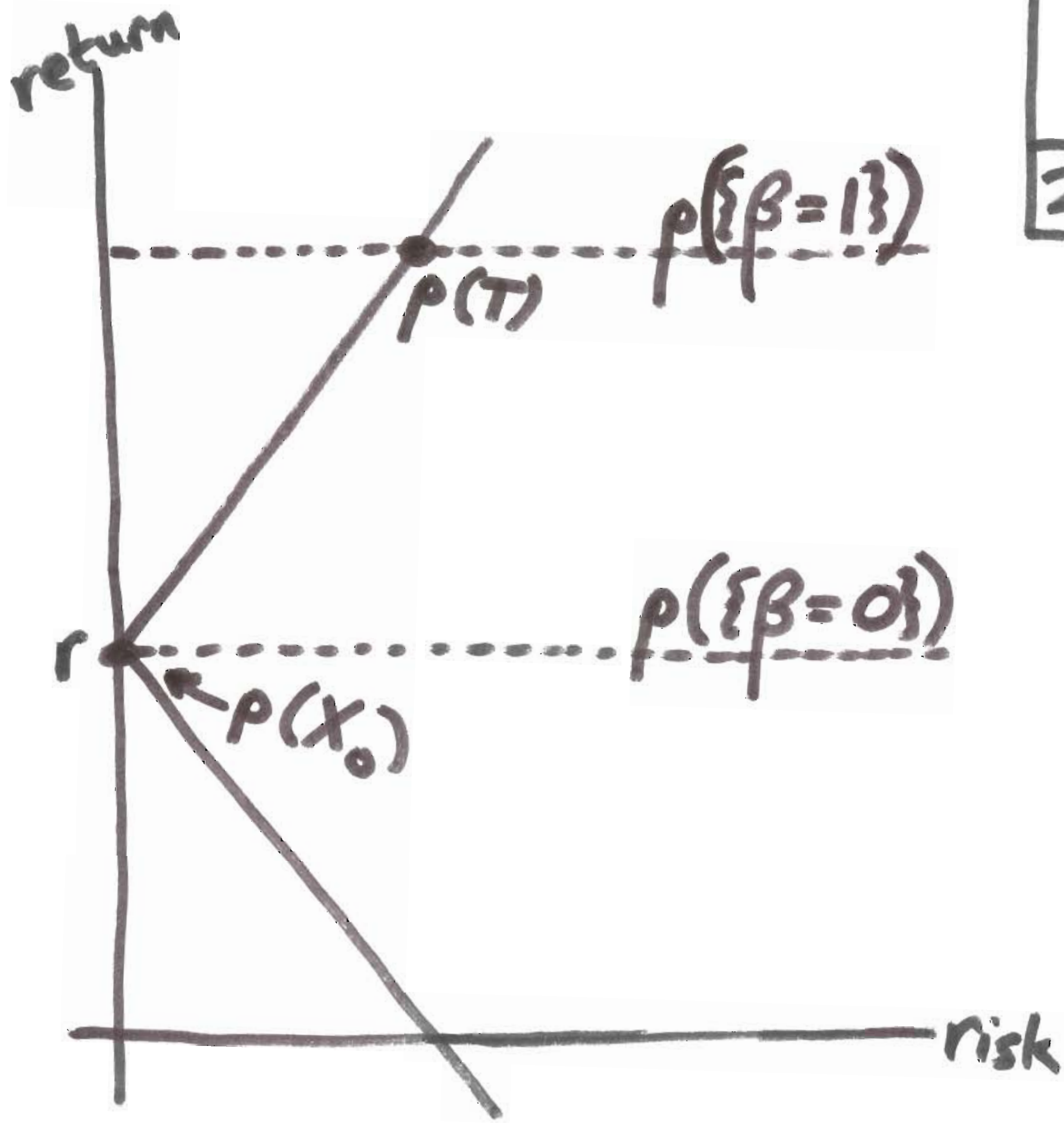


Any horizontal line is the p-image of a level set of

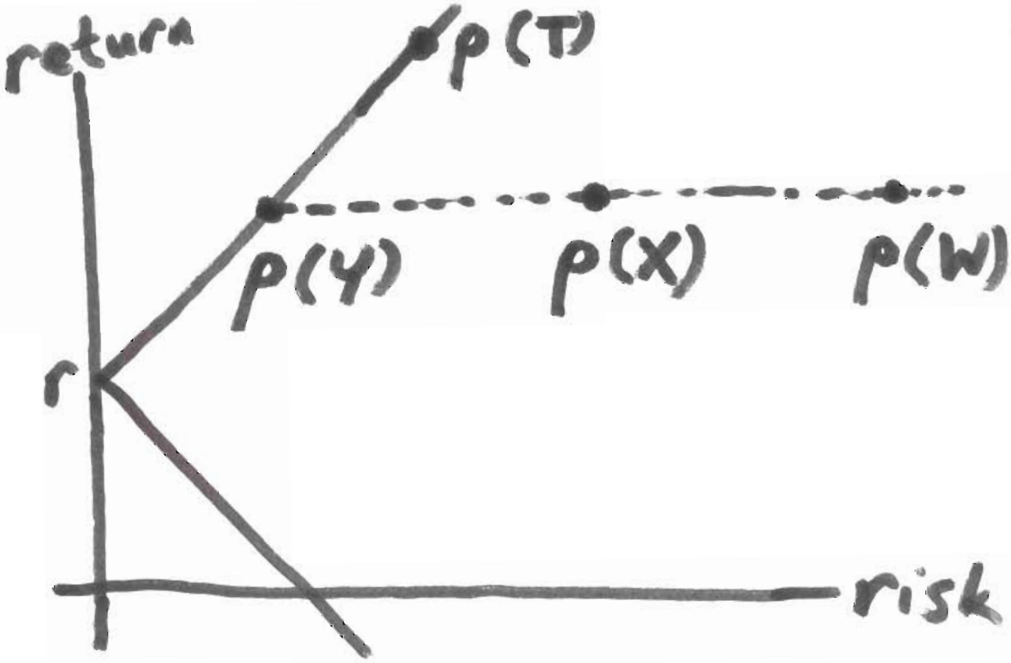
$$Z \mapsto \underbrace{\mathbb{E}[Z]}_{= K\beta_Z + r} : V_1' \longrightarrow \mathbb{R}$$

$$\beta_T = 1$$

$$\beta_{X_0} = 0$$



Horizontal lines are ρ-images of level sets of β



Harry, Jack want W, X resp.

Grace advises: Add "free" portfolios Y-W, Y-X for "diversification"

W, X have different risks but same "after-diversification" risk!

Suppose a new asset is introduced $\ni$: return on \$1 at T_{end} is $U(\omega)$

Say, after market analysis, we find

$(\beta_U, \mathbb{E}[U])$

is not on SML

Then we can obtain an "expectations arbitrage" opportunity:

$$U' := \frac{U - \beta_u T}{1 - \beta_u}$$

(return on a new \$1 asset)

$$\beta_{u'} = \frac{\beta_u - \beta_u \beta_T}{1 - \beta_u} = 0$$

$$\pi_{u'} = \frac{\pi_u - \beta_u \pi_T}{1 - \beta_u}$$

$$= \frac{\pi_u - \beta_u K \beta_T}{1 - \beta_u}$$

$$\neq \frac{K\beta_u - \beta_u K \beta_T}{1 - \beta_u} = 0$$

$\beta_{U'} = 0$ i.e. $\langle U', T \rangle = 0$

$\pi_{U'} \neq 0$ i.e. $\mathbb{E}[U'] \neq r$

Say e.g. $\mathbb{E}[U'] < r$

Shorting U' allows borrowing at < bank rate in <u>expectation</u>

Semi-guaranteed because

$$\langle U', T \rangle = 0$$