Numerical Methods for a Certain PIDE

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We assume the risk-neutral dynamics of some asset, S, are given by

$$S_t = S_0 \exp(rt + X_t), \tag{1}$$

where X is a Levy process with characteristic triplet (σ^2, ν, γ) under the (not necessarily unique!) measure $\mathbb{Q}$.

As usual, $\mathbb{Q}$ is chosen to make the discounted process $\hat{S}_t = e^{-rt} S_t$ a martingale

We further make a technical assumption that

$$\int_{|y| \geq 1} e^{2y} \nu(dy) < \infty.$$

We have that the dynamics of the discounted process are given by

$$\frac{d\hat{S}_t}{\hat{S}_{t-}} = \sigma dW_t + \int_{\mathbb{R}} (e^x - 1)\tilde{J}_X(dt\,dx).$$

And a European option, $C_t = C(t, S)$, whose payoff function is givnen by $H(S_T)$ has value

$$C_t = E[e^{-r(T-t)}H(S_T) \mid \mathcal{F}_t]$$

under the measure $\mathbb{Q}$. Equivalently, by the Markov property,

$$C(t, S) = E[e^{-r(T-t)}H(S_T) \mid S_t = S].$$

We will also may make use of a change of variables:

$$\begin{aligned} \tau &= T - t \\ x &= \ln(S/K) + r\tau, \end{aligned}$$

and define

$$\begin{aligned} u(\tau, x) &= e^{-r(T-t)} C(t, S)/K \\ h(x) &= H(Ke^{x})/K. \end{aligned}$$

So that

$$u(\tau, x) = E[h(x + X_\tau) \mid x = x_\tau].$$

u above is the option in log-moneyness coordinates. We will examine a PIDE for u in what follows.

We define the operator L^x by

$$L^x g(x) = rxg'(x) + \frac{\sigma^2 x^2}{2} g''(x) + \int_{\mathbb{R}} g(xe^y) - g(x) - x(e^y - 1)g'(x)\nu(dy)$$

We may derive (volunteers?...I think it would be worthwhile) that any European option with underlying following the dynamics in (1) satisfies the following PIDE:

$$\frac{\partial C}{\partial t}(t, S) + L^S C(t, S) - rC(t, S) = 0$$

on $[0, T) \times (0, \infty)$, with boundary condition

$$C(T, S) = H(S), \ \forall S > 0.$$

We may write the PIDE with the variable changes noted above (i.e., log-moneyness) as

$$-\frac{\partial u}{\partial \tau}(\tau, x) + \frac{\sigma^2}{2}\left(\frac{\partial^2 u}{\partial x^2}(\tau, x) - \frac{\partial u}{\partial x}(\tau, x)\right)$$

$$+ \int_{\mathbb{R}} u(\tau, x+y) - u(\tau, x) - (e^y - 1)\frac{\partial u}{\partial x}(\tau, x)\nu(dy) = 0,$$

or simply as

$$\frac{\partial u}{\partial \tau} = L^X u \tag{2}$$

with initial condition $u(0, x) = h(x)\ \forall x \in \mathbb{R}$.

So, disregarding calibration (that is, finding σ and ν from data), we are interested in solving (2) given a payoff function, h, and an operator of the form

$$
\begin{aligned}
Lf(x) &= a_1(x)\frac{\partial f}{\partial x} + a_2(x)\frac{\partial^2 f}{\partial x^2} \\
&+ \int f(x + a_0(x,y)) - f(x) - a_0(x,y)\frac{\partial f}{\partial x}(x)\nu(dy)
\end{aligned}
$$

The generality of terms here encompasses PIDE's that arise from European, American, and barrier options.

We may use

- Multinomial trees
- Finitie difference methods
- Galerkin methods
- etc., (these are all I know, but that doesn't mean much)

to solve (2) with a general operator L as above.

For today, we briefly discuss finite difference schemes for the problem:

$$\frac{\partial u}{\partial \tau} = L^X u,$$

with $u(0,x) = h(x)$ on the domain $[0,T] \times \mathcal{O}$.

For European options, we take $\mathcal{O} = \mathbb{R}$. For barrier options $\mathcal{O} = (a,b)$.

If the domain $\mathcal{O}$ is not bounded, we localize the problem by setting $\mathcal{O} = (-A, A)$:

$$\begin{aligned} \frac{\partial u_A}{\partial \tau} &= L^X u_A, \quad (0, T] \times (-A, A) \\ u_A(0, x) &= h(x) \ \ \forall x \in (-A, A) \end{aligned}$$

We also must impose condtions for u_A for all points outside of $(-A, A)$. We may reasonably set

$$u_A(\tau, x) = h(x) \ \ \forall x \notin (-A, A)$$

We also truncate large jumps. That is, the integral term must have finite bounds, say, B_l and B_r.

This amounts to replacing the Levy process, X_t, with a new process $\tilde{X}_\tau$ whose triplet is $(\bar{\gamma}, \sigma, \nu 1_{x \in [B_l, B_r]})$.

Truncating the integral will yield a solution $\tilde{u}$ defined by

$$\tilde{u}(\tau, x) = E[h(x + \tilde{X}_\tau) \mid x_\tau = x]$$

(Note that this step adds additional error in approximation due to the nonlocal nature of the PIDE)

We next discretize the space and time coordinates. We may do this uniformly (as a first run); viz.,

$$\tau_n = n\Delta t \qquad n = 0, \dots, M,\;\; \Delta t = T/M$$
$$x_i = -A + i\Delta x \qquad i = 0, \dots, N,\;\; \Delta x = 2A/N.$$

We define u_i^n to be the solution on the discretized grid extended by zero $u_i^n = h(x_i)$ for $i \notin [0, N]$

We next must approximate space and time derivative operators (easy-ish) and the integral operator for this grid.

We begin with the integral.

We will assume $\nu(\mathbb{R}) = \lambda < \infty$. This yields

$$L^X u = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - \left(\frac{\sigma^2}{2} + \alpha \right) \frac{\partial u}{\partial x} + \int_{B_l}^{B_r} u(\tau, x + y) \nu(dy) - \lambda u,$$

with $\alpha = \int_{B_l}^{B_r} (e^y - 1) \nu(dy)$.

Using a trapezoidal quadrature rule and the grid resolution of Δx, we choose K_l and K_r such that

$[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_r + 1/2)\Delta x]$.

Then the integral term is approximated by

$$
\begin{aligned}
\int_{B_l}^{B_r} u(\tau_n, x_i + y)\nu(dy) &\approx \sum_{j=K_l}^{K_r} \nu_j u_{i+j}^n \\
\lambda &\approx \sum_{j=K_l}^{K_r} \nu_j \\
\alpha &\approx \sum_{j=K_l}^{K_r} (e^j - 1)\nu_j \\
\nu_j &= \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu(dy)
\end{aligned}
$$

We use standard space derivatives:

$$
\begin{aligned}
\left(\frac{\partial^2 u}{\partial x^2}\right)_i &\approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \\
\left(\frac{\partial u}{\partial x}\right)_i &\approx \frac{u_{i+1} - u_i}{\Delta x} \text{ if } \sigma^2/2 + \hat{\alpha} < 0 \\
\left(\frac{\partial u}{\partial x}\right)_i &\approx \frac{u_i - u_{i-1}}{\Delta x} \text{ if } \sigma^2/2 + \hat{\alpha} \geq 0
\end{aligned}
$$

Denote by D and J the matrices representing the differential and integral parts of L^X.

The explicit scheme is given by

$$\frac{u^{n+1} - u^n}{\Delta t} = Du^n + Ju^n,$$

so that

$$u^{n+1} = [I + \Delta t(D + J)]u^n.$$

A sufficient condition for convergence is that $\Delta t \leq \min\left(\frac{1}{\hat{\lambda}}, \frac{(\Delta x)^2}{\sigma^2}\right)$

Similarly, we may construct the implicit scheme,

$$\frac{u^{n+1} - u^n}{\Delta t} = Du^{n+1} + Ju^{n+1},$$

where we solve

$$[I - \Delta t(D + J)]u^{n+1} = u^n$$

for u^{n+1}.

This scheme does not have a requirement on a small step size in the time dimension to obtain stability.

Finally, we may look at the explicit-implicit scheme:

$$\frac{u^{n+1} - u^n}{\Delta t} = Du^{n+1} + Ju^n,$$

where we solve

$$[I - \Delta t D]u^{n+1} = [I + \Delta t J]u^n.$$

Stability is assured when $\Delta t \leq 1/\hat{\lambda}$. We also have consistency with the PIDE above as $(\Delta t, \Delta x) \rightarrow 0$.

For these reasons, (today) we prefer the explicit-implicit scheme.

For the sake of concreteness, the operators D and J in the explicit-implicit scheme are given by

$$\begin{aligned}
\left(Du^{n+1}\right)_i &= \frac{\sigma^2}{2}\frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta x)^2} - \left(\frac{\sigma^2}{2}+\hat{\alpha}\right)\frac{u_{i+1}^{n+1}-u_i^{n+1}}{\Delta x} \\
(Ju^n)_i &= \sum_{j=K_l}^{K_r} \nu_j u_{i+j}^n - \hat{\lambda} u_i^n \\
\hat{\alpha} &= \sum_{j=K_l}^{K_r} (e^j - 1)\nu_j \\
\nu_j &= \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu(dy)
\end{aligned}$$

To implement the scheme we do the following:

Initialize:

$u_i^0 = h(x_i)$ for all i.

Solve:

$[I - \Delta t D]u^{n+1} = [I + \Delta t J]u^n$ for $n = 0, 1, \ldots, M-1$

Impose "Boundary" Conditions:

$u_i^{n+1} = h(x_i)$ for $i \notin \{0, \ldots, N\}$.

The solution of the explicit-implicit scheme converges uniformly on each compact subset of $(0, T] \times \mathbb{R}$ to the unique viscosity solution of the PIDE above.

So what's a viscosity solution? Basically a viscosity solution of a PDE (or PIDE) is a function which is not necessarily smooth, but satisfies the PDE in some appropriate sense.

The explicit-implicit scheme may be applied to an up and out barrier option, where the only modification is in the boundary condtions. In this case we use:

$$\forall S \in (0, B), \qquad C(t = T, S) = H(S)$$
$$\forall S \geq B, \forall t \in (0, T], \qquad C(t, S) = 0$$

Notice that we are concerned about $S \geq B$ and not just $S = B$ since the PIDE has a nonlocal nature.

We may also use an explicit-implicit scheme to price American options. However, this is more complicated than what has been shown! (Another Day.)

Another method we may use is the Galerkin method. There are technical issues that we do not address here, but the basic idea is as follows:

We fix a Hilbert space, H_N, with inner product $(\cdot, \cdot)_{H_N}$, and finite basis $\{e_1, \ldots, e_N\}$.

As before, we localize the problem to a bounded domain:

$$\begin{aligned} \frac{\partial u_A}{\partial \tau} &= L^X u_A, \quad (0, T] \times (-A, A) \\ u_A(0, x) &= h(x) \;\; \forall x \in (-A, A) \\ u_A(\tau, x) &= h(x) \;\; \forall x \notin (-A, A) \end{aligned}$$

The question arises...which Hilbert space should we use? Using the variable $U = u_A - h$, we obtain the equation

$$\begin{aligned} \frac{\partial U}{\partial \tau} - L^X U &= L^X h = F, \quad (0,T] \times (-A, A) \\ U(0,x) &= 0 \;\; \forall x \in (-A, A) \\ U(\tau,x) &= 0 \;\; \forall x \notin (-A, A) \end{aligned}$$

A good candidate for a Hilbert space is

$$H = \{f \in H^1(\mathbb{R}) \,:\, f(x) = 0 \,\forall x \notin (-A, A)\}.$$

Here $H^1(\Omega)$ is the Sobolev space:

$$\{f \in L^2(\Omega) : \frac{\partial f}{\partial x_i} \in L^2(\Omega)\}$$

We required a finite basis, though, and H above does not satisfy this. We discretize by fixing *a* finite basis of H, $\{e_1, \ldots, e_N\}$.

With this finite basis, we solve

$$(\frac{\partial U_N}{\partial \tau}, e_i) = (L^X U_N, e_i) + (F, e_i)$$

for $i = 1, \ldots, N$. The solution, U_N, is the solution for the finite basis Hilbert space of dimension N.

Heuristically, if the basis is 'big' enough, U_N should be 'close' to the solution over all of H.

Now

$$(\frac{\partial U_N}{\partial \tau}, e_i) = (L^X U_N, e_i) + (F, e_i)$$

yields the following equation

$$\sum_{j=1}^{N} K_{ij} \frac{d}{d\tau} a_i(\tau) = \sum_{j=1}^{N} L_{ij} a_i(\tau) + F_i$$

where $K_{ij} = (e_j, e_i)$, and $A_{ij} = (L^X e_j, e_i)$.

We may rewrite this initial value ODE as a matrix problem

$$K \frac{d}{d\tau} v_N + A v_N = F_N$$

with $v_N = (a_i(\tau),\ i = 1, \ldots, N)$.

This equation can now be solved using the same discretization in the time variable as before.

For example, we may use an explicit scheme again and solve the equation

$$K\frac{v_N^{n+1} - v_N^n}{\Delta t} = Av_N^n + F_N$$

for v_N^{n+1}.

The resulting solution is continuous, and is not simply defined on a grid. In fact, we get:

$$U_N^n = \sum_{i=1}^{N} v_{N,i}^n e_i(x)$$

and convergence to the solution U as the dimension of the basis is increased.

Some things I would like to do/see:

- A rigorous derivation of the PIDE we used.
- Go over the infinite activity case.
- A discussion of numerical methods for American Options.
- Code this stuff up and run it.