Advanced Credit Spread Models
Notes from Credit Derivatives Pricing Models by Philipp Schonbucher

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July 25, 2007

We would like to build a model to describe *default arrival risk* in an *intensity based* default risk model.

We use a Poisson process $N(t)$: a process that is increasing and takes values in $0, 1, 2, \ldots$, and assume N doesn't increase by more than 1.

And we will say that default occurs at

$$\tau = \inf\{t \in \mathbb{R} > 0 \,|\, N(t) > 0\}$$

We will look at processes $N(t)$ that are

- $\star$ Homogeneous ($\lambda(t) \equiv$ constant)
- $\star$ Inhomogeneous ($\lambda(t)$ a function of time)
- $\star$ Stochastic ($\lambda(t)$ is...you guessed it, stochastic)

Homogeneous

We assume that in the next (sufficiently) small time interval δt that

$$\mathbb{P}(N(t+\delta t) - N(t) > 0) = \lambda \delta t$$

and hence,

$$\mathbb{P}(N(t+\delta t) - N(t) = 0) = 1 - \lambda \delta t$$

We also assume that jumps in disjoint intervals are independent. With this,

$$\begin{aligned} &\mathbb{P}(N(t+2\delta t) - N(t) = 0) \\ = \mathbb{P}(N(t+2\delta t) - N(t+\delta t) = 0) \cdot &\mathbb{P}(N(t+\delta t) - N(t) = 0) \\ &= (1-\lambda\delta t)^2 \end{aligned}$$

And, continuing on, we conclude that the probability of no jumps in the interval $[t, T]$ is approximated by breaking up the interval into n pieces, and obtain

$$\mathbb{P}(N(T) - N(t) = 0) \approx (1 - \lambda\Delta t)^n$$

where $\Delta = \frac{1}{n}(T - t)$.

Therefore, we get, by taking $n \to 0$ (or $\Delta \to 0$), or

$$\mathbb{P}(N(T) - N(t) = 0) = \exp(-(T - t)\lambda)$$

To find the probability of exactly one jump in $[t, T]$ is

(discretely)

$$\mathbb{P}(N(T) - N(t) = 1) = n \cdot \Delta\lambda(1 - \Delta\lambda)^{n-1}$$

since we have to choose one interval where a jump happens and the rest must contain no jumps.

(continuously)

By multiplying and dividing by $(1 - \Delta\lambda)$, we get

$$\begin{aligned}
\mathbb{P}(N(T) - N(t) = 1) &= \frac{(T-t)\lambda}{1 - \Delta\lambda}(1 - \Delta\lambda)^n \\
&\to (T-t)\lambda \exp(-(T-t)\lambda)
\end{aligned}$$

In general, by using the binomial, theorem, you obtain that the probability of obtaining exactly m jumps is

(discretely)

$$\mathbb{P}(N(T) - N(t) = m) = \binom{n}{m} \cdot (\Delta\lambda)^m (1 - \Delta \cdot \lambda)^{n-m}$$

Which becomes in the limit (using the same ideas as before)

(continuously)

$$\mathbb{P}(N(T) - N(t) = m) \quad = \quad \frac{1}{m!}(T-t)^m \lambda^m \exp(-(T-t)\lambda)$$

And this is the definition of a homogeneous Poisson process. You just make sure that $N(0) = 0$.

A couple highlights with differentials:

- $\star$ $\mathbf{E}(dN) = \lambda dt$
- $\star$ The predictable compensator of $N(t)$ is λt
- $\star$ $dN \cdot dN = dN$, and so $\mathbf{E}(dN^2) = \lambda t$
- $\star$ N is uncorrelated with any martingale generated by a Brownian motion

Let's do something that actually pertains to credit:

The survival probability in our current model is given by

$$\begin{aligned} P(0,T) &= \mathbb{P}(N(T) - N(0) = 0) \\ &= e^{-\lambda T} \end{aligned}$$

If we assume default and interest rates are independent, we have hazard rates

$$\begin{aligned} H(t,T,T+\delta t) &= \frac{1}{\delta t}\left(\frac{P(t,T)}{P(t,T+\delta t) - 1}\right) \\ &= \frac{1}{\delta t}\left(\frac{e^{-\lambda(T-t)}}{e^{-\lambda(T+\delta t - t)}} - 1\right) \\ &= \frac{1}{\delta t}(e^{\lambda \delta t} - 1) \\ h(t,T) &= \lambda \end{aligned}$$

Inhomogeneous

The result of the last statement is that a Poisson process with constant intensity yields a flat term structure of spreads.

This is not sufficient.

We therefore broaden the model and consider

$$\mathbb{P}(N(t+\delta t) - N(t)) = \lambda(t)\delta t$$

The probability of no jumps in $[t, T]$ is

$$\begin{aligned}
\mathbb{P}(N(T) - N(t) = 0) &= \prod_{i=1}^{n} (1 - \lambda(t + i\delta t)\delta t) \\
\ln \mathbb{P}(N(T) - N(t) = 0) &= \sum_{i=1}^{n} \ln(1 - \lambda(t + i\delta t)\delta t) \\
\text{(by Taylor approximation of } \ln) &\approx \sum_{i=1}^{n} -\lambda(t + i\delta t)\delta t \\
&\to -\int_t^T \lambda(s)ds
\end{aligned}$$

So that

$$\mathbb{P}(N(T) - N(t) = 0) = \exp\left(-\int_t^T \lambda(s)ds\right)$$

Just as before, we can determine that

$$\mathbb{P}(N(T) - N(t) = m) = \frac{1}{m!}\left(-\int_t^T \lambda(s)ds\right)^m \exp\left(-\int_t^T \lambda(s)ds\right)$$

We can show that the compensator of the inhomogeneous Poisson process is $-\int_t^t \lambda(s)ds$

Applying this to survival probabilities and hazard rates:

$$P(0,T) = \exp\left(-\int_0^T \lambda(s)ds\right)$$

$$\begin{aligned}
H(t,T,T+\delta t) &= \frac{1}{\delta t}\left(\frac{e^{-\int_t^T \lambda(s)ds}}{e^{-\int_t^{T+\delta t} \lambda(s)ds)}} - 1\right) \\
&= \frac{1}{\delta t}\left(e^{\int_T^{T+\delta t} \lambda(s)ds} - 1\right) \\
h(t,T) &= \lambda(T)
\end{aligned}$$

We have therefore obtained enough flexibility to capture the term structure.

Continuing, we price the building blocks $\overline{B}$ and e.

If we assume independence of the default free rate and the arrival time of of default, we have

$$\begin{aligned}
\overline{B}(0,T) &= \mathbb{E}\left(e^{-\int_0^T r(s)ds}\mathbf{1}_{N(T)=0}\right) \\
&= \mathbb{E}\left(e^{-\int_0^T r(s)ds}\right)\mathbb{E}\left(\mathbf{1}_{N(T)=0}\right) \\
&= B(0,T)e^{-\int_0^T \lambda(s)ds}
\end{aligned}$$

Payoff of 1 at default

(discrete) Default happens in $[T_k, T_{k+1}]$, and same independence assumption

$$\begin{aligned} e(0,T_k,T_{k+1}) &= \mathbb{E}\left(e^{-\int_0^{T_{k+1}} r(s)ds}\left(\mathbf{1}_{N(T_k)=0} - \mathbf{1}_{N(T_{k+1})=0}\right)\right) \\ &= B(0,T_{k+1})\left(e^{-\int_0^{T_k}\lambda(s)ds} - e^{-\int_0^{T_{k+1}}\lambda(s)ds}\right) \\ &= B(0,T_{k+1})e^{-\int_0^{T_k}\lambda(s)ds}\left(1 - e^{-\int_{T_k}^{T_{k+1}}\lambda(s)ds}\right) \\ &= \overline{B}(0,T_{k+1})\left(1 - e^{-\int_{T_k}^{T_{k+1}}\lambda(s)ds}\right) \end{aligned}$$

Payoff of 1 at default

(continuous) Taking the limit as $T_{k+1} \to T_k = T$

$$\begin{aligned} e(0,T) &= \lim_{\delta t \to 0} e(0,T,T+\delta t) \\ &= \lim_{\delta t \to 0} \overline{B}(0,T+\delta t)\left(1 - e^{-\int_T^{T+\delta t} \lambda(s)ds}\right) \\ &= \overline{B}(0,T) \lim_{\delta t \to 0} \left(1 - e^{-\int_T^{T+\delta t} \lambda(s)ds}\right) \\ &= \overline{B}(0,T)\lambda(T) \end{aligned}$$

Stochastic

We'd be happy if that was enough. The market shows yield spreads that are not smooth, but stochastic.

We therefore need a stochastic process built in to our intensity based model.

Here we still want

$$\mathbb{P}(N(t+\delta t) - N(t)) = \lambda(t)\delta t$$

but now we see that we need dynamics for λ, and assume λ is the stochastic process given by

$$d\lambda(t) = \mu_\lambda(t)dt + \sigma_\lambda(t)dW(t)$$

There is a bit of detail into getting the appropriate (i.e., what we want) model in place here.

One of the requirements is that we have a *background driving filtration*, $\{\mathcal{G}_t\}$. Essentially this is a filtration generated by a *background driving process* (think economic factors).

All default free processes are adapted to $\{\mathcal{G}_t\}$.

The (stochastic) intensity, λ, is adapted to $\{\mathcal{G}_t\}$.

And the full filtration, $\{\mathcal{F}_t\}$ is made up of $\{\mathcal{G}_t\}$ and the filtration generated by the jump process, N.

Given $\mathcal{G}$, we know what λ has been like so far, so we can compute the probability of m jumps in $[t, T]$ exactly as before, but with a conditional expectation:

$$\begin{aligned}\mathbb{P}(N(T) - N(t) = m) &= \mathbb{E}(\mathbb{P}(N(T) - N(t) = m)|\lambda) \\ &= \mathbb{E}\left(\frac{1}{m!}\left(-\int_t^T \lambda(s)ds\right)^m e^{-\int_t^T \lambda(s)ds}\right)\end{aligned}$$

which is a bit more complicated, but not bad.

Notice that in the inhomogeneous case we smoothed with integration, and now with expectation.

We would like to price the building blocks again.

Before we do this, we need to recall the law of iterated expectations:

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\mathcal{G}))$$

For us, the intuition is that if we condition on the background driving filtration (think economic factors), our jump process has an inhomogeneous jump intensity, and things become tractable. We then take the average of all of these tractable things.

We'll price the building blocks (again!)

$$\begin{aligned}\overline{B}(0,T) &= \mathbb{E}(e^{-\int_0^T r(s)ds}\mathbf{1}_{\tau>T}) \\ \text{(by conditioning)} &= \mathbb{E}(\mathbb{E}(e^{-\int_0^T r(s)ds}\mathbf{1}_{\tau>T}|\mathcal{G}))\end{aligned}$$

Since we assume that default free processes are $\mathcal{G}$ measurable, we get

$$\begin{aligned}\mathbb{E}(e^{-\int_0^T r(s)ds}\mathbf{1}_{\tau>T}|\mathcal{G}) &= e^{-\int_0^T r(s)ds}\mathbb{E}(\mathbf{1}_{\tau>T}|\mathcal{G}) \\ &= e^{-\int_0^T r(s)ds}e^{-\int_0^T \lambda(s)ds} \\ &= = e^{-\int_0^T r(s)+\lambda(s)ds}\end{aligned}$$

And therefore

$$\overline{B}(0,T) = \mathbb{E}(e^{-\int_0^T r(s)+\lambda(s)ds})$$

A payoff of 1 at T_{k+1} if default happens in $[T_k, T_{k+1}]$

(discrete)

$$e(0, T_k, T_{k+1}) = \mathbb{E}\left(e^{-\int_0^{T_{k+1}} r(s)ds}\left(\mathbf{1}_{N(T_k)=0} - \mathbf{1}_{N(T_{k+1})=0}\right)\right)$$
$$= \mathbb{E}\left(\mathbb{E}\left(e^{-\int_0^{T_{k+1}} r(s)ds}\left(\mathbf{1}_{N(T_k)=0} - \mathbf{1}_{N(T_{k+1})=0}\right)|\mathcal{G}\right)\right)$$

The inner expectation is

$$e^{-\int_0^{T_{k+1}} r(s)ds}\mathbb{E}\left(\left(\mathbf{1}_{N(T_k)=0} - \mathbf{1}_{N(T_{k+1})=0}\right)|\mathcal{G}\right)$$
$$= e^{-\int_0^{T_{k+1}} r(s)ds}\left(e^{-\int_0^{T_k} \lambda(s)ds} - e^{-\int_0^{T_{k+1}} \lambda(s)ds}\right)$$

and the end result after taking the final expectation is

$$e(0, T_k, T_{k+1}) = \mathbb{E}\left(e^{-\int_0^{T_{k+1}} r(s)ds} e^{-\int_0^{T_k} \lambda(s)ds}\right) - \overline{B}(0, T_k)$$

A payoff of 1 at default; taking $T_{k+1} \to T_k = T$

(continuous)

$$e(0,T) = \lim_{\delta t \to 0} \frac{1}{\delta t} \left[\mathbb{E} \left(e^{-\int_0^{T+\delta t} r(s)ds} e^{-\int_0^{T} \lambda(s)ds} \right) - \overline{B}(0, T+\delta t) \right]$$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \left[\mathbb{E} \left(e^{-\int_0^{T+\delta t} r(s)+\lambda(s)ds} e^{\int_T^{T+\delta t} \lambda(s)ds} \right) - \overline{B}(0, T+\delta t) \right]$$

which becomes

$$\mathbb{E}(\lambda(T) e^{-\int_0^T r(s)+\lambda(s)ds})$$

The problem with what we've done...

We didn't solve the pricing problem.

The coupling of interest rates and intensity means we have to do more work.

Another day.

Also, I would like to see how the intensity is calibrated.

I think this is for a much, much later day.