Robust Replication of Default Contingent Claims

Carr/Flesaker

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Outline

Problem: Replicate the payoff on a default-contingent claim, using CDSs. We consider a claim with maturity at time T, which may default at a random time $\tau \in [0, T]$. The claim pays coupon interest at rate $c(t)$ until time τ or time T whichever comes first. At default time τ, the claim has a recovery value $R(\tau)$. If there is no default, the claim is worth $R(T)$ at time T. This is the target claim.

Ingredients for replication: The CDS spread $S_0(t)$ for different maturities t is assumed to be known at the intial time $t = 0$. The investor replicating the payoff can take positions in defaultable bonds with all maturity dates $u \in [0, T]$. These positions are credit default swaps. At time 0, the investor sells protection on an amount $Q(u)du$ of bonds with maturity dates in the interval $[u, u + du]$. This causes premium income at rate $S_0(u)Q(u)du$ during the time interval $[0, u]$. These bonds have a fixed loss factor $L \in (0, 1]$, so that a unit bound is worth $1 - L$ of the notional amount at the time of default.

The investor also has a money-market account that pays interest at a deterministic rate $r(t)$.

We have two controls on our strategy $M(.)$ and $Q(u)du$.

At time $t = 0$, the investor has a money-market amount $M(0)$. He also sells protection on bonds as described above. As time goes on, he receives interest revenue and premium revenue, and gives up money at rate $c(t)$ to replicate the coupon payments of the claim. As covered last week, the defining equations are (for $t \in [0, T]$): (balance equation)

$$M(t) - L \int_t^T Q(u)\,du = R(t), \tag{1}$$

and (revenue equation)

$$M'(t) = r(t)M(t) + \int_t^T S_0(u)\,Q(u)\,du - c(t). \tag{2}$$

This leads to:

$$Q(t) = \frac{1}{L}\left[R'(t) - M'(t)\right] \tag{3}$$

and the key differential equation

$$\mathcal{L}M = f, \tag{4}$$

where

$$\mathcal{L} = M''(t) - \left[r(t) + \frac{S_0(t)}{L}\right] M'(t) - r'(t)M(t)$$

and

$$f(t) = -c'(t) - \frac{S_0(t)}{L} R'(t).$$

$\mathcal{L}$ is a linear second-order differential operator with (in general) non-constant coefficients.

M satisfies the boundary conditions $M(T) = 0$ (no waste) and $\lim_{t \uparrow T} M'(t) = -c(T)$. The latter equation is obtained from the revenue equation as $t \uparrow T$.

For general f, the key equation (4) can be solved either numerically or in some cases using explicit formulas. The paper seems to stress the need to provide solutions for many choices of f. For this purpose the technique of using the Green's function of the equation in introduced. This technique is typical for linear differential equation, where we obtain the solution as a superposition of special-case solutions.

It will be convenient to modify M, so that it has simpler boundary conditions. Let V denote the solution of an associated problem, namely

$$\mathcal{L}V(t) = 0$$

for all $t \in [0, T]$ and $V(T) = 0$ and $\lim_{t \uparrow T} V'(t) = -c(T)$. Let $\tilde{M} = M - V$, so that

$$\mathcal{L}\tilde{M}(t) = f$$

and $\tilde{M}(T) = 0$ and $\lim_{t \uparrow T} \tilde{M}'(t) = 0$.
Note that if we find V once and for all, we can find M from $\tilde{M}$ immediately for any choice of f.

The special-case solutions alluded to earlier will be denoted by $g(t;u)$, where $g(\cdot;u)$ satisfies

$$\mathcal{L}g(t;u) = \delta(t-u) \tag{5}$$

and $\delta(t-u)$ is the Dirac delta function with appropriate boundary conditions specified shortly. By using the $g(t;u)$ solutions we can obtain the solution for any f. $\delta(t-u)$ is a generalized function representing a "spike" concentrated in a small interval around the time u. It is said to have zero width, infinite height, and integral equal to 1. The solution $\tilde{M}$ to the equation $\mathcal{L}\tilde{M} = f$ is given by

$$\tilde{M}(t) = \int_0^T f(u)\, g(t;u)\, du. \tag{6}$$

Our equation looks different from the representation of $\tilde{M}$ obtained in the paper, because we must specify boundary conditions to get a unique solution for $g(t;u)$. We will follow the paper and choose the conditions to be $g(T;u) = 0$ and $\lim_{t\uparrow T} \frac{\partial}{\partial t} g(t;u) = 0$. These condition plus the differential equation for $g(t;u) = 0$ for all $t \geq u$. (And, usefully, we then see that $g(u;u) = 0$ and $\lim_{t\uparrow u} \frac{\partial}{\partial t} g(t;u) = -1$.) As a consequence, the equation for $\tilde{M}(t)$ becomes

$$\tilde{M}(t) = \int_t^T f(u)\, g(t;u)\, du. \tag{7}$$

What more could we ask for?

Suppose our goal is to price the original contingent claim. Then we need to find the value $\tilde{M}(0)$ from out solution. Note that we apply the boundary for $\tilde{M}(T)$ and $\tilde{M}'(T)$ to determine $\tilde{M}(0)$, so this is is a *backwards* procedure. Our representation for $\tilde{M}(t)$ using the Green's function $g(t;u)$ is also a backwards procedure, because we solve for $g(t;u)$ $t \in [0,u]$ working backwards from $t = u$. This is perfectly feasible but note that if we wish to use the integral representation of f in terms of $g(t;u)$, we must know $g(0;u)$ for *every* value of u. This means we must solve a differential equation in t (for $g(t;u)$) for each value of u. This is inefficient. We need a better approach.

It turns out that as a function of u, the quantity $g(t;u)$ satisfies a second order linear differential equation (we use u as a subscript to indicate that derivative are with respect to u):

$$\mathcal{L}_u^* g(t;u) = \delta(t-u), \tag{8}$$

where for any N,

$$\mathcal{L}_u^* N = N''(u) + \left[r(u) + \frac{S_0(u)}{L} \right] N'(u) + \frac{S_0'(u)}{L} N(u). \tag{9}$$

The operator $\mathcal{L}^*$ is called the adjoint of L. We will call the differential equation using $\mathcal{L}^*$ the *forward* equation, since it involves value of u later than t.

The operator $\mathcal{L}^*$ is cooked up so that for any functions W and Z,

$$\int_0^T (\mathcal{L}W)\, Z = \int_0^T W\, (\mathcal{L}^* Z)\,,$$

provided that the values of W and Z at the endpoints 0 and T are compatible (so that all endpoint terms resulting from integrations by parts vanish).

The appropriate boundary conditions to determine $g(t;u)$ as a function of u turn out to be $g(u;u) = 0$ and $\lim_{u \downarrow t} \frac{\partial}{\partial u} g(t;u) = 1$. Using the forward equation, we can now find, say, $g(0;u)$ for all u by solving a single differential equation. We can then represent the solution $\tilde{M}$ for any choice of f using our integral representation in terms of $g(0;u)$.

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