Option Replication and Model Risk

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Black-Scholes Problematic Assumptions

- Zero Transaction costs
- Continuous hedging
- Simplistic stochastic process
  - deterministic and known volatility
  - existence of volatility Skew and Smile

Transaction Costs

- Black-Scholes equation with an effective volatility (Leland 1985)

- $\% TC \propto$ the number of shares transacted

- $E[TC] = \frac{k}{\sigma\sqrt{2\pi\Delta t}}\frac{\partial C}{\partial \sigma}$

- continuous hedging is too expensive

- vast literature mostly in GBM framework and primarily without regard to other options

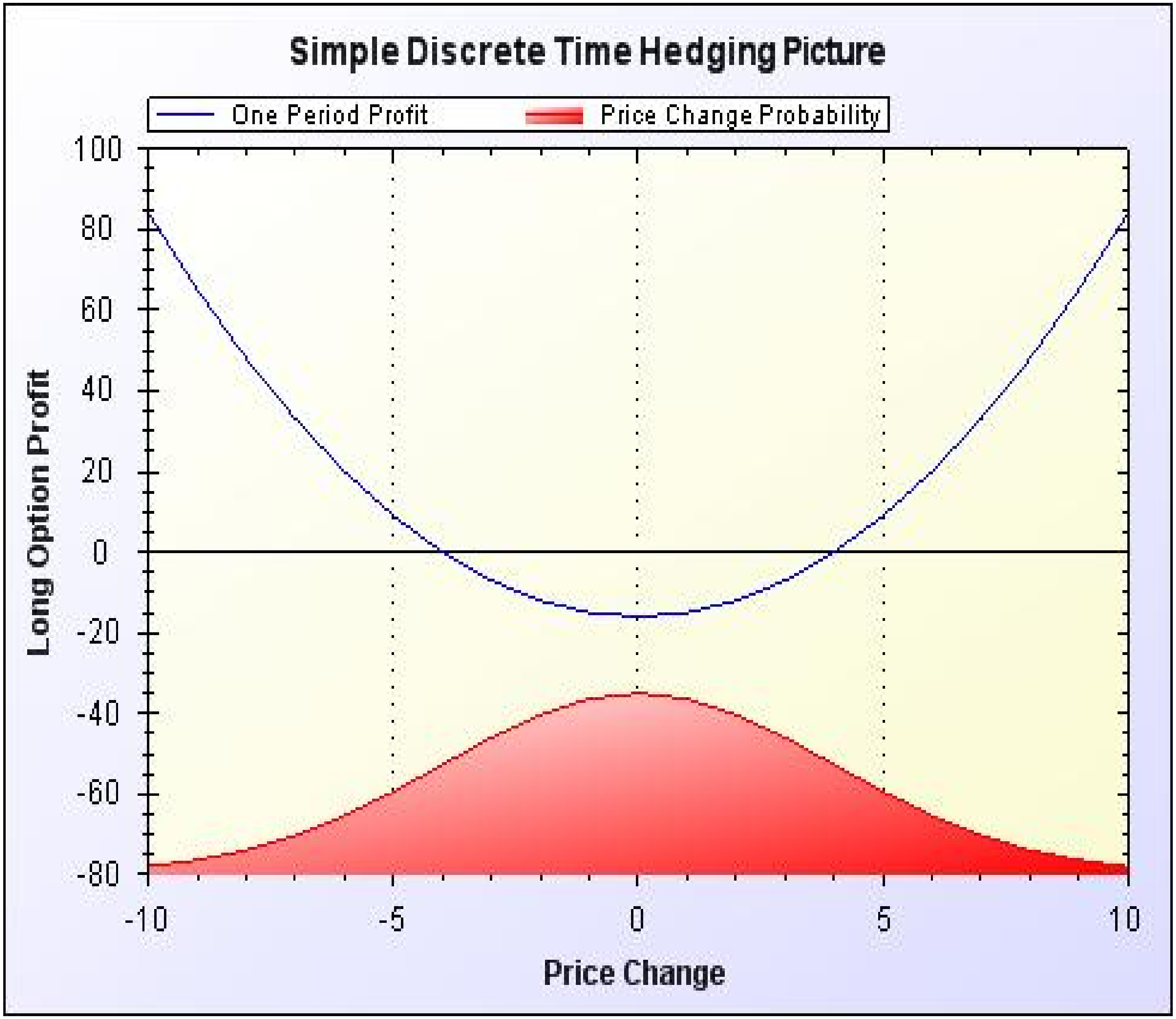
Discrete Hedging Error over Δt

- $\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$ and $\Pi = C - \frac{\partial C}{\partial S} S$

- $\Delta \Pi \approx \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \sigma^2 (\epsilon^2 - 1) \Delta t + \mathcal{O} \Delta t^{3/2}$ where $\epsilon \sim$ N(0,1)

- Chi-Square Distribution of order 1

- discrete hedging presents risk skewed against the seller


Simple Discrete Time Hedging Picture
One Period Profit
Price Change Probability
Long Option Profit
100
80
60
40
20
0
-20
-40
-60
-80
-10
-5
0
5
10
Price Change

Discrete Hedging Error over T

- $n(=\frac{T}{\Delta t})$ Chi-squares of order 1 $\sim$ Chi-square of order n. Normal as $n \to \infty$

- $\sqrt{E[\Pi^2]} = \sqrt{\frac{\pi \Delta t}{4T}} \sigma \frac{\partial C}{\partial \sigma}$ (Kamal and Derman)

- attributed to discrete sampling error of continuous process

- One could perform mean-variance optimization to find optimal Δt if k and σ are known

σ estimation assuming no other options to hedge

- probe historical distributions and statistical relationships
- $E[\sigma] = \bar{\sigma}$ $\text{Var}[\sigma] = \kappa^2$ which implies $\sqrt{E[\Pi^2]} \approx \kappa \frac{\partial C}{\partial \sigma}$
- in most cases $\kappa >> \sqrt{\frac{\pi \Delta t}{4T}} \sigma > \frac{k}{\sigma \sqrt{2 \pi \Delta t}}$
- GBM assumption introduces the most risk
- motivation for more descriptive processes

Stochastic Volatility

- Designed to account for volatility fluctuations, volatility clustering and correlations with the underlying. Incompleteness due to additional source of randomness.

- Heston Model

$$
\begin{aligned}
dS &= \mu S dt + \sqrt{v} S dW_1 \\
dv &= -\lambda(v - \bar{v}) dt + \eta \sqrt{v} dW_2 \\
\rho dt &= \langle dW_1 dW_2 \rangle
\end{aligned}
$$

- Differential equation formulation assumes $\Pi = C - \delta_1 V - \delta_2 S$ where V is another volatility dependent instrument

- What if there is no V ($\delta_1 = 0$)? What is δ_2 and what does the discrete hedging risk look like?

Heston hedging error

- Risk minimizing hedge sets $\delta_2 = \frac{\partial C}{\partial S} + \frac{\rho\eta}{S}\frac{\partial C}{\partial v}$

- $\Delta\Pi \approx \sqrt{v\Delta t}\eta\sqrt{1-\rho^2}\frac{\partial C}{\partial v}\epsilon_2 + \mathcal{O}\Delta t(\text{B-S terms})$

- $\sqrt{E[\Pi^2]} \propto \eta\sqrt{T(1-\rho^2)}\frac{\partial C}{\partial \sigma}$

- Note the connection to σ estimation in B-S framework: uncertainty in vol, hedge-ability term, generally greater in magnitude

- What about leptokurtic and skewed returns?

Levy Processes

- Adding jumps allows a higher moments in return distribution
- Discontinuities cause market incompleteness
- Levy Process X_t has characteristic function $\Phi_t(u) = E[e^{iuX_t}] = e^{t\psi(u)}$
- $\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\gamma u + \int (e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1})\nu(dx)$
- The Levy measure, ν, allows much rich and interesting behavior
- Brownian motion ($\nu = 0$), Merton jump diffusion, variance gamma, CGMY, NIG are all examples of Levy Processes

Hedging under a Levy Process

- Delta hedge with jump risk diversified through the portfolio (Merton)

- Utility Maximization

  - Global mean-variance optimization popular due to tractability

- Local MV and Super-hedging are not discussed below

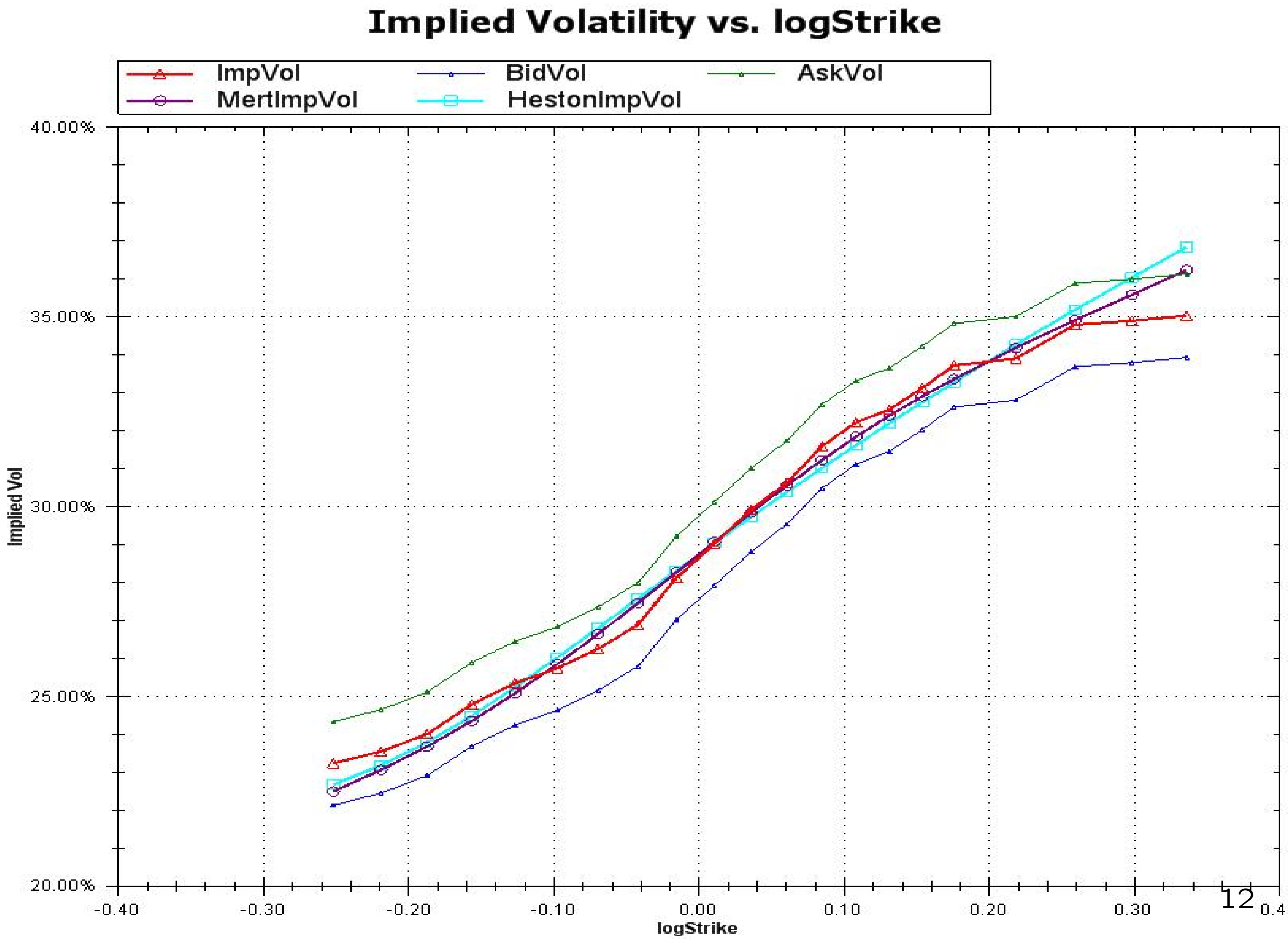
Option Pricing using Fourier Transforms

- Knowledge of risk-neutral characteristic function $\Phi(u)$ allows integral transform representation of European style options

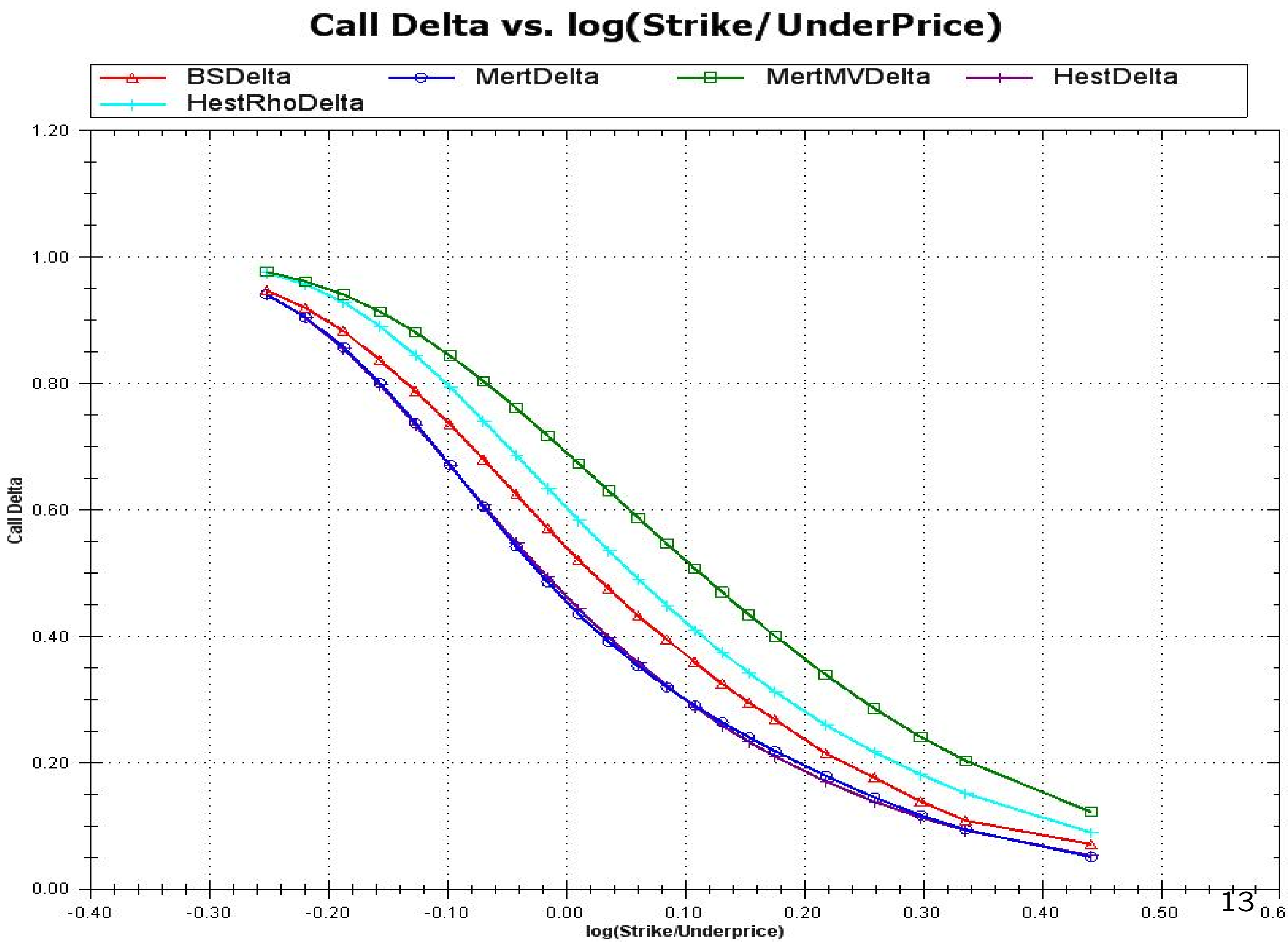
- $C(S,K,T) = S - \sqrt{SK}\frac{1}{\pi}\int_0^\infty \frac{du}{u^2 + 1/4}\mathrm{Re}[e^{-iuk}\Phi_T(u - i/2)]$

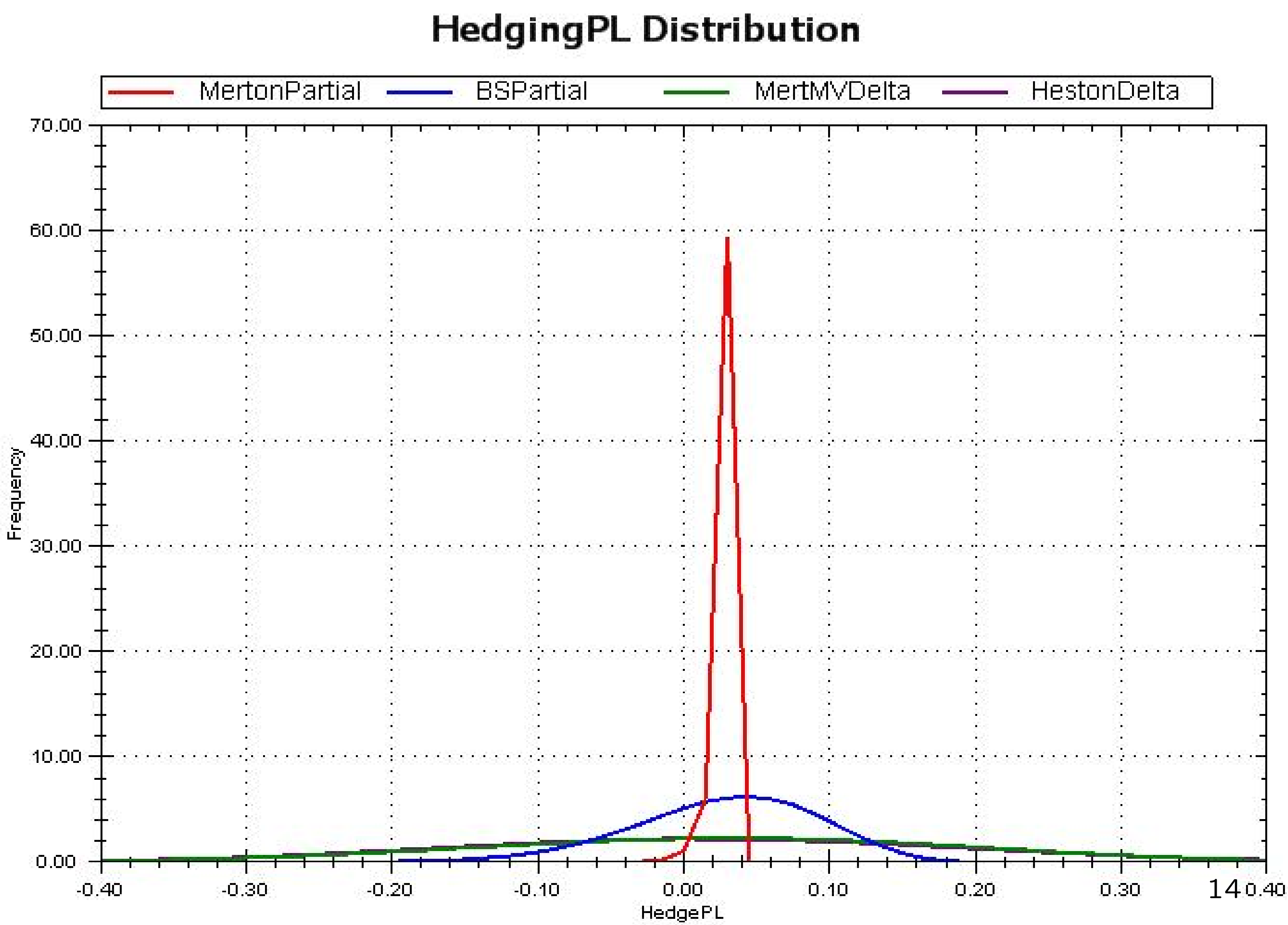
- Stochastic vol models and Levy processes have analytical Φ

- quadrature or FFT methods for efficient computation



Implied Volatility vs. logStrike
ImpVol
BidVol
AskVol
MertImpVol
HestonImpVol
40.00%
35.00%
30.00%
25.00%
20.00%
Implied Vol
-0.40
-0.30
-0.20
-0.10
0.00
0.10
0.20
0.30
0.40
logStrike


Call Delta vs. log(Strike/UnderPrice)
BSDelta
MertDelta
MertMVDelta
HestDelta
HestRhoDelta
Call Delta
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.40
-0.30
-0.20
-0.10
0.00
0.10
0.20
0.30
0.40
0.50
0.60
log(Strike/Underprice)


HedgingPL Distribution
MertonPartial
BSPartial
MertMVDelta
HestonDelta
Frequency
70.00
60.00
50.00
40.00
30.00
20.00
10.00
0.00
-0.40
-0.30
-0.20
-0.10
0.00
0.10
0.20
0.30
0.40
HedgePL

Hedging stats over Δt

stat	Merton(red)	B-S(blue)	MV(purple)	Hest(green)
StdDev/C_0	4.95%	3.77%	2.70%	2.23%
P($< -C_0/4$)	0.123%	0.115%	0.063%	0.010%
P($< -C_0$)	**0.105%**	0.088%	0.048%	0.004%
P($< -2C_0$)	**0.015%**	**0.004%**	0.000%	0.000%

Conclusions and Further topics

- Risks assuming perturbed GBM underestimate real risks
- Hedge strategy/ratio is model dependent. You must have faith in your stochastic process.
- Global MV (local MV too) and partial derivative hedge under Levy process all have their own idiosyncracies
- Jumps with stochastic vol and stochastic Levy processes offer better market description but more parameters
- Many models can reasonably fit option prices and (separately) historical return distributions. Model comparisons probing replication error are more practical