Black-Scholes Problematic Assumptions

• Zero Transaction costs

• Continuous hedging

• Simplistic stochastic process
  – deterministic and known volatility
  – existence of volatility Skew and Smile
Transaction Costs

• Black-Scholes equation with an effective volatility (Leland 1985)

• \( \%TC \propto \) the number of shares transacted

\[
E[TC] = \frac{k}{\sigma \sqrt{2\pi \Delta t}} \frac{\partial C}{\partial \sigma}
\]

• continuous hedging is too expensive

• vast literature mostly in GBM framework and primarily without regard to other options
Discrete Hedging Error over $\Delta t$

- $\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$ and $\Pi = C - \frac{\partial C}{\partial S} S$

- $\Delta \Pi \approx \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \sigma^2 (\epsilon^2 - 1) \Delta t + O(\Delta t^{3/2})$ where $\epsilon \sim N(0,1)$

- Chi-Square Distribution of order 1

- discrete hedging presents risk skewed against the seller
Discrete Hedging Error over $T$

- $n(= \frac{T}{\Delta t})$ Chi-squares of order 1 $\sim$ Chi-square of order $n$. Normal as $n \to \infty$

- $\sqrt{E[\Pi^2]} = \sqrt{\frac{\pi \Delta t}{4T}} \sigma \frac{\partial C}{\partial \sigma}$ (Kamal and Derman)

- attributed to discrete sampling error of continuous process

- One could perform mean-variance optimization to find optimal $\Delta t$ if $k$ and $\sigma$ are known
σ estimation assuming no other options to hedge

- probe historical distributions and statistical relationships

- \( E[σ] = \bar{σ} \) \( \text{Var}[σ] = κ^2 \) which implies \( \sqrt{E[Π^2]} \approx κ \frac{∂C}{∂σ} \)

- in most cases \( κ >> \sqrt{\frac{πΔt}{4T}}σ > \frac{k}{σ\sqrt{2πΔt}} \)

- GBM assumption introduces the most risk

- motivation for more descriptive processes
Stochastic Volatility

- Designed to account for volatility fluctuations, volatility clustering and correlations with the underlying. Incompleteness due to additional source of randomness.

- Heston Model

\[
\begin{align*}
    dS &= \mu S dt + \sqrt{v} S dW_1 \\
    dv &= -\lambda (v - \bar{v}) dt + \eta \sqrt{v} dW_2 \\
    \rho dt &= \langle dW_1 dW_2 \rangle
\end{align*}
\]

- Differential equation formulation assumes \( \Pi = C - \delta_1 V - \delta_2 S \) where \( V \) is another volatility dependent instrument

- What if there is no \( V \) (\( \delta_1 = 0 \))? What is \( \delta_2 \) and what does the discrete hedging risk look like?
Heston hedging error

- Risk minimizing hedge sets $\delta_2 = \frac{\partial C}{\partial S} + \frac{\rho \eta}{S} \frac{\partial C}{\partial v}$

- $\Delta \Pi \approx \sqrt{v \Delta t} \eta \sqrt{1 - \rho^2} \frac{\partial C}{\partial v} \epsilon_2 + O \Delta t (B-S \text{ terms})$

- $\sqrt{E[\Pi^2]} \propto \eta \sqrt{T(1 - \rho^2)} \frac{\partial C}{\partial \sigma}$

- Note the connection to $\sigma$ estimation in B-S framework: uncertainty in vol, hedge-ability term, generally greater in magnitude

- What about leptokurtic and skewed returns?
Levy Processes

- Adding jumps allows a higher moments in return distribution
- Discontinuities cause market incompleteness
- Levy Process $X_t$ has characteristic function $\Phi_t(u) = E[e^{iuX_t}] = e^{t\psi(u)}$

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\gamma u + \int (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx)$$

- The Levy measure, $\nu$, allows much rich and interesting behavior
- Brownian motion ($\nu = 0$), Merton jump diffusion, variance gamma, CGMY, NIG are all examples of Levy Processes
Hedging under a Levy Process

• Delta hedge with jump risk diversified through the portfolio (Merton)

• Utility Maximization
  – Global mean-variance optimization popular due to tractability

• Local MV and Super-hedging are not discussed below
Option Pricing using Fourier Transforms

- Knowledge of risk-neutral characteristic function $\Phi(u)$ allows integral transform representation of European style options

$$C(S, K, T) = S - \sqrt{SK} \frac{1}{\pi} \int_0^\infty \frac{du}{u^2 + 1/4} \text{Re}[e^{-iuk} \Phi_T(u - i/2)]$$

- Stochastic vol models and Levy processes have analytical $\Phi$

- Quadrature or FFT methods for efficient computation
Call Delta vs. log(Strike/UnderPrice)
### Hedging stats over $\Delta t$

<table>
<thead>
<tr>
<th>stat</th>
<th>Merton(red)</th>
<th>B-S(blue)</th>
<th>MV(purple)</th>
<th>Hest(green)</th>
</tr>
</thead>
<tbody>
<tr>
<td>StdDev/$C_0$</td>
<td>4.95%</td>
<td>3.77%</td>
<td>2.70%</td>
<td>2.23%</td>
</tr>
<tr>
<td>$P(&lt; -C_0/4)$</td>
<td>0.123%</td>
<td>0.115%</td>
<td>0.063%</td>
<td>0.010%</td>
</tr>
<tr>
<td>$P(&lt; -C_0)$</td>
<td><strong>0.105%</strong></td>
<td>0.088%</td>
<td>0.048%</td>
<td>0.004%</td>
</tr>
<tr>
<td>$P(&lt; -2C_0)$</td>
<td><strong>0.015%</strong></td>
<td><strong>0.004%</strong></td>
<td>0.000%</td>
<td>0.000%</td>
</tr>
</tbody>
</table>
Conclusions and Further topics

- Risks assuming perturbed GBM underestimate real risks

- Hedge strategy/ratio is model dependent. You must have faith in your stochastic process.

- Global MV (local MV too) and partial derivative hedge under Levy process all have their own idiosyncracies

- Jumps with stochastic vol and stochastic Levy processes offer better market description but more parameters

- Many models can reasonably fit option prices and (separately) historical return distributions. Model comparisons probing replication error are more practical