BS, by Scot Adams
PART A: EASY OPTIONS

Assume:

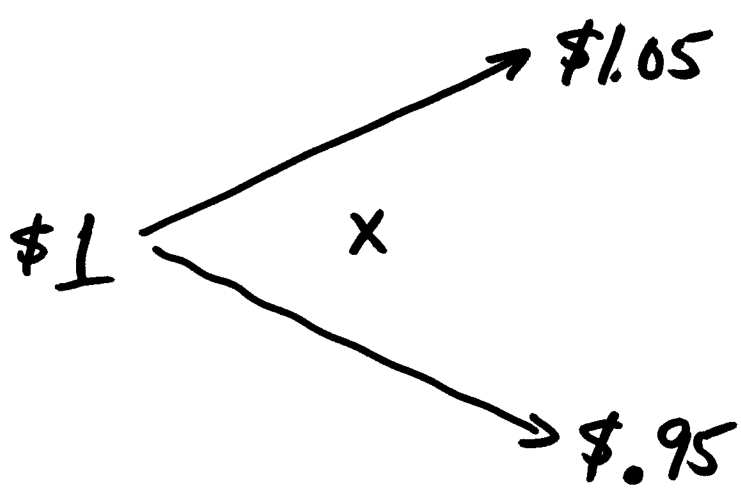
\$/Euro either ↗ 5% or ↘ 5% each month

Bank loan rate for 1 month is 1%

$$\begin{tikzcd}[row sep=small] & 1.05\,E \\ E \arrow[ur] \arrow[dr] \arrow[r, phantom, "\times"] & {} \\ & .95\,E \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} L \arrow[r, "1.01"'] & 1.01L \end{tikzcd}$$

Dan wants to buy
100 Euros for $100
one month from now

Current: 1 Euro = $ 1.00



Alice charges Dan
$2.97
takes out a loan of
$47.03
and buys 50 Euro for
$50

Profit : $0

One

month

later ...

Scenario A1: 1 Euro = \$1.05

Alice's profit in \$ is

Dan: $100 - 105 = -5$

Bank: $-(47.03)(1.01) = -47.50$

Euros: $(50)(1.05) = 52.50$

Profit A1 = \$0

Scenario A2: 1 Euro = $.95

Alice's profit in $ is

Dan: 0

Bank: $-(47.03)(1.01) = -47.50$

Euros: $(50)(.95) = 47.50$

Profit A2 = $0

Alice charges Dan

$\$u$

takes out a loan of

$\$v$

buys $u+v$ Euros for

$\$(u+v)$

Profit = $0

Alice's profits in \$:

	Dan	Bank	Euros
A1:	-5	$-\,v(1.01)$	$+\,(u+v)(1.05)$
A2:	0	$-\,v(1.01)$	$+\,(u+v)(.95)$

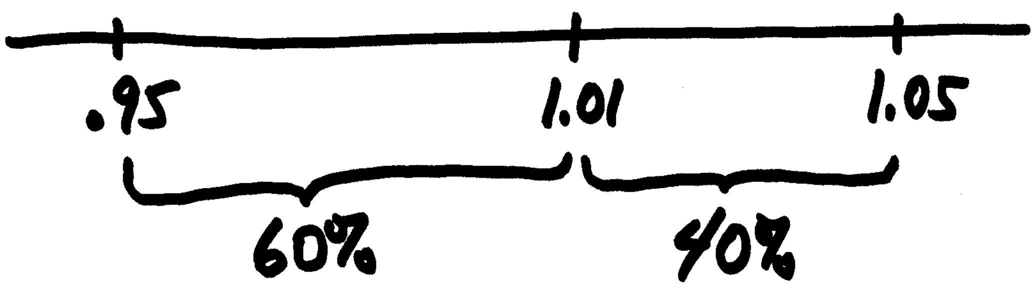
Profit A1 = Profit A2 = 0 $\Longrightarrow$

$$-5 \quad + \quad 0 \quad + (u+v)(.1) = 0$$

$$\Longrightarrow \quad u+v = \frac{5}{.1} = 50$$

(Subtract 2nd eqn from first)

$$p := \frac{1.01 - .95}{1.05 - .95} = .6$$



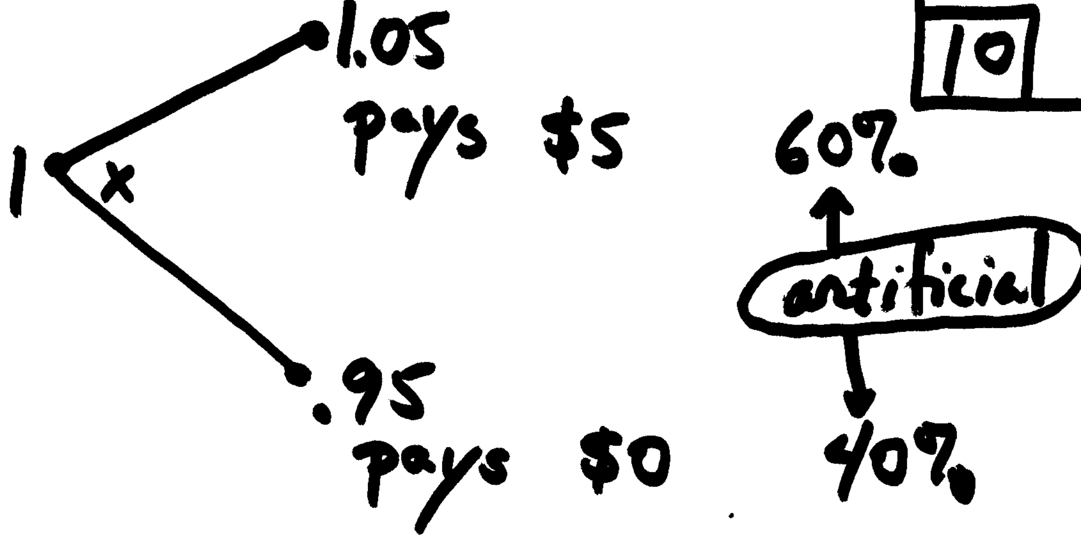
$$\begin{aligned} &\left[\ -5 - v(1.01) + (u+v)(1.05) = 0\ \right](.6) \\ +&\left[\ 0 - v(1.01) + (u+v)(.95) = 0\ \right](.4) \end{aligned}$$

$$-3 - v(1.01) + (u+v)(1.01) = 0$$

$$-3 \qquad + u\,(1.01) = 0$$

$$u = \frac{3}{1.01} = 2.970297$$

$$v = 50 - u = 47.0297$$



$$\sum \text{payment} \times \text{artif. prob}$$

$$= 5(.6) + 0(.4) = 3$$

$$\text{Cost} = \frac{3}{1.01} = 2.970297$$

Earl wants to buy
100 Euros for \$100
one month from now

Current: 1 Euro = \$1.1052631

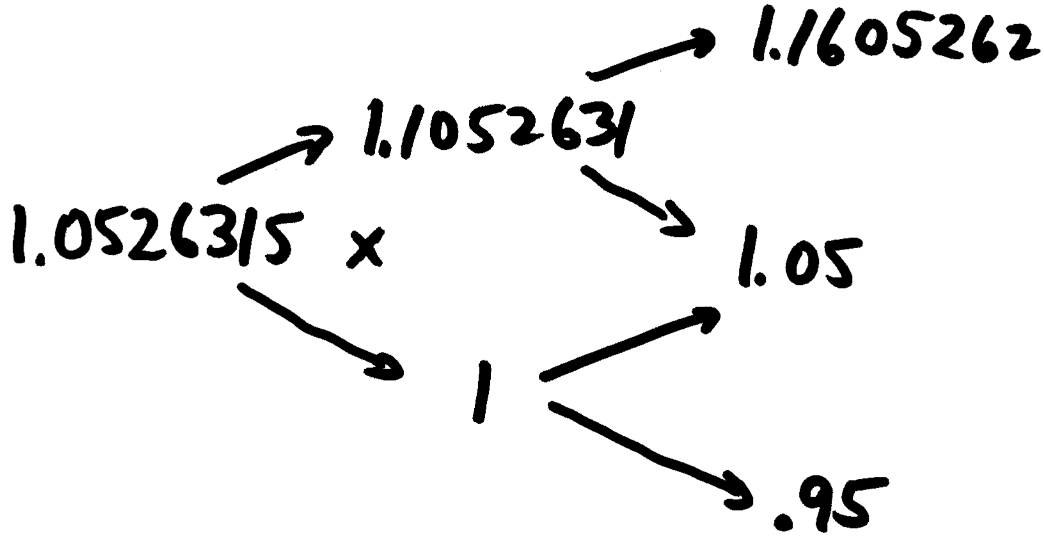
$$
1.1052631 \times \begin{cases} \nearrow 1.1605262 & \text{pays } \$16.05262 \\ \searrow 1.05 & \text{pays } \$5 \end{cases}
$$

$$
\text{Cost: } \frac{(16.05262)(.6) + 5(.4)}{1.01} = 11.516406
$$

Beth sells this option

Fred wants to buy
100 Euros for \$100
<u>two</u> months from now

Current: 1 Euro = \$1.0526315



Cathy sells this option

$$1.0526315 \begin{cases} \nearrow 1.1052631 \\ \searrow 1 \end{cases}$$

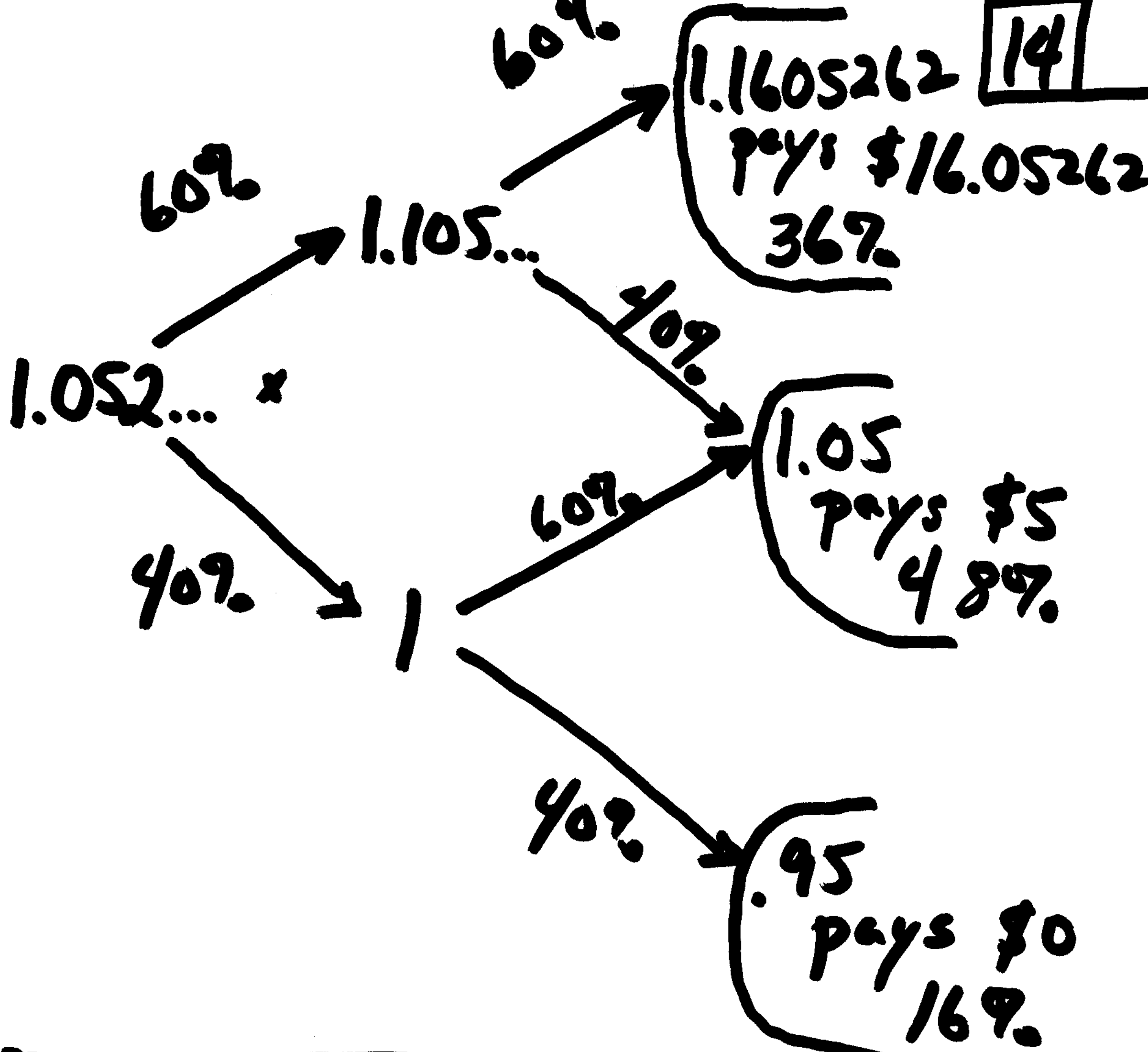
x

1.1052631: pays Beth \$11.516406

1: pays Alice \$2.970297

Cost: $\dfrac{(11.516406)(.6)+(2.970297)(.4)}{1.01}$

$= \$\,8.0177845$



$$\frac{\sum \text{payment} \times \text{end. artif. prob.}}{(1.01)^2}$$

$= \$8.0177845$

H 60%, T 40%

HH pays \$16.05262 (36%)

HT or TH pays \$5 (48%)

TT pays \$0 (16%)

$$\frac{\text{expected payment}}{(1.01)^2}$$

$= \$8.0177845$

PART B: COIN FLIPPING

$GP := 10^{10^{100}}$

Q: After GP flips of a fair coin, find prob. that

$$-\sqrt{GP} \le (\# H) - (\# T) \le \sqrt{GP}$$

$$\binom{n}{3}\frac{1}{n^3} = \frac{n(n-1)(n-2)}{3!\; n^3} \longrightarrow \frac{1}{3!}$$

$$\binom{n}{k}\frac{1}{n^k} \longrightarrow \frac{1}{k!} \quad \text{as } n \to \infty$$

$$(1-x^2)^n \Big|_{x \to x/\sqrt{n}}$$

$$= 1 - nx^2 + \binom{n}{2}x^4 - \binom{n}{3}x^6 + \cdots \Big|_{x \to x/\sqrt{n}}$$

$$= 1 - n\frac{x^2}{n} + \binom{n}{2}\frac{x^4}{n^2} - \binom{n}{3}\frac{x^6}{n^3} + \cdots$$

$$\longrightarrow 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \cdots$$

$$= e^{-x^2}$$

$$\left(1-x^2+5x^3\right)^n\Big|_{x\to x/\sqrt{n}}$$

$$= \cdots + 5nx^3 + \cdots \Big|_{x\to x/\sqrt{n}}$$

$$= \cdots + 5n\frac{x^3}{n^{3/2}} + \cdots \longrightarrow e^{-x^2}$$

<u>Fact</u>: $f(x) = 1 - x^2 + \cdots$

$$\Rightarrow \left(f(x)\right)^n\Big|_{x\to x/\sqrt{n}} \longrightarrow e^{-x^2}$$

<u>Fact</u>: $f(x) = 1 - \frac{1}{2}x^2 + \cdots$

$$\Rightarrow \left(f(x)\right)^n\Big|_{x\to x/\sqrt{n}} \longrightarrow e^{-x^2/2}$$

e.g. $\cos x = 1 - \frac{1}{2}x^2 + \cdots$

$$\therefore \cos^n x \Big|_{x \to x/\sqrt{n}} \longrightarrow e^{-x^2/2}$$

$$\cos^{GP} x \Big|_{x \to x/\sqrt{GP}} \approx e^{-x^2/2}$$

$X_1 := (\#H) - (\#T)$

after 1 flip (fair coin)

$\frac{1}{2}$	$\frac{1}{2}$
-1	1

$\frac{1}{2} z^{-1} + \frac{1}{2} z$ | (gen. fn.)

$z \longrightarrow e^{-ix}$, $i = \sqrt{-1}$

$\frac{1}{2} e^{ix} + \frac{1}{2} e^{-ix}$ | (F. transf.)

$= \cos x$

$\exists$ inverse F. transf.

$$X_1/7 = [(\#H) - (\#T)]/7$$

after 1 flip

$$\frac{1}{2} z^{-1/7} + \frac{1}{2} z^{1/7} \Big| \text{(g. fn.)}$$

$$\frac{1}{2} e^{ix/7} + \frac{1}{2} e^{-ix/7} \Big| \text{(F. tn.)}$$

$$\shortparallel$$

$$\cos(x/7)$$

$$\shortparallel$$

$$\cos x \Big|_{x \to x/7}$$

$X_2 := (\#H) - (\#T)$

after 2 flips

$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
-2	0	2

$\frac{1}{4}z^{-2} + \frac{1}{2} + \frac{1}{4}z^{2}$ (g. fn.)

$$\| \quad \text{(i.e. } = \text{)}$$

$$\left(\frac{1}{2}z^{-1} + \frac{1}{2}z\right)^2$$

$\cos^2 x$ (F. fn.)

$$X_{GP} := (\#H) - (\#T)$$

after GP flips

NO WAY!

$\left(\frac{1}{2} z^{-1} + \frac{1}{2} z\right)^{GP}$	(g. fn.)
$\cos^{GP} x$	(F. tn.)

F. tn. of $X_{GP}/\sqrt{GP}$ is

$$\cos^{GP} x \Big|_{x \longrightarrow x/\sqrt{GP}}$$

$$\approx e^{-x^2}$$

inverse F. tn.?

$$h(w) = e^{-w^2/2}/\sqrt{2\pi}$$

Calculus: $\int_{-\infty}^{\infty} h(w)\,dw = 1$

Y with distn. h

$h(w)\,dw \qquad \forall w$

w

$$\int_{-\infty}^{\infty} z^w\, h(w)\,dw \quad \text{(g. fn.)}$$

$$\int_{-\infty}^{\infty} e^{-iwx}\, h(w)\,dw \quad \text{(F. tn.)}$$

$$= e^{-x^2/2}$$

Calculus
e.g., $x=0$

$$\Pr\left[-1 \le \frac{X_{GP}}{\sqrt{GP}} \le 1\right]$$

$$\approx \Pr\left[-1 \le Y \le 1\right]$$

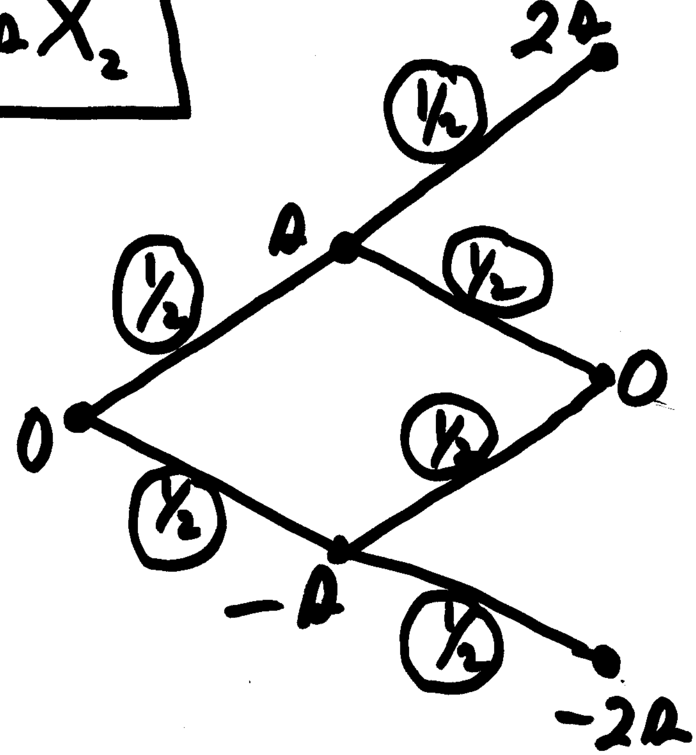
$$= \int_{-1}^{1} h(w)\,dw$$

$$\underset{\text{Calculus}}{=} .6826$$

$$\Delta = 1/\sqrt{GP}$$

ΔX_2

end pr

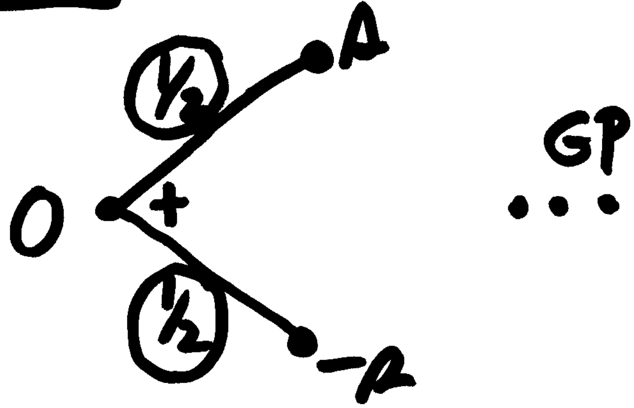


1/4

1/2

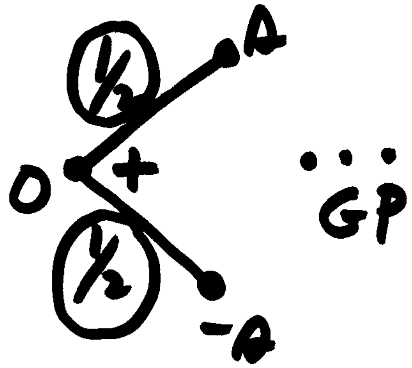
1/4

ΔX_{GP}



NO WAY!

$\approx h(w)\,dw \quad \forall w$



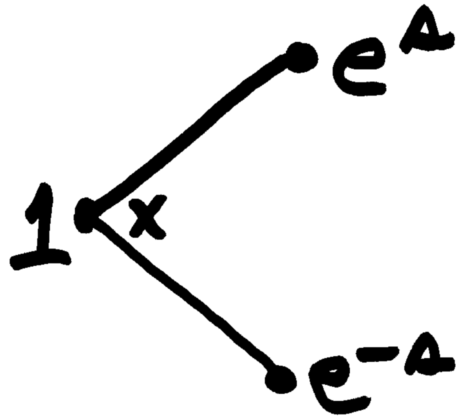
$$\frac{\text{if end at } w, \text{ pay } f(w)}{\text{end. prob.} = \text{NO WAY!} \approx h(w)\,dw, \ \forall w}$$

$$E[\text{payment}] = \sum_{w} [f(w)] \begin{bmatrix} \text{NO} \\ \text{WAY!} \end{bmatrix}$$

$$\approx \int_{-\infty}^{\infty} [f(w)][h(w)]\,dw$$

PART C: STOCK OPTIONS

stock price:



$\cdots$ GP

$a = 1/\sqrt{GP}$

int. i

$\underbrace{\qquad}_{1/GP \text{ year} = \text{"split second"}}$

Assume e^{-a} ——— $1+i$ ——— e^{a}

i.e. artificial prob = 50%

Let $g(\omega) := \begin{cases} \omega - 5, & \text{if } \omega \geq 5 \\ 0, & \text{if } \omega \leq 5. \end{cases}$

Assume: final stock price $= \omega$

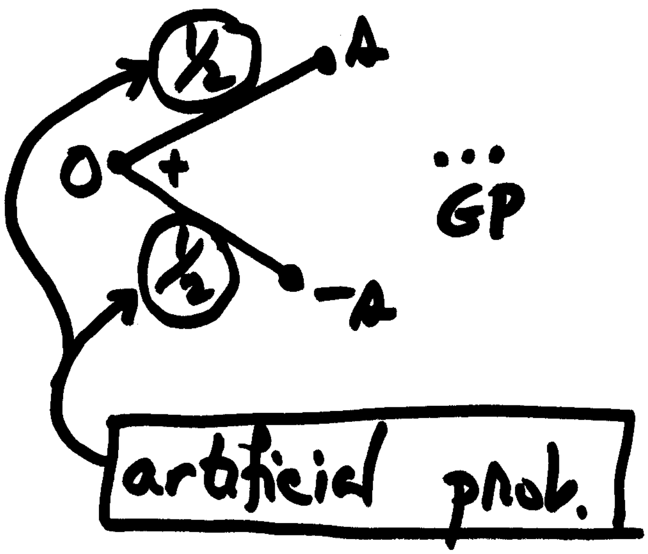
$\Rightarrow$ option pays $g(\omega)$

Guarantees holder of option can ~~by~~ buy 1 share of stock for $\leq \$5$

Equiv: ln(final stock price) $= w$

$\Rightarrow$ option pays $g(e^w)$

ln(stock price):



... GP

if end at w, pay $g(e^w)$

end artif. prob $=$ NO WAY! $\approx h(w)\,dw,\ \forall w$

$\sum$ payment $\times$ end. artif. prob

$$= \sum_w \left[g(e^w)\right] \left[\begin{matrix}\text{NO}\\ \text{WAY!}\end{matrix}\right]$$

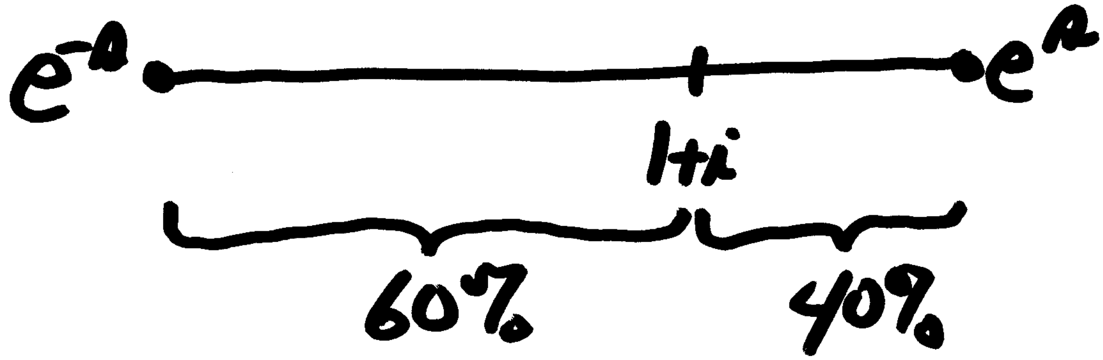
$$\approx \int_{-\infty}^{\infty} \left[g(e^w)\right]\left[h(w)\right] dw$$

Cost of option $\approx$

$$\frac{1}{(1+i)^{GP}} \int_{-\infty}^{\infty} [g(e^{w})][h(w)]\,dw$$

CALCULUS

<u>Q</u>: What if



?

PART D: CENTRAL LIMIT THM

Def $\forall \sigma > 0$,

$$h^{\sigma}(w) := \frac{1}{\sigma}\left[h\left(\frac{w}{\sigma}\right)\right]$$

$$\int_{-\infty}^{\infty} h^{\sigma}(w)\, dw = 1$$

Def $\forall \mu \in \mathbb{R}$, $\forall \sigma > 0$

$$h^{\sigma}_{\mu}(w) := h^{\sigma}(w-\mu)$$

$$\int_{-\infty}^{\infty} h^{\sigma}_{\mu}(w)\, dw = 1$$

$$h^{1}_{0}(w) = h(w)$$

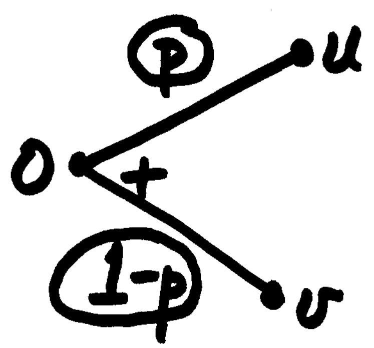
Central Limit Thm

Say $0 \le p \le 1$, $u \geq v$, $N \geq 1$, $N \in \mathbb{Z}$

Let $\sigma := \sqrt{N p (1-p)}\,(u-v)$

$$\mu := N(pu + (1-p)v)$$

Then



$\cdots$ N

end prob. $\approx$

$h_\mu^\sigma(w)\,dw, \ \forall w$

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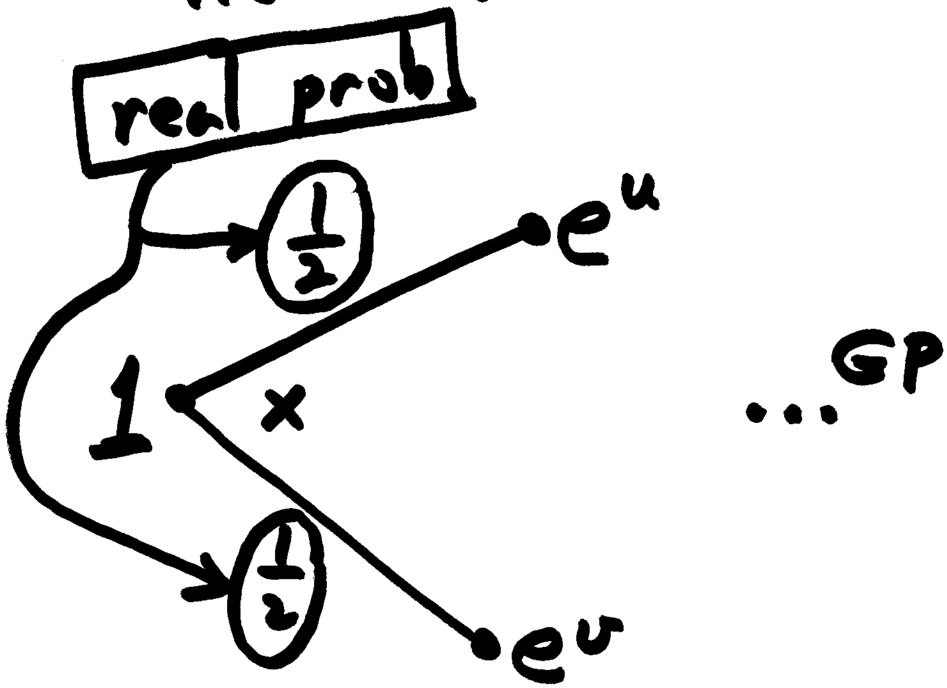
e.g. $u = A$, $v = -A$, $p = 1/2$, $N = GP$

$\Rightarrow \sigma = 1, \mu = 0 \Longrightarrow h_\mu^\sigma = h$

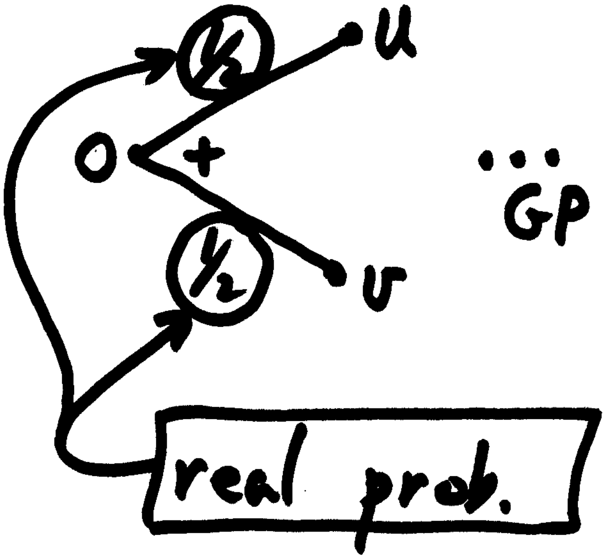
PART E: MORE ON STOCK OPTIONS

new stock:



$u \geqslant v$, $\quad u, v$ unknown

ln(stock price):



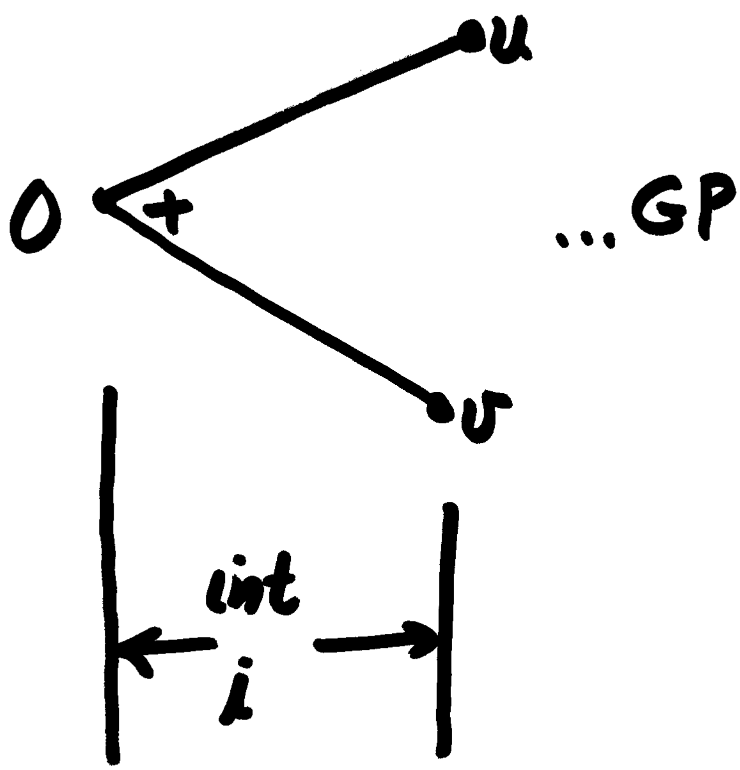
$\therefore$ GP

ext. real prob.
= NO WAY!
$\approx h_\mu^\sigma(w)\,dw,\ \forall w$

Assume σ, μ known

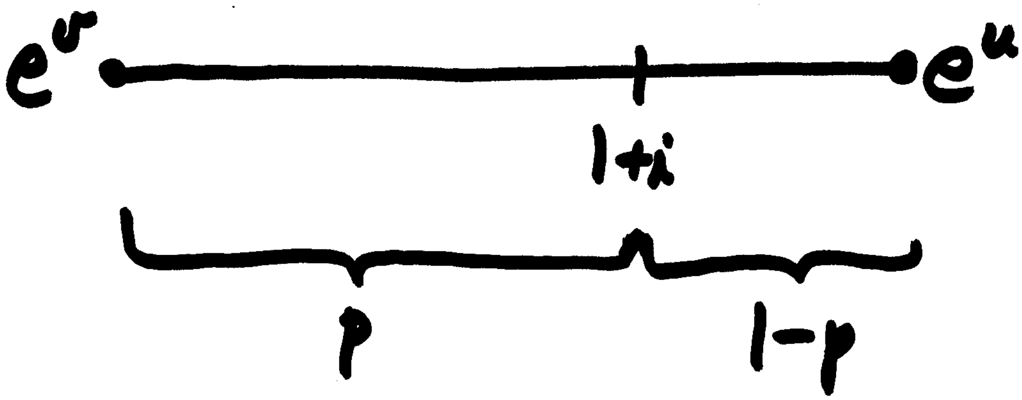
I. Find u, v from σ, μ via Central Limit Thm

ln(stock price):



$$(1+i)^{GP} = e^{r}; \quad r \text{ known}$$

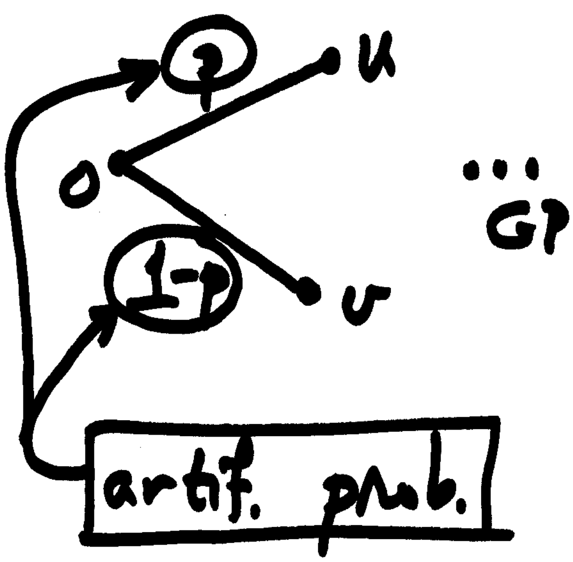
$$i = e^{r/GP} - 1$$



$p =$ the artificial probability

$$= \frac{1+i-e^v}{e^u-e^v}$$

ln(stock price):



end. artif. prob. = NO WAY! $\approx h_v^{\tau}(w)\, dw$, $\forall w$

II. Find τ, ν from u, v, p via Central Limit Thm

K known

$$g(\omega) = \begin{cases} \omega - K & \text{if } \omega \geq K \\ 0 & \text{if } \omega \leq K \end{cases}$$

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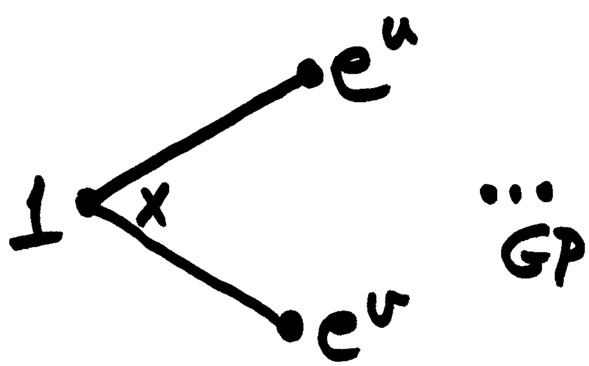
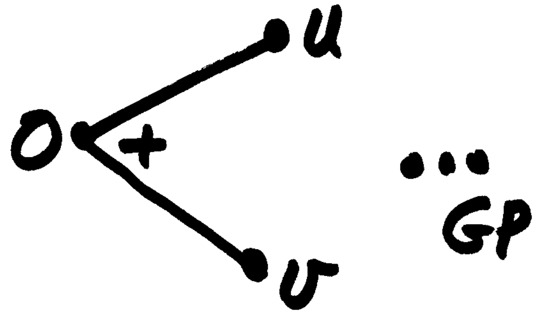
stock
price:



Assume:
if final stock
price $= \omega$
option pays
$g(\omega)$

Purchaser of this option can get 1 share of stock for $\leq \$K$ at end of time period

ln(stock price):

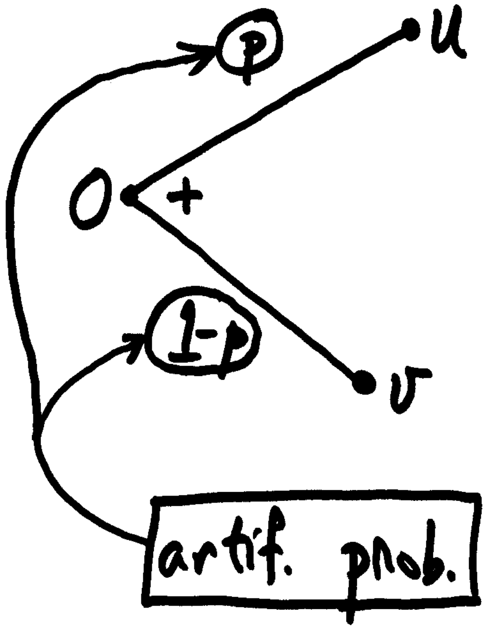


ln(final stock price) $= w$

$\Rightarrow$ option pays $g(e^w)$

$$g(e^w)=\begin{cases} e^w-K & \text{if } w\ge \ln K\\ 0 & \text{if } w\le \ln K\end{cases}$$

ln (stock price):



$\cdots$

GP

if ends at w, pays $g(e^w)$

end. artificial probability = NO WAY! $\approx$ $h_v^\tau(w)\,dw$ $\forall w$

$\sum$ payment $\times$ end. artif. prob.

$$\approx \int_{-\infty}^{\infty} [g(e^w)][h_v^\tau(w)]\,dw$$

Cost of option $\approx$

$$\frac{1}{(1+i)^{GP}} \int_{-\infty}^{\infty} [g(e^w)][h_\nu^\tau(w)]\,dw$$

CALCULUS

III. Find approx. cost of option in terms of σ, μ, B, K

Result is BS!

PART F: BS

I. $\mu = GP\left(\frac{1}{2}u + \frac{1}{2}v\right)$

σ [redacted] $= \sqrt{GP\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(u-v)}$ [redacted]

$$u = \frac{\mu}{GP} + \frac{\sigma}{\sqrt{GP}}$$

$$v = \frac{\mu}{GP} - \frac{\sigma}{\sqrt{GP}}$$

$u_N := \frac{\mu}{N} - \frac{\sigma}{\sqrt{N}}$	$u = u_{GP}$
$v_N := \frac{\mu}{N} - \frac{\sigma}{\sqrt{N}}$	$v = v_{GP}$

$N = 1, 2, 3 \ldots$

Recall $i = e^{r/GP} - 1$

$$i_N := e^{r/N} - 1, \quad N = 1, 2, 3, \ldots$$

$$i = i_{GP}$$

Recall $p = \dfrac{1 + i - e^{v}}{e^{u} - e^{v}}$

$$p_N := \frac{1 + i_N - e^{v_N}}{e^{u_N} - e^{v_N}}, \quad N = 1, 2, 3, \ldots$$

$$p = p_{GP}$$

II. $\nu = GP\,(pu + (1-p)v)$

$$\tau = \sqrt{GP\cdot p(1-p)}\;(u-v)$$

$$\nu_N := \cdots \xrightarrow{N\to\infty} r - \frac{\sigma^2}{2}$$

$$\boxed{\text{CALC \& ALG}}$$

$$\tau_N := \cdots \xrightarrow[N\to\infty]{} \sigma$$

$$\nu = \nu_{GP} \approx r - \frac{\sigma^2}{2}$$

$$\tau = \tau_{GP} \approx \sigma$$

$$\text{III.} \int_{-\infty}^{\infty} [g(e^w)][h_\nu^\tau(w)]\,dw$$

$$\approx \int_{\ln K}^{\infty} [e^w - K]\Big[\underbrace{h^{\sigma}_{r-(\sigma^2/2)}(w)}_{\dfrac{e^{-(w-r+(\sigma^2/2))^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}}\Big]\,dw$$

$$\Phi(x) := \int_{-\infty}^{x} \underbrace{h(w)}_{e^{-w^2/2}/\sqrt{2\pi}}\,dw$$

Calculus

Price

$$\left[\Phi\left(\frac{r-(\ln K)+(\sigma^2/2)}{\sigma}\right)\right]$$

$$-Ke^{-r}\left[\Phi\left(\frac{r-(\ln K)-(\sigma^2/2)}{\sigma}\right)\right]$$

Black-Scholes Formula

<u>Note</u>: $\nexists\, \mu$ in the formula

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in terms of σ, r, K

PART G: DETAILS

$$P_N = \frac{e^{r/N} - e^{(\mu/N)-(\sigma/\sqrt{N})}}{e^{(\mu/N)+(\sigma/\sqrt{N})} - e^{(\mu/N)-(\sigma/\sqrt{N})}}$$

$$\overset{?}{=} \frac{\frac{r}{N} - \frac{\mu}{N} + \frac{\sigma}{\sqrt{N}} - \frac{\sigma^2}{2N} + o_1\left(\frac{1}{N}\right)}{2\frac{\sigma}{\sqrt{N}} + o_2\left(\frac{1}{N}\right)}$$

$$= \frac{1}{2}\left(1 + \frac{r - \mu - (\sigma^2/2)}{\sigma\sqrt{N}}\right) + o_3\left(\frac{1}{\sqrt{N}}\right)$$

$$\tau_N^{\bullet} = \sqrt{N \cdot p_N(1-p_N)}\left(\frac{2\sigma}{\sqrt{N}}\right)^{\bullet}$$

$$= \sqrt{p_N(1-p_N)}\; 2\sigma$$

$$\longrightarrow \sqrt{\frac{1}{2}\cdot\frac{1}{2}}\; 2\sigma = \sigma$$

$$\nu_N = N\left(p_N\left(\frac{\mu}{N}+\frac{\sigma}{\sqrt{N}}\right)+(1-p_N)\left(\frac{\mu}{N}-\frac{\sigma}{\sqrt{N}}\right)\right)$$

$$= N\left(2p_N\frac{\sigma}{\sqrt{N}}+\frac{\mu}{N}-\frac{\sigma}{\sqrt{N}}\right)$$

$$= (2p_N-1)\sigma\sqrt{N} + \mu$$

$$\stackrel{\bullet}{=} r-\mu-\frac{\sigma^2}{2}+\mu \bullet r-\frac{\sigma^2}{2}$$

$+ \longrightarrow$

$$\left[O_3\left(\frac{1}{\sqrt{N}}\right)\right]\sqrt{N}$$

$$e^{r}(\text{Price}) \doteq \int_{\ln K}^{\infty} \left[e^{w}-K\right]\left[h_{\nu}^{\tau}(w)\right]dw$$

$$\overset{\boxed{w \to w+\nu}}{=} \int_{-\nu+\ln K}^{\infty} \left[e^{w+\nu}-K\right]\left[h^{\tau}(w)\right]dw$$

$$\overset{\boxed{w \to \tau w}}{=} \int_{a}^{\infty} \left[e^{\tau w+\nu}-K\right]\left[h(w)\right]dw$$

$$a := \frac{-\nu+\ln K}{\tau}$$

$$\Phi(-x) = \int_{x}^{\infty} h(w)\,dw$$

$$\left[e^{r}(\text{Price}) + K(\Phi(a))\right]e^{-\nu}$$

$$\doteq \int_a^\infty e^{\tau w}\, e^{-w^2/2}\, \frac{dw}{\sqrt{2\pi}}$$

$\boxed{w \to w+\tau}$

$$= \int_{a-\tau}^{\infty} e^{\tau w+\tau^2}\, e^{-(w^2+2\tau w+\tau^2)/2}\, \frac{dw}{\sqrt{2\pi}}$$

$$= e^{\tau^2/2} \int_{a-\tau}^\infty e^{-w^2/2}\, \frac{dw}{\sqrt{2\pi}}$$

$$= e^{\tau^2/2}\left(\Phi(\tau-a)\right)$$

$$e^{r}(\text{Price}) \doteq$$

$$e^{\nu+(\tau^2/2)}\left(\Phi(\tau-a)\right) - K\left(\Phi(-a)\right)$$

$$\nu + \frac{\tau^2}{2} \approx r - \frac{\sigma^2}{2} + \frac{\sigma^2}{2} = r$$

$$-a \approx \frac{r - (\ln K) - (\sigma^2/2)}{\sigma}$$

$$\tau - a \approx \frac{r - (\ln K) + (\sigma^2/2)}{\sigma}$$

Price $\approx$

$$\left[\Phi\left(\frac{r-(\ln K)+(\sigma^2/2)}{\sigma}\right)\right]$$

$$-Ke^{-r}\left[\Phi\left(\frac{r-(\ln K)-(\sigma^2/2)}{\sigma}\right)\right]$$

Black-Scholes!