BS, by Scot Adams

PART A: EASY OPTIONS

Assume:

\$/Euro either $\nearrow$ 5% or $\searrow$ 5% each month

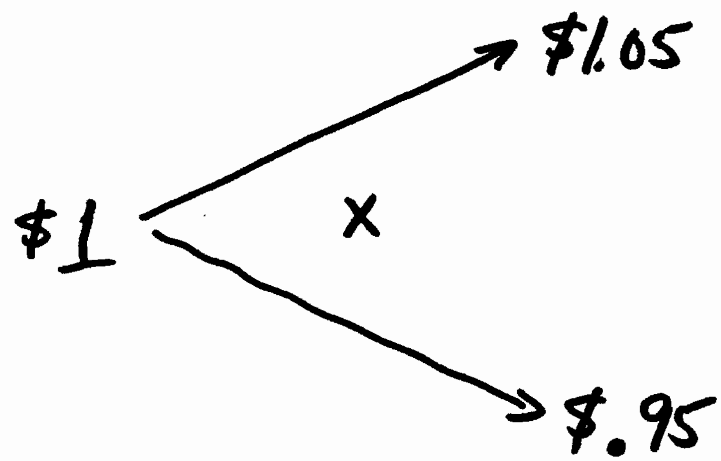
Bank loan rate for 1 month is 1%

$$E \xrightarrow{\times} \begin{cases} 1.05\,E \\ .95\,E \end{cases}$$

$$L \xrightarrow{1.01} 1.01L$$

Dan wants to buy
100 Euros for $100
one month from now

Current: 1 Euro = $ 1.00



Alice charges Dan

$2.97

takes out a loan of

$47.03

and buys 50 Euros for

$50

Profit : $0

One
month
later ...

$\boxed{5}$

Scenario A1: 1 Euro = \$1.05

Alice's profit in \$ is

Dan: $100 - 105 = -5$

Bank: $-(47.03)(1.01) = -47.50$

Euros: $(50)(1.05) = 52.50$

Profit A1 = \$0

Scenario A2: 1 Euro = \$.95

Alice's profit in \$ is

Dan: 0

Bank: $-(47.03)(1.01) = -47.50$

Euros: $(50)(.95) = 47.50$

Profit A2 = \$0

Alice charges Dan

$u

takes out a loan of

$v

buys u+v Euros for

$(u+v)

Profit = $0

Alice's profits in \$:

	<u>Dan</u>	<u>Bank</u>	<u>Euros</u>
A1:	-5	$-\ v(1.01)$	$+\ (u+v)(1.05)$
A2:	0	$-\ v(1.01)$	$+\ (u+v)(.95)$

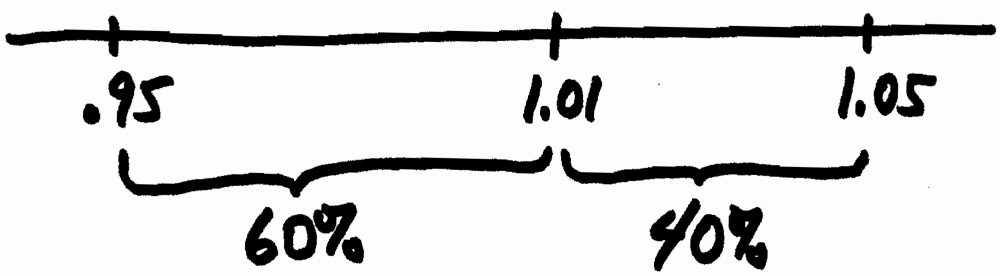
Profit A1 = Profit A2 = 0 $\Longrightarrow$

$$-5 \quad + \quad 0 \quad + (u+v)(.1) = 0$$

$$\Longrightarrow \quad u+v = \frac{5}{.1} = 50$$

(Subtract 2nd eqn from first)

$$p := \frac{1.01 - .95}{1.05 - .95} = .6$$



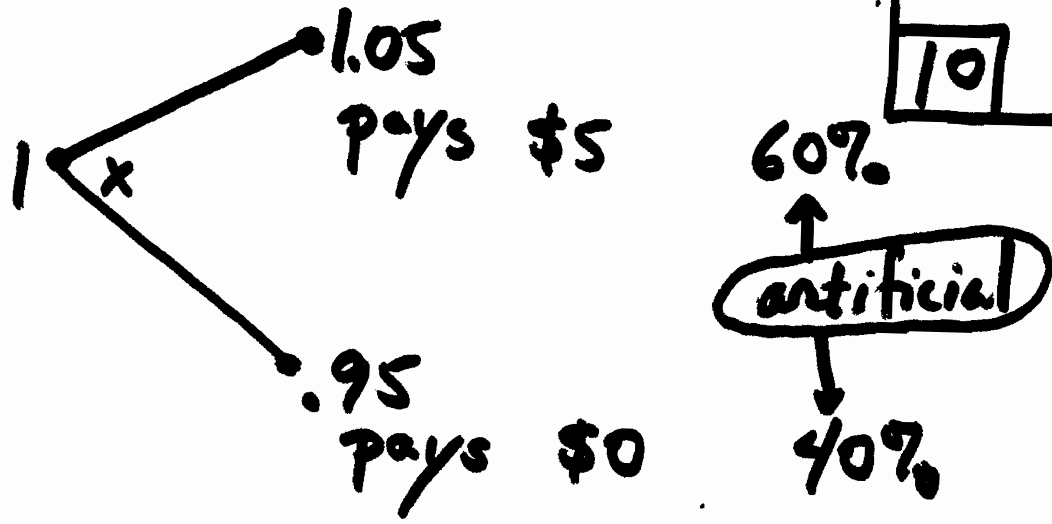
$$
\begin{aligned}
&[\ -5 - v(1.01) + (u+v)(1.05) = 0\ ]\ (.6) \\
+&[\ 0 - v(1.01) + (u+v)(.95) = 0\ ]\ (.4)
\end{aligned}
$$

$$-3 - v(1.01) + (u+v)(1.01) = 0$$

$$-3 + u\,(1.01) = 0$$

$$u = \frac{3}{1.01} = 2.970297$$

$$v = 50 - u = 47.0297$$



$$\sum \text{payment} \times \text{artif. prob}$$

$$= 5(.6) + 0(.4) = 3$$

$$\text{Cost} = \frac{3}{1.01} = 2.970297$$

Earl wants to buy
100 Euros for \$100
one month from now

Current: 1 Euro = \$1.1052631

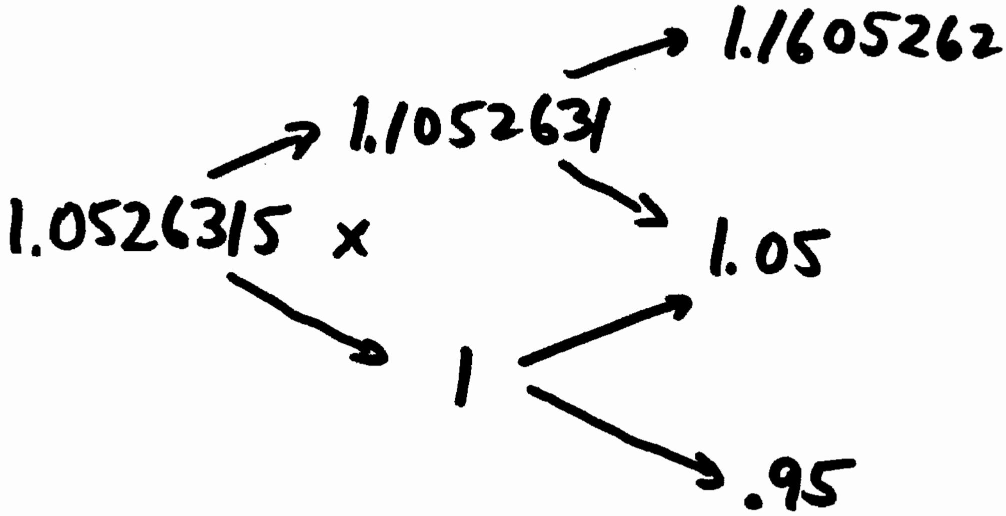
$$\begin{array}{lll} & \nearrow 1.1605262 & \text{pays } \$16.05262 \\ 1.1052631 \; x & & \\ & \searrow 1.05 & \text{pays } \$5 \end{array}$$

$$\text{Cost: } \frac{(16.05262)(.6) + 5(.4)}{1.01} = 11.516406$$

Beth sells this option

Fred wants to buy
100 Euros for \$100
<u>two</u> months from now

Current: 1 Euro = \$1.0526315



Cathy sells this option

$$
\begin{tikzcd}[row sep=small]
 & 1.1052631 \\
1.0526315 \arrow[ur] \arrow[dr] & \\
 & 1
\end{tikzcd}
$$

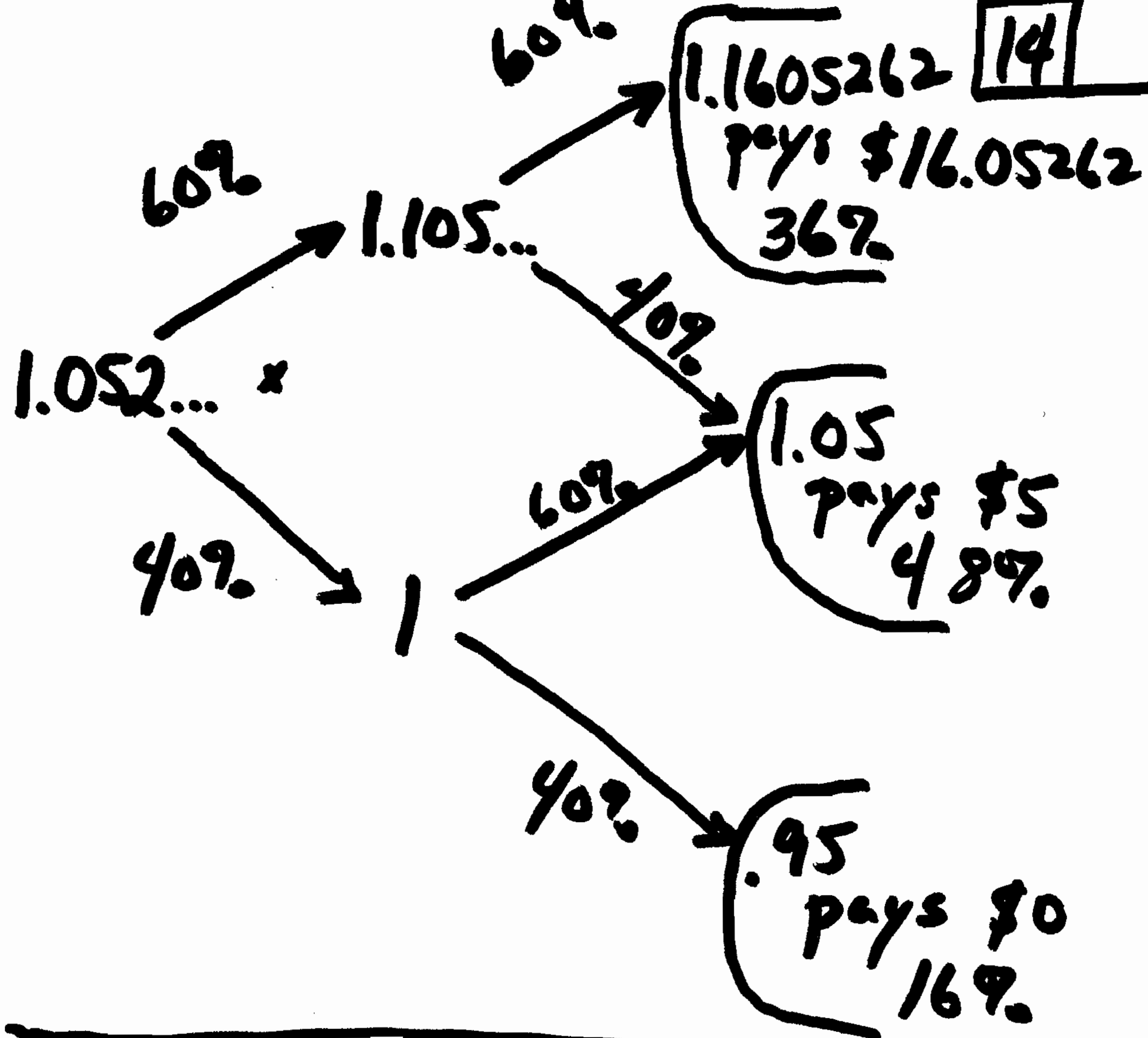
x

1.1052631 pays Beth $11.516406

1 pays Alice $2.970297

$$\text{Cost: } \frac{(11.516406)(.6)+(2.970297)(.4)}{1.01}$$

$$= \$\,8.0177845$$



$$\frac{\sum \text{payment} \times \text{end. artif. prob.}}{(1.01)^2}$$

$$= \$8.0177845$$

H 60%, T 40%

HH pays \$16.05262 (36%)

HT or TH pays \$5 (48%)

TT pays \$0 (16%)

$$\frac{\text{expected payment}}{(1.01)^2}$$

$= \$8.0177845$

PART B: COIN FLIPPING

$$GP := 10^{10^{100}}$$

Q: After GP flips of a fair coin, find prob. that

$$-\sqrt{GP} \leq (\# H) - (\# T) \leq \sqrt{GP}$$

$$\binom{n}{3}\frac{1}{n^3} = \frac{n(n-1)(n-2)}{3!\ n^3} \longrightarrow \frac{1}{3!}$$

$$\binom{n}{k}\frac{1}{n^k} \longrightarrow \frac{1}{k!} \quad \text{as } n \to \infty$$

$$(1-x^2)^n \Big|_{x \to x/\sqrt{n}}$$

$$= 1 - nx^2 + \binom{n}{2}x^4 - \binom{n}{3}x^6 + \cdots \Big|_{x \to x/\sqrt{n}}$$

$$= 1 - n\frac{x^2}{n} + \binom{n}{2}\frac{x^4}{n^2} - \binom{n}{3}\frac{x^6}{n^3} + \cdots$$

$$\longrightarrow 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \cdots$$

$$= e^{-x^2}$$

$$\left(1-x^2+5x^3\right)^n\Big|_{x\to x/\sqrt{n}}$$

$$= \cdots + 5nx^3 + \cdots \Big|_{x\to x/\sqrt{n}}$$

$$= \cdots + 5n\frac{x^3}{n^{3/2}} + \cdots \longrightarrow e^{-x^2}$$

<u>Fact</u>: $f(x) = 1 - x^2 + \cdots$

$$\Rightarrow \left(f(x)\right)^n\Big|_{x\to x/\sqrt{n}} \longrightarrow e^{-x^2}$$

<u>Fact</u>: $f(x) = 1 - \frac{1}{2}x^2 + \cdots$

$$\Rightarrow \left(f(x)\right)^n\Big|_{x\to x/\sqrt{n}} \longrightarrow e^{-x^2/2}$$

e.g. $\cos x = 1 - \frac{1}{2}x^2 + \cdots$

$$\therefore \cos^n x \Big|_{x \to x/\sqrt{n}} \longrightarrow e^{-x^2/2}$$

$$\cos^{GP} x \Big|_{x \to x/\sqrt{GP}} \approx e^{-x^2/2}$$

$X_1 := (\# H) - (\# T)$

after 1 flip (fair coin)

$\frac{1}{2} z^{-1} + \frac{1}{2} z$ | (gen. fn.)

$z \longrightarrow e^{-ix}$, $i = \sqrt{-1}$

$\frac{1}{2} e^{ix} + \frac{1}{2} e^{-ix}$ | (F. transf.)

$= \cos x$

$\exists$ inverse F. transf.

$$X_1/7 = [(\#H)-(\#T)]/7$$

after 1 flip

$\frac{1}{2}$	$\frac{1}{2}$
$-1/7$	$1/7$

$$\frac{1}{2} z^{-1/7} + \frac{1}{2} z^{1/7} \quad \text{(g. fn.)}$$

$$\frac{1}{2} e^{ix/7} + \frac{1}{2} e^{-ix/7} \quad \text{(F. fn.)}$$

$$= \cos(x/7)$$

$$= \cos x \Big|_{x \to x/7}$$

$X_2 := (\#H) - (\#T)$

after 2 flips

$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
-2	0	2

$$\frac{1}{4}z^{-2} + \frac{1}{2} + \frac{1}{4}z^{2} \quad \text{(g. fn.)}$$

$$\|$$

$$\left(\frac{1}{2}z^{-1} + \frac{1}{2}z\right)^2$$

$$\cos^2 x \quad \text{(F. fn.)}$$

$$X_{GP} := (\#H) - (\#T)$$

after GP flips

NO WAY!

$\left(\frac{1}{2}z^{-1} + \frac{1}{2}z\right)^{GP}$	(g. fn.)
$\cos^{GP} x$	(F. tr.)

F. tr. of $X_{GP}/\sqrt{GP}$ is

$$\cos^{GP} x \Big|_{x \longrightarrow x/\sqrt{GP}}$$

$\approx e^{-x^2/2}$ inverse F. tr.?

$$h(w) = e^{-w^2/2} \Big/ \sqrt{2\pi}$$

Calculus: $\int_{-\infty}^{\infty} h(w)\,dw = 1$

Y with distn. h

$h(w)\,dw \qquad \forall w$

w

$\int_{-\infty}^{\infty} z^w\, h(w)\,dw \Big|$ (g. fn.)

$\int_{-\infty}^{\infty} e^{-iwx}\, h(w)\,dw \Big|$ (F. tr.)

$= e^{-x^2/2}$

Calculus e.g., $x=0$

$$\Pr\left[-1 \le \frac{X_{GP}}{\sqrt{GP}} \le 1\right]$$

$$\approx \Pr\left[-1 \le Y \le 1\right]$$

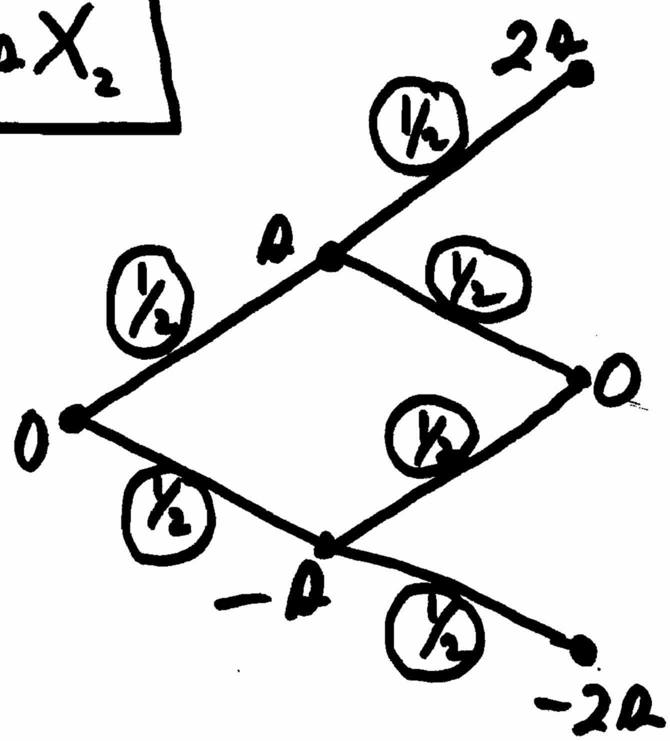
$$= \int_{-1}^{1} h(w)\,dw$$

$$= .6826$$

Calculus

$\Delta = 1/\sqrt{GP}$

ΔX_2



end pr

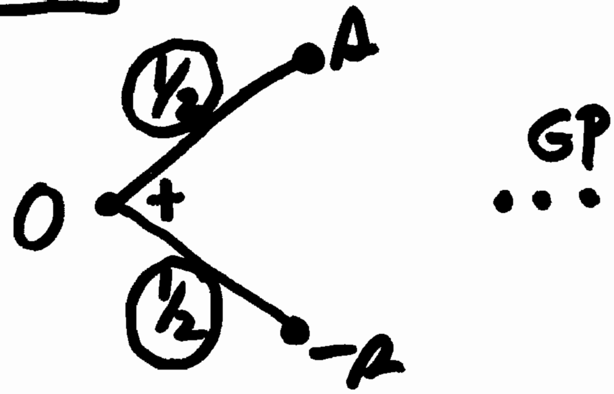
1/4

1/2

1/4

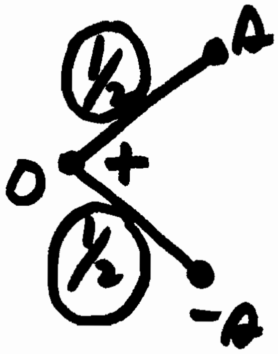
26

ΔX_{GP}



NO WAY!

$\approx h(w)\,dw \quad \forall w$



... GP

if end at w, pay $f(w)$

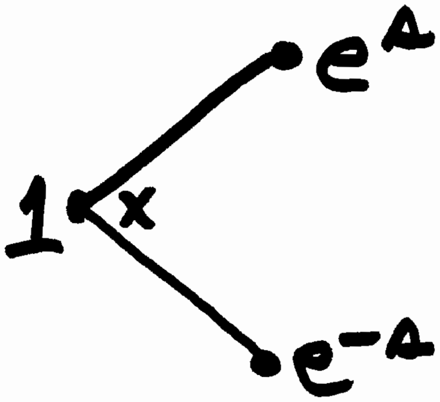
end. prob. = NO WAY! $\approx$ $h(w)\,dw$, $\forall w$

$$E[\text{payment}] = \sum_{w} [f(w)]\left[\begin{matrix}\text{NO}\\ \text{WAY!}\end{matrix}\right]$$

$$\approx \int_{-\infty}^{\infty} [f(w)][h(w)]\,dw$$

PART C: STOCK OPTIONS

stock price:



$\dots GP$

$\Delta = 1/\sqrt{GP}$

$\underbrace{\text{int. } i}$

$1/GP$ year = "split second"

Assume $e^{-\Delta} \;\text{———}\; \underset{1+i}{|} \;\text{———}\; e^{\Delta}$

i.e. artificial prob = 50%

Let $g(\omega) := \begin{cases} \omega - 5, & \text{if } \omega \geq 5 \\ 0, & \text{if } \omega \leq 5. \end{cases}$

Assume: final stock price $= \omega$

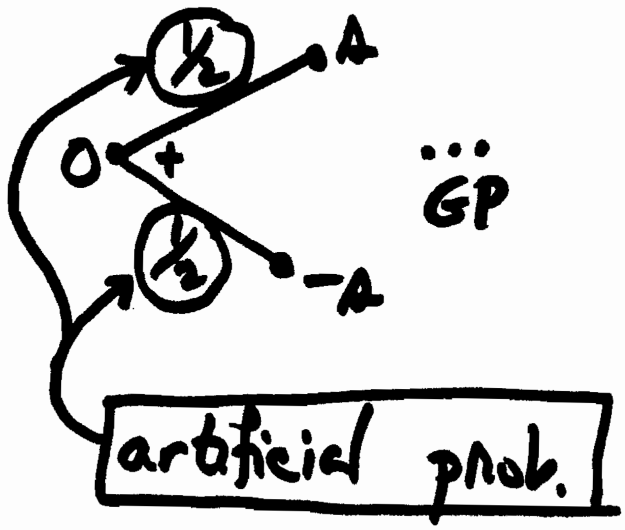
$\Longrightarrow$ option pays $g(\omega)$

Guarantees holder of option can ~~buy~~ buy 1 share of stock for $\leq \$5$

Equiv: $\ln$(final stock price) $= w$

$\Longrightarrow$ option pays $g(e^w)$

ln(stock price):



if end at w, pay $g(e^w)$

end artif. prob $=$ NO WAY! $\approx h(w)\,dw$, $\forall w$

$$\sum \text{payment} \times \text{end. artif. prob}$$

$$= \sum_{w} \left[g(e^w)\right] \left[\begin{matrix}\text{NO} \\ \text{WAY!}\end{matrix}\right]$$

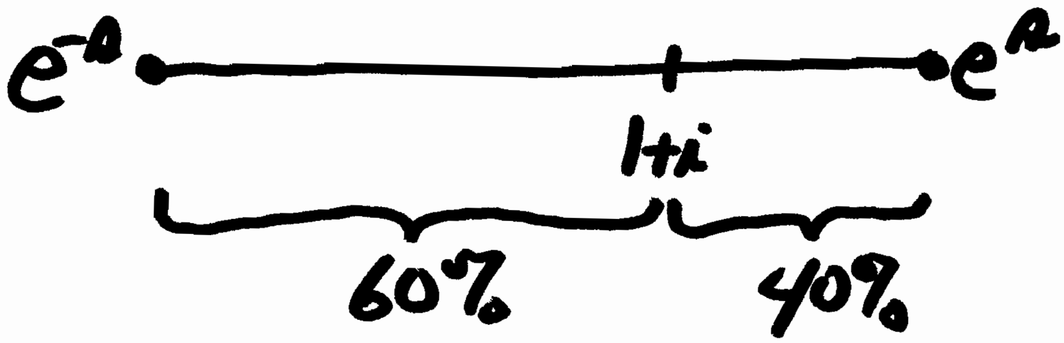
$$\approx \int_{-\infty}^{\infty} \left[g(e^w)\right]\left[h(w)\right] dw$$

Cost of option ≈

$$\frac{1}{(1+i)^{GP}} \int_{-\infty}^{\infty} [g(e^{w})][h(w)]\,dw$$

CALCULUS

Q: What if



?

PART D: CENTRAL LIMIT THM

Def $\forall \sigma > 0$,

$$h^{\sigma}(\omega) := \frac{1}{\sigma}\left[h\left(\frac{\omega}{\sigma}\right)\right]$$

$$\int_{-\infty}^{\infty} h^{\sigma}(\omega)\, d\omega = 1$$

Def $\forall \mu \in \mathbb{R}$, $\forall \sigma > 0$

$$h^{\sigma}_{\mu}(\omega) := h^{\sigma}(\omega - \mu)$$

$$\int_{-\infty}^{\infty} h^{\sigma}_{\mu}(\omega)\, d\omega = 1$$

$$h^{1}_{0}(\omega) = h(\omega)$$

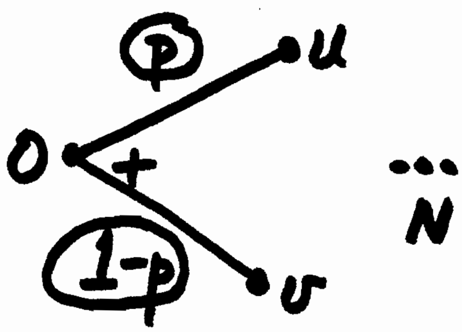
Central Limit Thm

Say $0 \leq p \leq 1$, $u \geq v$, $N \geq 1$, $N \in \mathbb{Z}$

Let $\sigma := \sqrt{N p (1-p)}\,(u-v)$

$$\mu := N(pu + (1-p)v)$$

Then



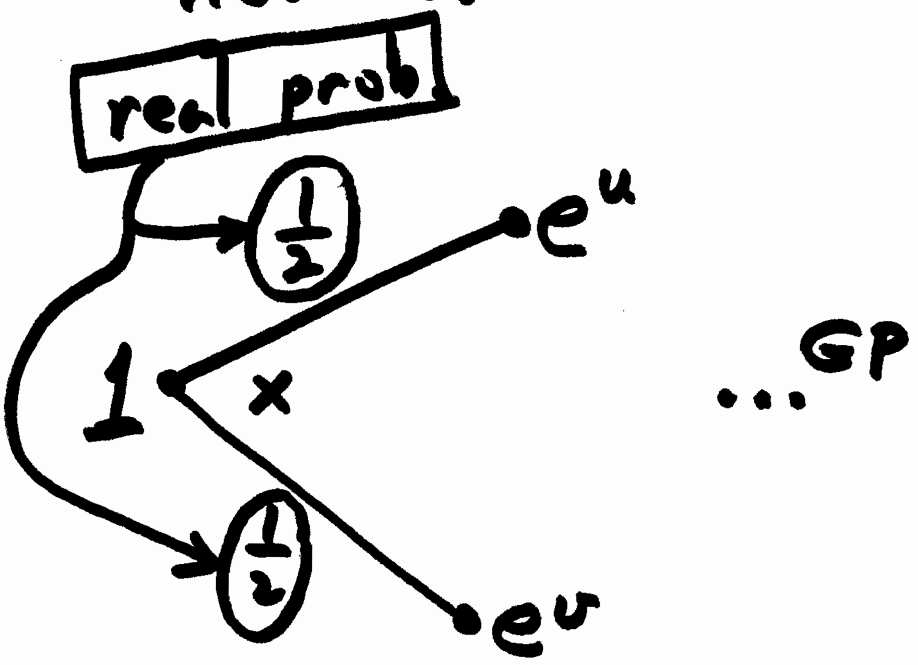
end prob. $\approx$ $h^{\sigma}_{\mu}(w)\,dw$, $\forall w$

e.g. $u = A$, $v = -A$, $p = 1/2$, $N = GP$

$\Rightarrow \sigma = 1$, $\mu = 0 \Longrightarrow h^{\sigma}_{\mu} = h$

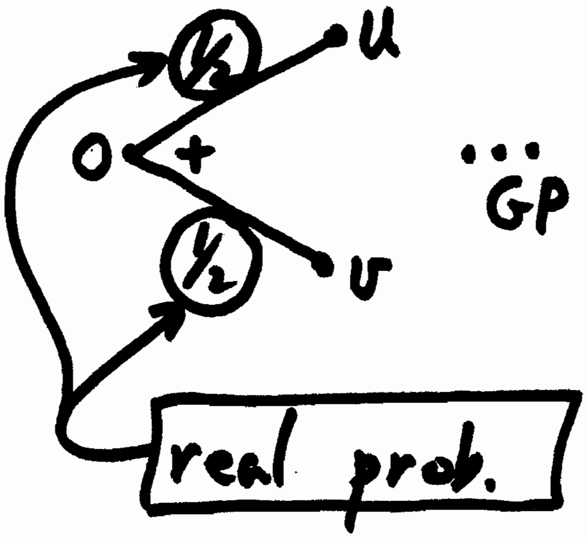
PART E: MORE ON STOCK OPTIONS

new stock:



$u \geq v$, u, v unknown

ln(stock price):



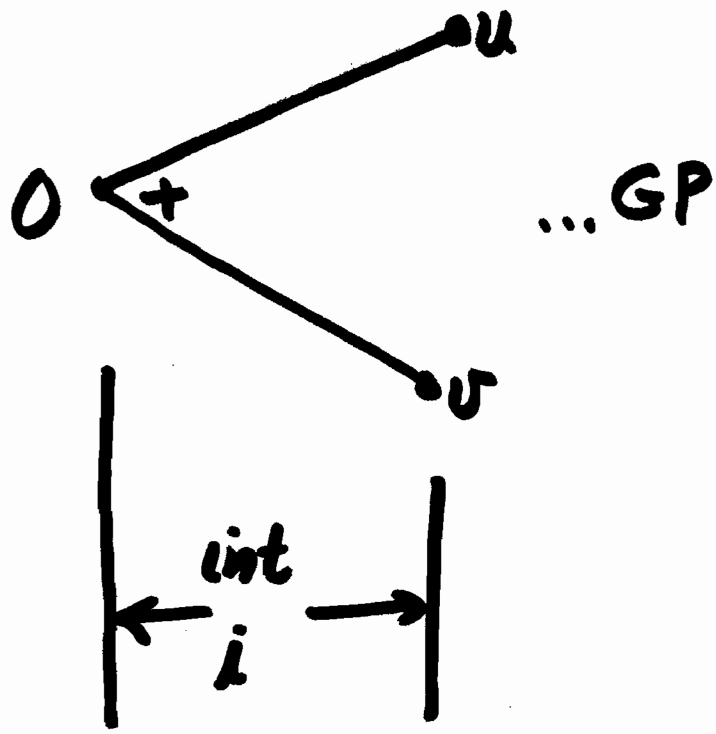
... GP

exact. real prob. = NO WAY! $\approx h_\mu^\sigma(\underline{w})\,d\underline{w},\ \forall \underline{w}$

Assume $\underline{\sigma}, \mu$ known

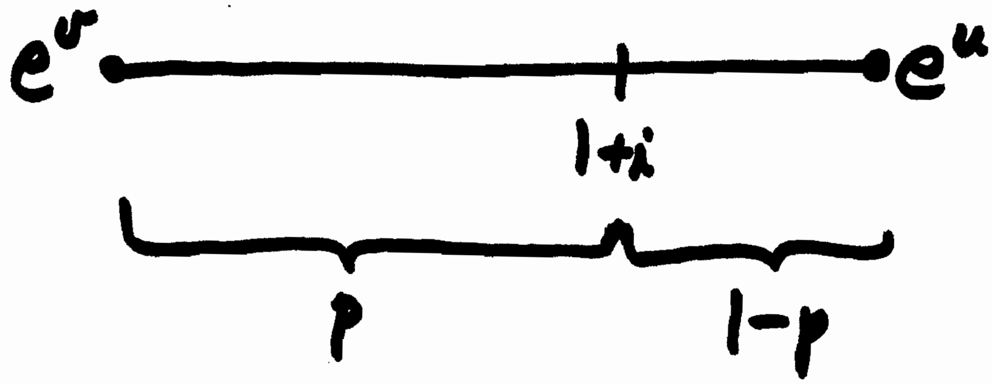
I. Find $\underline{u}, v$ from $\underline{\sigma}, \mu$ via Central Limit Thm

ln(stock price):



$$(1+i)^{GP} = e^{r}, \quad r \text{ known}$$

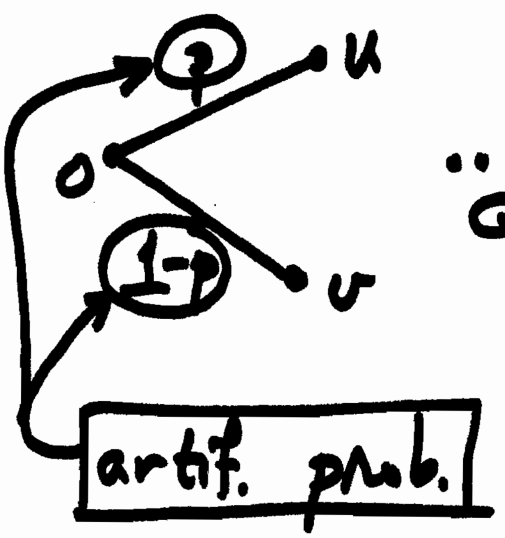
$$i = e^{r/GP} - 1$$



$p =$ the artificial probability

$$= \frac{1+i-e^v}{e^u-e^v}$$

ln(stock price):



$\ddot{\cdot\cdot\cdot}$ GP

end. artif. prob.

= NO WAY!

$\approx h_{\nu}^{\tau}(w)\, dw,$

$\forall w$

II. Find τ, ν from u, v, p via Central Limit Thm

K known

$$g(\omega) = \begin{cases} \omega - K & \text{if } \omega \geq K \\ 0 & \text{if } \omega \leq K \end{cases}$$

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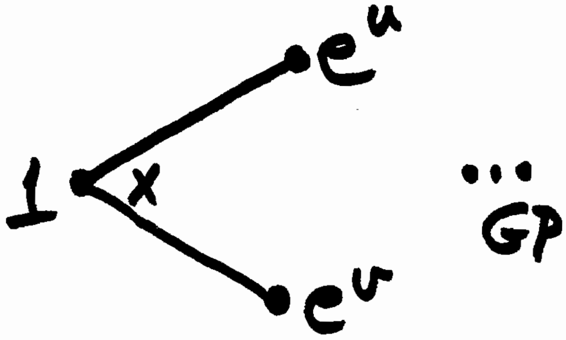
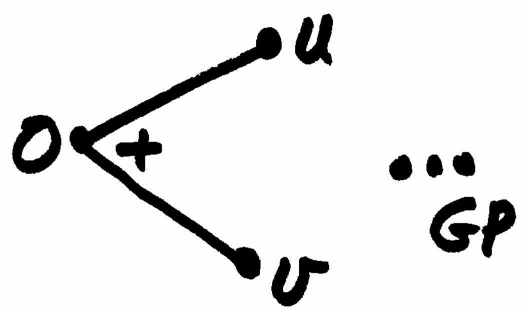
stock price:



Assume: if final stock price $= \omega$ option pays $g(\omega)$

Purchaser of this option can get 1 share of stock for $\leq \$K$ at end of time period

ln(stock price):

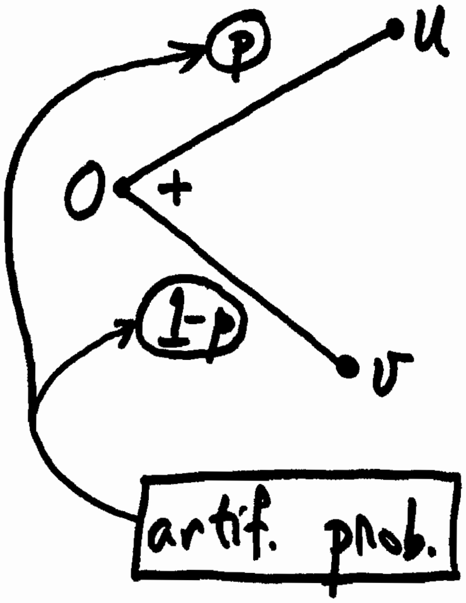


ln(final stock price) $= w$

$\Rightarrow$ option pays $g(e^w)$

$$g(e^w) = \begin{cases} e^w - K & \text{if } w \geq \ln K \\ 0 & \text{if } w \leq \ln K \end{cases}$$

ln (stock price):



... GP

if ends at w, pays $g(e^w)$

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end. artificial probability = NO WAY! $\approx$ $h_v^\tau(w)\, dw$ $\forall w$

$\sum$ payment $\times$ end. artif. prob.

$$\approx \int_{-\infty}^{\infty} [g(e^w)][h_v^\tau(w)]\, dw$$

Cost of option $\approx$

$$\frac{1}{(1+i)^{GP}} \int_{-\infty}^{\infty} [g(e^{w})][h_{\nu}^{\tau}(w)]\,dw$$

CALCULUS

III. Find approx. cost of option in terms of σ, μ, B, K

Result is BS!

PART F: BS

I. $\mu = GP\left(\frac{1}{2}u + \frac{1}{2}v\right)$

$$\sigma = \sqrt{GP\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}(u-v)$$

$$u = \frac{\mu}{GP} + \frac{\sigma}{\sqrt{GP}}$$

$$v = \frac{\mu}{GP} - \frac{\sigma}{\sqrt{GP}}$$

$u_N := \frac{\mu}{N} - \frac{\sigma}{\sqrt{N}}$	$u = u_{GP}$
$v_N := \frac{\mu}{N} - \frac{\sigma}{\sqrt{N}}$	$v = v_{GP}$

$N = 1, 2, 3 \ldots$

Recall $i = e^{r/GP} - 1$

$$i_N := e^{r/N} - 1, \quad N = 1, 2, 3, \ldots$$

$$i = i_{GP}$$

Recall $p = \dfrac{1 + i - e^{v}}{e^{u} - e^{v}}$

$$p_N := \frac{1 + i_N - e^{v_N}}{e^{u_N} - e^{v_N}}, \quad N = 1, 2, 3, \ldots$$

$$p = p_{GP}$$

II. $\nu = GP\,(pu + (1-p)v)$

$$\tau = \sqrt{GP \cdot p(1-p)}\;(u-v)$$

$$\nu_N := \cdots \xrightarrow{N\to\infty} r - \frac{\sigma^2}{2}$$

$$\boxed{\text{CALC \& ALG}}$$

$$\tau_N := \cdots \xrightarrow[N\to\infty]{} \sigma$$

$$\nu = \nu_{GP} \approx r - \frac{\sigma^2}{2}$$

$$\tau = \tau_{GP} \approx \sigma$$

$$\text{III. } \int_{-\infty}^{\infty} [g(e^w)][h_\nu^\tau(w)]\,dw$$

$$\approx \int_{\ln K}^{\infty} [e^w - K]\Big[\underbrace{h^{\sigma}_{r-(\sigma^2/2)}(w)}_{\frac{e^{-(w-r+(\sigma^2/2))^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}}}\Big]\,dw$$

$$\Phi(x) := \int_{-\infty}^{x} \underbrace{h(w)}_{e^{-w^2/2}/\sqrt{2\pi}}\,dw$$

Calculus

Price

$$\left[\Phi\left(\frac{r-(\ln K)+(\sigma^2/2)}{\sigma}\right)\right]$$

$$-Ke^{-r}\left[\Phi\left(\frac{r-(\ln K)-(\sigma^2/2)}{\sigma}\right)\right]$$

Black-Scholes Formula

Note: $\nexists\,\mu$ in the formula

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in terms of σ, r, K

PART G: DETAILS

$$P_N = \frac{e^{r/N} - e^{(\mu/N)-(\sigma/\sqrt{N})}}{e^{(\mu/N)+(\sigma/\sqrt{N})} - e^{(\mu/N)-(\sigma/\sqrt{N})}}$$

$$\overset{?}{=} \frac{\frac{r}{N} - \frac{\mu}{N} + \frac{\sigma}{\sqrt{N}} - \frac{\sigma^2}{2N} + o_1\left(\frac{1}{N}\right)}{2\frac{\sigma}{\sqrt{N}} + o_2\left(\frac{1}{N}\right)}$$

$$= \frac{1}{2}\left(1 + \frac{r - \mu - (\sigma^2/2)}{\sigma\sqrt{N}}\right) + o_3\left(\frac{1}{\sqrt{N}}\right)$$

$$\tau_N = \sqrt{N \cdot p_N(1-p_N)}\left(\frac{2\sigma}{\sqrt{N}}\right)$$

$$= \sqrt{p_N(1-p_N)}\; 2\sigma$$

$$\longrightarrow \sqrt{\frac{1}{2}\cdot\frac{1}{2}}\; 2\sigma = \sigma$$

$$\nu_N = N\left(p_N\left(\frac{\mu}{N}+\frac{\sigma}{\sqrt{N}}\right)+(1-p_N)\left(\frac{\mu}{N}-\frac{\sigma}{\sqrt{N}}\right)\right)$$

$$= N\left(2p_N\frac{\sigma}{\sqrt{N}}+\frac{\mu}{N}-\frac{\sigma}{\sqrt{N}}\right)$$

$$= (2p_N-1)\sigma\sqrt{N} + \mu$$

$$\stackrel{?}{=} r-\mu-\frac{\sigma^2}{2}+\mu \quad r-\frac{\sigma^2}{2}$$

$$+ \longrightarrow$$

$$\left[O_3\left(\frac{1}{\sqrt{N}}\right)\right]\sqrt{N}$$

$$e^{r}(\text{Price}) \doteq \int_{\ln K}^{\infty} \left[e^{w}-K\right]\left[h_{\nu}^{\tau}(w)\right]dw$$

$$\overset{\boxed{w \to w+\nu}}{\doteq} \int_{-\nu+\ln K}^{\infty} \left[e^{w+\nu}-K\right]\left[h^{\tau}(w)\right]dw$$

$$\overset{\boxed{w \to \tau w}}{\doteq} \int_{a}^{\infty} \left[e^{\tau w+\nu}-K\right]\left[h(w)\right]dw$$

$$a := \frac{-\nu+\ln K}{\tau}$$

$$\Phi(-x) = \int_{x}^{\infty} h(w)\,dw$$

$$\left[e^{r}(\text{Price}) + K(\Phi(a))\right] e^{-r}$$

$$\doteq \int_{a}^{\infty} e^{\tau w} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}}$$

$$\boxed{\begin{matrix} w \longrightarrow \\ w+\tau \end{matrix}} \doteq \int_{a-\tau}^{\infty} e^{\tau w + \tau^2} e^{-(w^2 + 2\tau w + \tau^2)/2} \frac{dw}{\sqrt{2\pi}}$$

$$= e^{\tau^2/2} \int_{a-\tau}^{\infty} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}}$$

$$= e^{\tau^2/2} \left(\Phi(\tau - a)\right)$$

$$e^{r}(\text{Price}) \doteq$$

$$e^{\nu+(\tau^2/2)}(\Phi(\tau-a))$$

$$-K(\Phi(-a))$$

$$\nu + \frac{\tau^2}{2} \approx r - \frac{\sigma^2}{2} + \frac{\sigma^2}{2} = r$$

$$-a \approx \frac{r-(\ln K)-(\sigma^2/2)}{\sigma}$$

$$\tau - a \approx \frac{r-(\ln K)+(\sigma^2/2)}{\sigma}$$

Price $\approx$

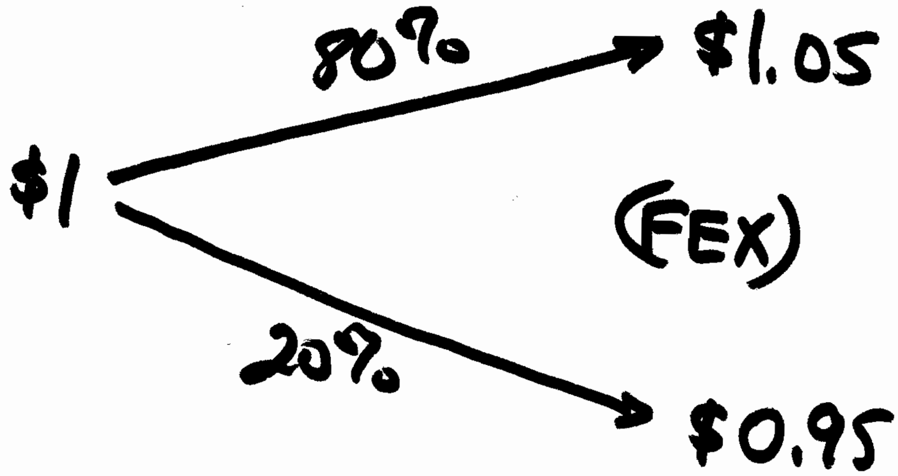
$$\left[\Phi\left(\frac{r-(\ln K)+(\sigma^2/2)}{\sigma}\right)\right]$$

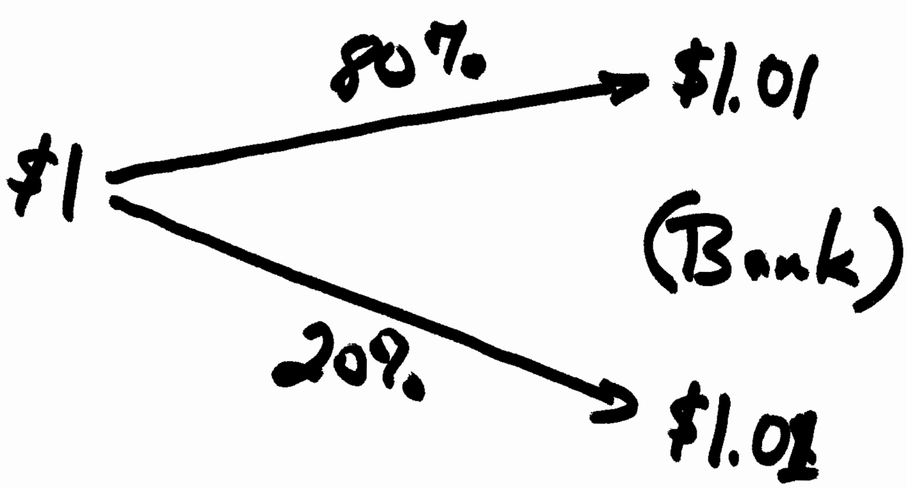
$$-Ke^{-r}\left[\Phi\left(\frac{r-(\ln K)-(\sigma^2/2)}{\sigma}\right)\right]$$

Black - Scholes!

A

Say, in our world





(Expected FEX > Expected Bank)

Want: Pay claim

$5 if up
$0 if down

Know $\exists$ hedging strategy
i.e. $\exists$ FEX/Bank portfolio

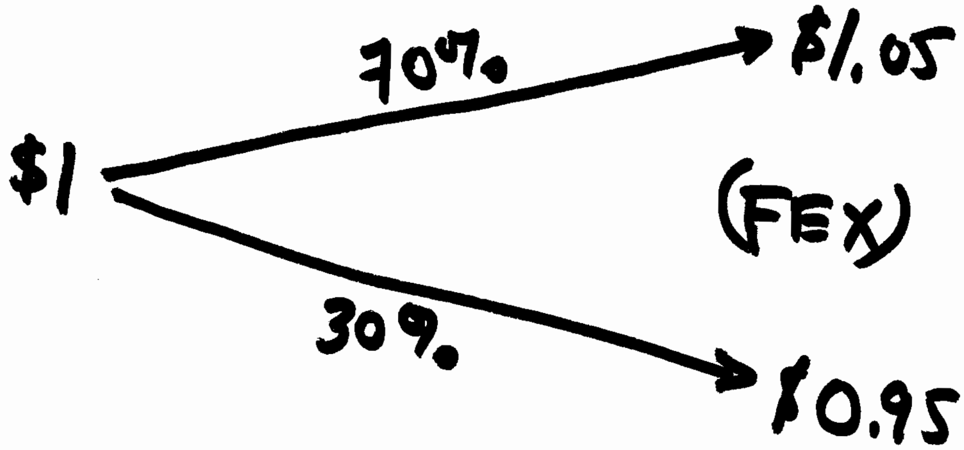
$(50, -47.03)$

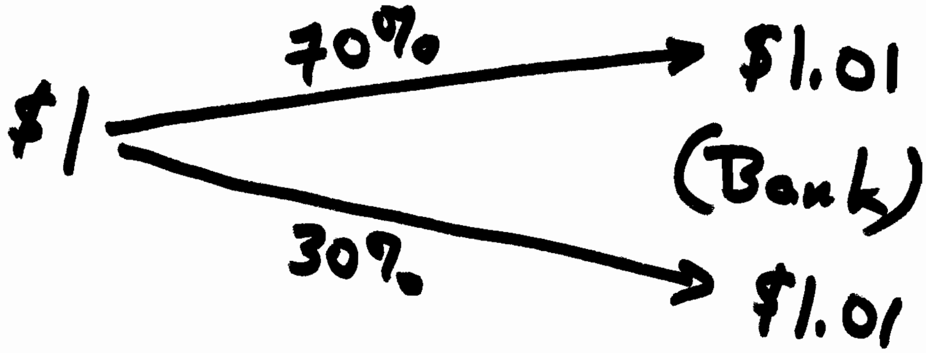
that "replicates" the claim

Suppose we forget these #s

Goal: Portfolio value at
time 0
= cost of option

Another world



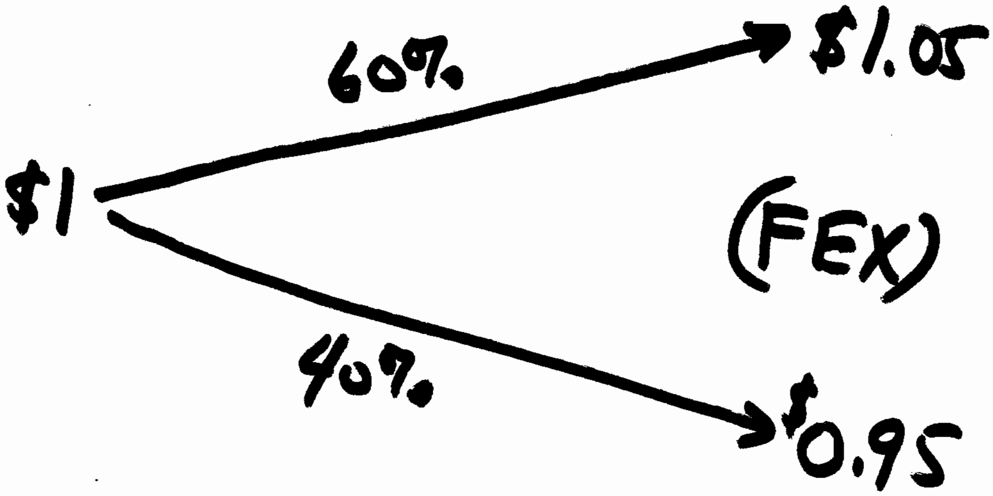


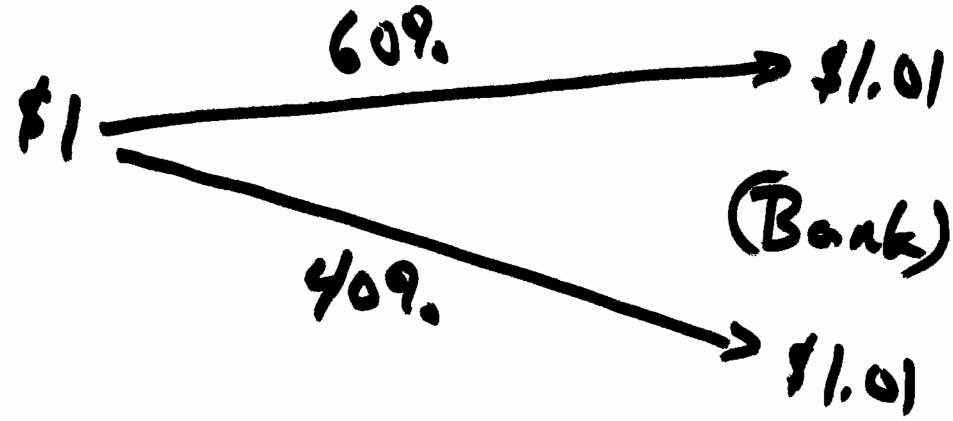
(Expected FEX > Expected Bank)

Hedge still works!

D

Risk-neutral world





(Expected FEX = Expected Bank)

Hedge still works

E

FEX, Bank
$7, −$4 } e.g. of portfolio

Value at $t=0$ is $7-4$ dollars

Value at $t=1$:

up $\Rightarrow$ $(1.05)7-(1.01)4$ dollars

down $\Rightarrow$ $(0.95)7-(1.01)4$ dollars

Expected value at $t=1$ is

$(1.01)7-(1.01)4$ dollars

in risk-neutral world

(Value at $t=0$) $\times$ 1.01

= Expected value at $t=1$

(Risk-neutral world!)

F

Hedge is a portfolio whose value at $t=1$ is

up $\Longrightarrow$ \$5

down $\Longrightarrow$ \$0

<u>Want</u>: Value at $t=0$

(Value at $t=0$) $\times$ 1.01

= Expected value at $t=1$

= (60%) 5 + (40%) 0 dollars

= \$3

G

Price of option =

Value at time 0 =

\$3 / 1.01 = \$2.97

Q: Find hedge!

$$\left.\begin{array}{l}(\text{FEX},\ \text{Bank})\\ (x,\ y)\end{array}\right\}\ \text{replicating portfolio}$$

Replication means

up $\Longrightarrow$ \$5

down $\Longrightarrow$ \$0

Find x, y

H

$$\begin{tikzcd}[row sep=small] & 1.05 \\ 1 \arrow[ur] \arrow[dr] & \\ & .95 \end{tikzcd}$$

FEX diff = .1

$$\begin{tikzcd}[row sep=small] & 1.01 \\ 1 \arrow[ur] \arrow[dr] & \\ & 1.01 \end{tikzcd}$$

Bank diff = 0

$$\begin{tikzcd}[row sep=small] & 5 \\ 2.97 \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd}$$

Optim diff = 5

Optim = x × FEX

$+\ y$ × Bank

$x(.1) + y(0) = 5$ diff

$x + y = 2.97$

I

x usually called Δ

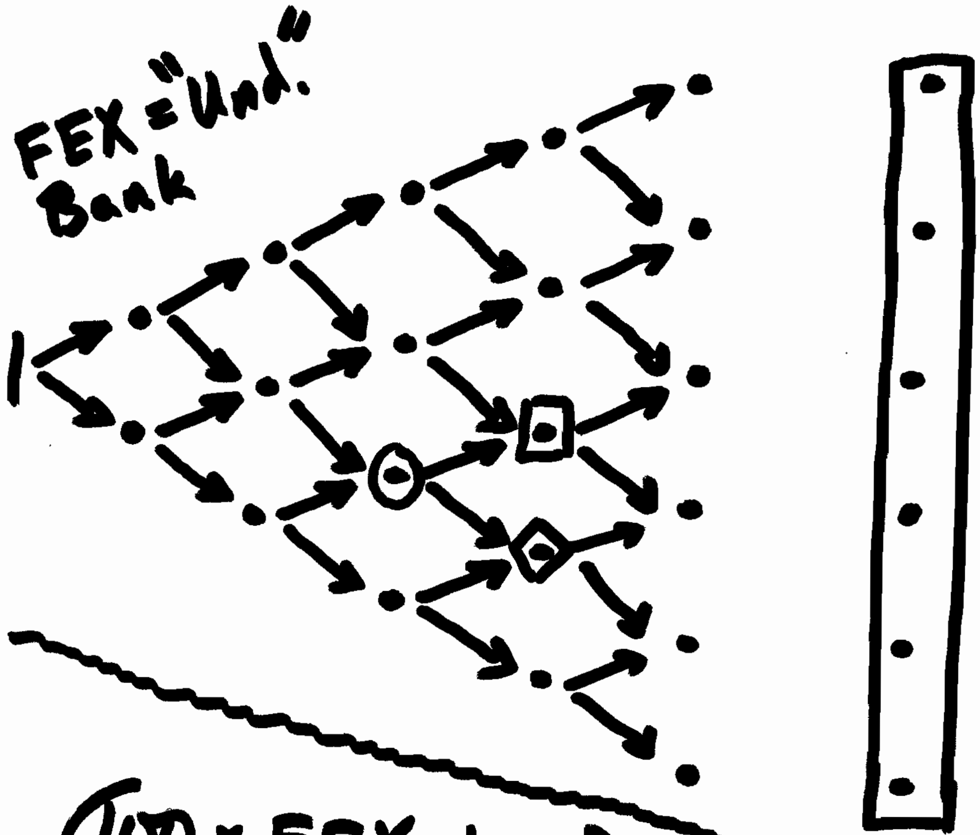
$$\Delta = [redacted] \frac{\text{option diff}}{\text{FEX diff}}$$

= change in option value

per

change in FEX value

(the "underlying")



$$\left(100 \times \text{FEX at end} - 100\right)^+ = \text{Claim}$$

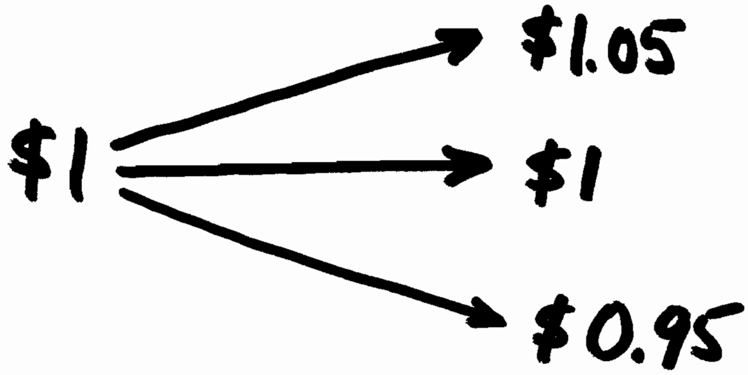
$\exists$ portfolio that replicates the claim

Δ = amt in "Und." at $\odot$

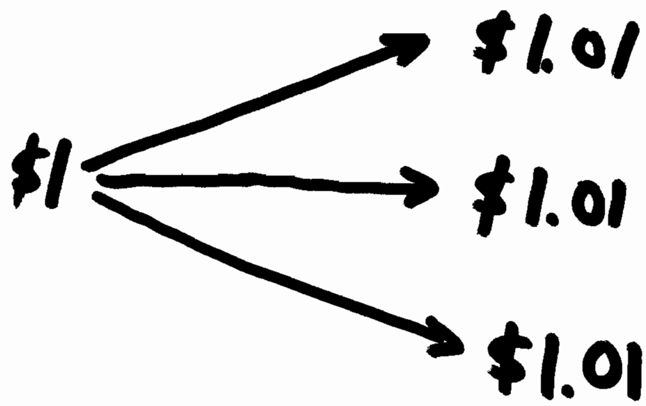
$$\Delta = \frac{(\text{Opt. at } \boxdot) - (\text{Opt at } \diamond)}{(\text{Und. at } \boxdot) - (\text{Und at } \diamond)}$$

"Option pricers got hedge"

K



FEX



Bank

$\exists$ unhedgeable claims!

INCOMPLETE

MARKET

L

$$\begin{tikzcd}[row sep=small] & \$1.05 & \\ \$1 \arrow[ur] \arrow[dr] & & \text{FEX} \\ & \$0.95 & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & \$1.01 & \\ \$1 \arrow[ur] \arrow[dr] & & \text{Bank} \\ & \$1.01 & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & \$5 \\ \$4 \arrow[ur] \arrow[dr] & \\ & \$0 \end{tikzcd}$$

$\exists$ arbitrage !

MARKET NOT ARBITRAGE-FREE

Ⓜ

$1 → $1.05
FOX
$1 → $0.95

$1 → $1.01
Bank
$1 → $1.01

COMPLETE; ASSUME
ARBITRAGE - FREE

Then

$? → $1
Arrow-Debreux
$? → $0

$$\frac{.6}{1.01}$$

---

and

$$\$\,\frac{.4}{1.01}$$ → $0
Arrow-Debreux
$$\$\,\frac{.4}{1.01}$$ → $1

A0

[redacted]

Q: Compute e.g.

$$\int_{-\infty}^{\infty} (e^{w}-7)^{+} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}}$$

Q: Compute

$$I := \int_a^b e^{cw+d-w^2/2}\, dw$$

$$I = \int_{a-c}^{b-c} e^{c^2/2 + d - w^2/2}\, dw$$

$$= e^{c^2/2+d}\left(\Phi(b-c) - \Phi(a-c)\right)$$

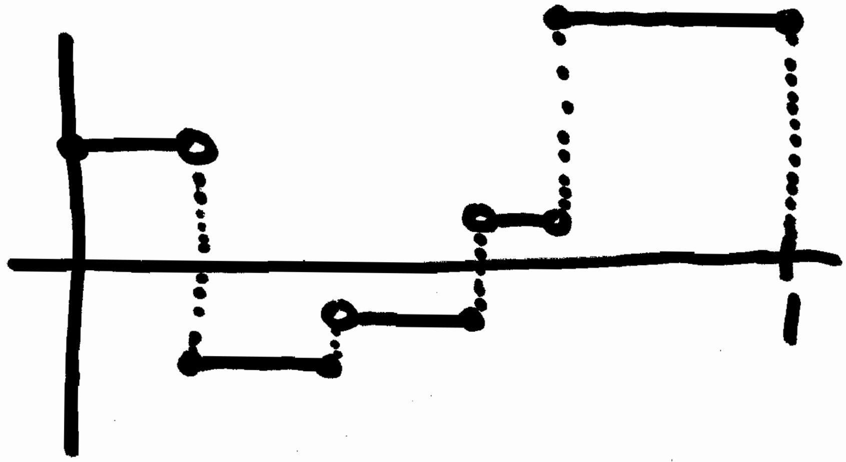
A1

$\Omega := [0,1]$

A $\boxed{\text{simple random variable}}$ or $\boxed{\text{SRV}}$ is a piecewise constant fn $\Omega \longrightarrow \mathbb{R}$

---

e.g.



A2

$\forall$ SRV $X : \Omega \to \mathbb{R}$

$\forall f : \mathbb{R} \to \mathbb{R}$,

$\boxed{f(X)} := f \circ X : \Omega \to \mathbb{R}$

is a [redacted] SRV

$\boxed{|I|} := \sup I - \inf I$

$\forall$ intvls I

$\boxed{|I_1 \cup \dots \cup I_n|} := \sum |I_j|$

if $I_1, \dots, I_n$ pw-dj intvls

A3

$\boxed{\Pr[X=x]}$:=

$|\{\omega \ni: X(\omega)=x\}|$

$\boxed{E[X]}$:=

$\sum x \cdot \Pr[X=x]$

$\mathcal{F} = \{I_1, \ldots, I_n\}$ is

an $\boxed{\text{intvl-}\cup\text{ partition}}$

if $I_1, \ldots, I_n$ pw dj $\overbrace{\text{intvl}\overline{\text{s}}}^{\cup \text{ intvls}}$ $^{\omega \Omega}$

& $I_1 \cup \cdots \cup I_n = \Omega$

A4

$$\boxed{\chi_A} := \begin{cases} 1 & \text{on } A \\ 0 & \text{on } \Omega \setminus A \end{cases}$$

$$\boxed{E[X|\mathcal{F}]} : \Omega \longrightarrow \mathbb{R}$$

defd by : $\forall I \in \mathcal{F}$, $\forall x \in I$,

$$\big(E[X|\mathcal{F}]\big)(x) = E[X\chi_I] / |I|$$

X is $\boxed{\mathcal{F}\text{-msble}}$ if

$\forall I \in \mathcal{F}$, X const on I

A4a

$$E[X+Y \mid \mathcal{F}] =$$
$$E[X \mid \mathcal{F}] + E[Y \mid \mathcal{F}]$$

X $\mathcal{F}$-msbl $\Longrightarrow$
$$E[XY \mid \mathcal{F}] = X \cdot E[Y \mid \mathcal{F}]$$

X $\mathcal{F}$-msbl $\Longleftrightarrow$
$$E[X \mid \mathcal{F}] = X$$

A4b

$\forall$SRV X

$$\boxed{\mathcal{F}(X)} := \{X^{-1}(x) \ni: x \in \mathbb{R}\}$$

$$\boxed{\mathcal{F}(X_1,\dots,X_N)} :=$$

OK

$$\{I_1 \cap \cdots \cap I_N \ni: I_1 \in \mathcal{F}(X_1), \dots, I_N \in \mathcal{F}(X_N)\}$$

$$\boxed{E[Y|X_1,\dots,X_N]} :=$$

$$E[Y|\mathcal{F}(X_1,\dots,X_N)]$$

$\boxed{\Pr[X=x \ \& \ Y=y]}$ $\boxed{A5}$

$$\parallel$$

$$\left|\{\omega \ni: X(\omega)=x \ \& \ Y(\omega)=y\}\right|$$

X and Y are $\boxed{\text{independent}}$, write $\boxed{X \perp Y}$ if $\forall x, y \in \mathbb{R}$

$$\Pr[X=x \ \& \ Y=y] = \left(\Pr[X=x]\right)\left(\Pr[Y=y]\right)$$

[redacted] A6

$GP = $ [redacted] $10^{10^{100}}$

$GPP := 10^{GP}$

$GPPP := 10^{GPP}$

$G\text{-}4\text{-}P := 10^{GPPP}$

$\vdots$

$BP := G\text{-}GP\text{-}P$

$ds = dt = du = dv = 1/BP$

$I := \{0, dt, 2dt, \ldots, GP\}$

$\text{size} = BP \times GP + 1$

$I^{\times} := I \setminus \{GP\}$

[redacted] | A7

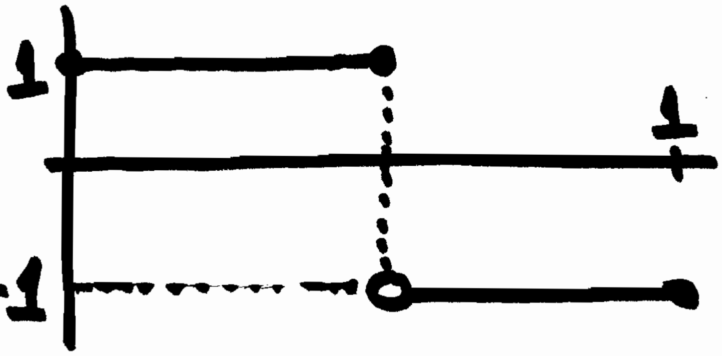
A $\boxed{\text{simple process}}$ on SP is an I-indexed seq $X_0, X_{dt}, \ldots, X_{GP}$ of SRVs

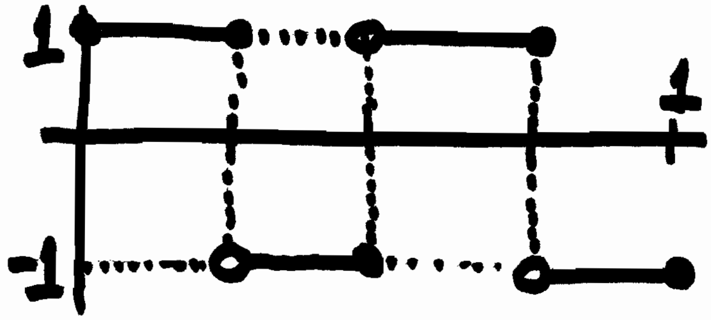
---

A $\boxed{\text{differential simple process}}$ or $\boxed{\text{dSP}}$ is an I^*-indexed seq. of SRVs

---

AP

$CF_0 =$ 

$CF_{dt} =$ 

etc. to CF_{GP-dt} (dSP)

$\forall t \in I^{\times}, \qquad CF_t{}^2 = 1$

$\forall t \in I, \quad \boxed{W_t} := \dfrac{CF_0 + \cdots + CF_{t-dt}}{\sqrt{BP}}$

<u>Q</u>: What "is" W_1?

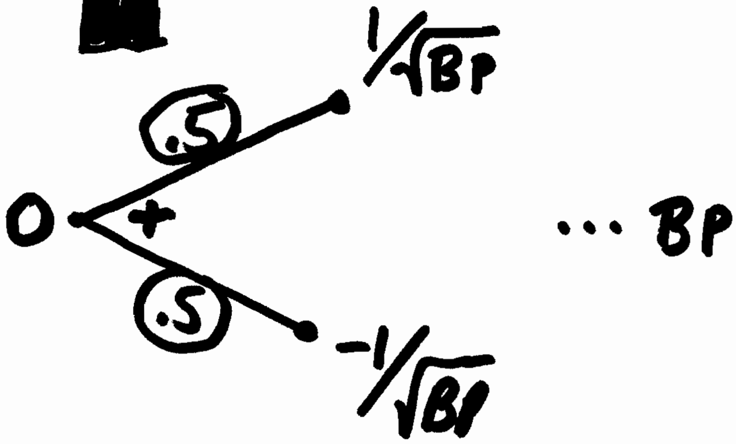
[redacted] | A8a

$\forall$ "reasonable" f

compute $E[f(W_1)]$

$$= \sum_{w} [f(w)] \underbrace{Pr[W_1 = w]}_{\text{NO WAY!}}$$

[redacted]



NO WAY! $\approx h(w)\,dw \quad \forall w$

$$h(w) = e^{-w^2/2} / \sqrt{2\pi}$$

[redacted] A86

$$E[f(W_1)] =$$

$$\sum_{w} [f(w)]\,[\text{NO WAY!}]$$

$$\approx \int_{-\infty}^{\infty} [f(w)]\,[h(w)]\,dw$$

Def $p : \mathbb{R} \longrightarrow \mathbb{R}$ is an $\boxed{\text{approx. PDF}}$ for a SRV X if $\forall$ reasonable f,

$$E[f(X)] \approx \int_{-\infty}^{\infty} [f(x)]\,[p(x)]\,dx$$

H8c

Approx PDF for W_2?

$$W_2 = \underbrace{\int_0^2}_{\sum_{\substack{0 \le s < 2 \\ s \in I}}} \underbrace{dW_s}$$

mean $0 \cdot ds$
var $ds \rightarrow E[dW_s^2] = ds$
i.i.d.

<u>CLT</u>: Approx., W_2 is normal w/ mean $\int_0^2 0\, ds = 0$

var $\int_0^2 ds = 2$

Approx PDF is $f^{\sqrt{2}}(w) = \frac{1}{\sqrt{2}} f\left(\frac{w}{\sqrt{2}}\right)$

A9

[redacted] [redacted]

$\forall$SP (X_t)

the dSP $\boxed{(dX_t)}$

is defined by

$$dX_t = X_{t+dt} - X_t$$

$$\forall t \in I^x$$

---

e.g. $dW_t = CF_t \sqrt{dt}$

$$\therefore dW_t^2 = dt$$

A10

(X_t) is a

$\boxed{\text{Brownian motion}}$ or $\boxed{BM}$

if $\forall s, t \in I^x, s \neq t,$

$$dX_s \perp dX_t$$

& $\forall t \in I^x,$

$$E[dX_t] = 0 \text{ \&}$$

$$E[dX_t^2] = dt$$

"sum of iid w/ mean 0 & var. dt"

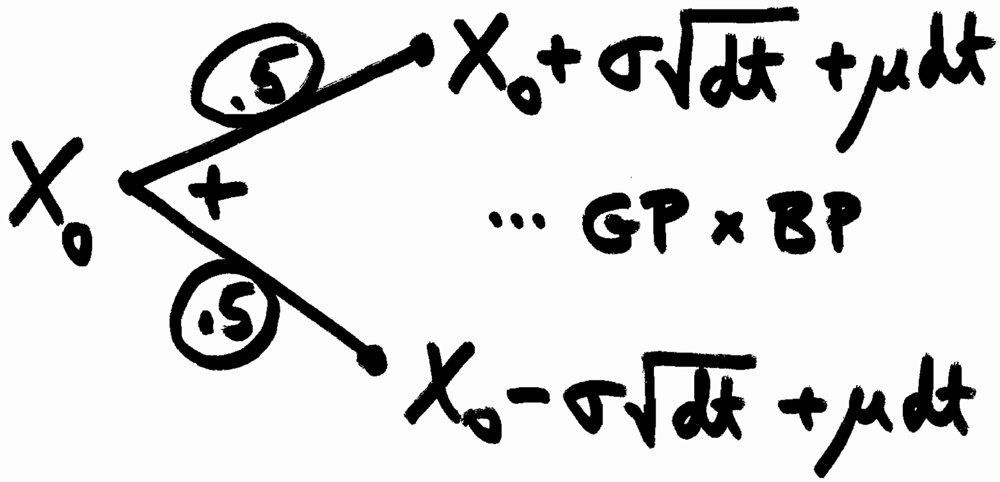
---

e.g. (W_t)

A10a

$$dX_t = \sigma\, dW_t + \mu\, dt$$

assoc. to:



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dX_t has mean $\mu\, dt$

var. $\sigma^2 dt$

B.M. iff $\mu = 0$, $\sigma = 1$

$\boxed{\text{Approx B.M.}}$ if $\mu = 0$

$\sigma \approx 1$

AIII

$\forall SP\ (\sigma_t)(\mu_t),$

$\forall a, b \in I,\quad a \le b,$

$$\boxed{\int_a^b \sigma_s\, dW_s + \mu_s\, ds} :=$$

$$\sum_{\substack{a \le s < b \\ s \in I}} \sigma_s\, dW_s + \mu_s\, ds$$

Same w/ $s \longrightarrow t$, u OR v

$ds = dt = du = dv = 1/BP$

[redacted]	A12
[redacted]	[redacted]

$$dX_t = \sigma_t \, dW_t + \mu_t \, dt$$

$$\forall t \in I^{[\text{redacted}]X} \Longrightarrow$$

$$X_t = X_0 + \int_0^t \sigma_s \, dW_s + \mu_s \, ds$$

$$\forall t \in I$$

$$dX_t \approx \sigma_t \, dW_t + \mu_t \, dt \Longrightarrow$$

$$X_t \approx X_0 + \int_0^t \sigma_s \, dW + \mu_s \, ds$$

but error magnified by

[redacted] [redacted] $t \cdot BP \leq GP \cdot BP$

e.g. $f(x) = x^2$

$$X_t := f(W_t), \quad \forall t \in I$$

$$dX_t = ?$$

$$f(W_{t+dt}) - f(W_t) =$$

$$f(W_t + dW_t) - f(W_t) =$$

$$f'(W_t)\, dW_t + \frac{1}{2!} f''(W_t)\, dW_t^2 =$$

$$2W_t\, dW_t + \frac{1}{2!}\, 2 \;\; dt$$

$$dX_t = 2W_t\, dW_t + dt$$

A141

<u>e.g.</u> $f(x) = x^3$

$$X_t = f(W_t)$$

$$dX_t = 3W_t^2\, dW_t$$
$$+ 6W_t\, dt \;/ 2!$$
$$+ 6\; dt \cdot dW_t \;/ 3!$$

Last term negligible
even after summation

$$dX_t \approx 3W_t^2\, dW_t + 6W_t\, dt$$

A15

e.g. $f(x) = x^4$

$$X_t = f(W_t)$$

$$\begin{aligned} dX_t &= 4W_t^3 \, dW_t \\ &\quad + 12 W_t^2 \, dt \,/\, 2! \\ &\quad + 24 W_t \, dt \cdot dW_t \,/\, 3! \\ &\quad + 24 \, dt^2 / 4! \end{aligned}$$

---

last term summably negligible

second to last also, ~~on~~

~~all but~~ $\cancel{\frac{1}{GP}}$ ~~of~~ $\cancel{\Omega}$.

A76

Ito-1:

$\forall$ reasonable f,

$$X_t = f(W_t)$$

$$\Longrightarrow dX_t \approx f'(W_t)\, dW_t + \frac{1}{2} f''(W_t)\, dt$$

$Y_t = f(X_t)$, $dX_t \approx \sigma_t\, dW + \mu_t\, dt$

$$\Longrightarrow dY_t \approx f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, dX_t^2$$

A17

$\underline{Q}$: Can we "solve"

$$dX_t = 3\,dW_t + 4\,dt$$

$$X_0 = 2 \qquad ?$$

$\underline{Q}$: Can we "solve"

$$\frac{dX_t}{X_t} = 6\,dW_t + 7\,dt$$

$$X_0 = 5 \qquad ?$$

$\underline{Q}$: Meaning of

"solve"?

[redacted] A17

$$dX_t = 3\,dW_t + 4\,dt$$

$$X_0 = 2$$

$$X_t = 2 + \int_0^t dX_s$$

$$= 2 + \int_0^t 3\,dW_s + 4\,ds$$

$$= 2 + 3W_t + 4t$$

~~Q: D_0 w/ "knob"?~~

$$\cancel{W_1 = \frac{CF_0 + \dots + CF_1}{\sqrt{BR}}}$$

[redacted] A20
[redacted] [redacted]

Do we "know" X_2 ?

$$X_2 = 2 + 3W_2 + 4 \cdot 2$$
$$= 10 + 3W_2$$

$$E[f(X_2)] = \sum_x [f(x)] \Pr[X_2 = x]$$

$$= \sum_w [f(10+3w)] \underbrace{\Pr[X_2 = 10+3w]}_{\Pr[W_2 = w]}$$

$$\approx \int_{-\infty}^{\infty} [f(10+3w)] [h^{\sqrt{2}}(w)] \, dw$$

[redacted] A21 [redacted]

$$E[f(X_2)] \approx \int_{-\infty}^{\infty} \left[f(10+3w)\right]\left[h^{\sqrt{2}}(w)\right] dw$$

$$= \int_{-\infty}^{\infty} \left[f(10+w)\right]\left[h^{\sqrt{2}}\left(\frac{w}{3}\right)\right] \frac{dw}{3}$$

$$= \int_{-\infty}^{\infty} \left[f(10+w)\right]\left[h^{3\sqrt{2}}(w)\right] dw$$

$$= \int_{-\infty}^{\infty} \left[f\left(\frac{x}{\text{[redacted]}}\right)\right] \underbrace{\left[h_{10}^{3\sqrt{2}}\left(\frac{x}{\text{[redacted]}}\right)\right]}_{\text{Approx PDF for } X_2} d\frac{x}{\text{[redacted]}}$$

[redacted] A22

Solve $\dfrac{dX_t}{X_t} = 6\,dW_t + 7\,dt$

$$X_0 = 5$$

E.g. Find $E[f(X_3)]$ approx, $\forall$ reasonable f

$$Y_t = \ln X_t \Rightarrow dY_t \approx \frac{dX_t}{X_t} - \frac{1}{2}\left(\frac{dX_t}{X_t}\right)^2$$

$$dY_t \approx 6\,dW_t + 7\,dt - \frac{1}{2}\,36\,dt$$

$$Y_0 = \ln 5$$

[redacted] A23

$$dY_t \approx 6\,dW_t - 11\,dt$$

$$Y_0 = \ln 5$$

$$Y_t \approx \ln 5 + 6W_t - 11t$$

$$X_t \approx 5e^{6W_t - 11t}$$

$$E[f(X_3)] \approx E\left[f\left(5e^{6W_3 - 11\cdot 3}\right)\right]$$

$$\approx \int_{-\infty}^{\infty} \left[f\left(5e^{6w-33}\right)\right]\left[h^{\sqrt{3}}(w)\right]dw$$

A24

[redacted] [redacted]

A $\boxed{\text{c.o.m.}}$ or

$\boxed{\text{change of measure}}$

is a piecewise linear

bijection $\Omega \longrightarrow \Omega$

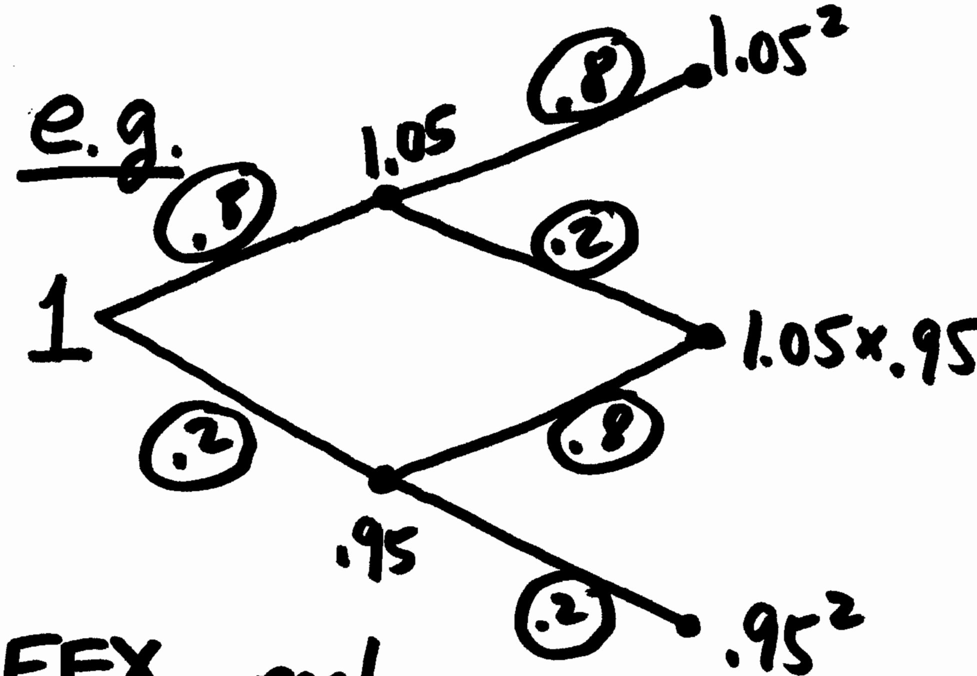
$$\boxed{X^Q} := X \circ Q$$

$\forall$ SRVs X

$\forall$ c.o.m. Q

A25

e.g.



FEX real

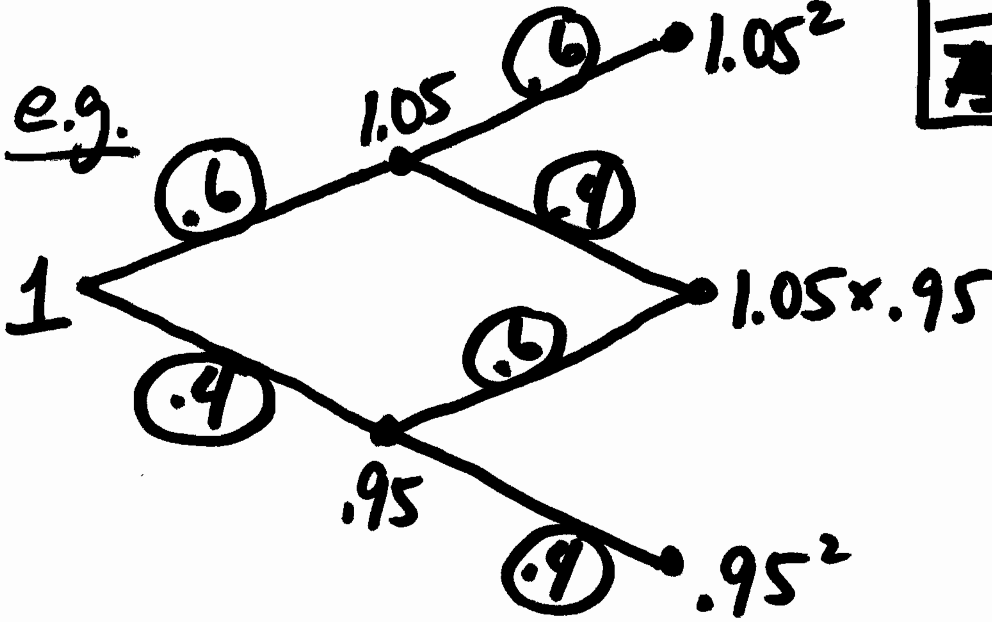
$$E_0 = 1 : \Omega \longrightarrow \mathbb{R}$$

$$E_1 = \begin{cases} 1.05 & \text{on } [0, .8] \\ .95 & \text{on } (.8, 1] \end{cases}$$

$$E_2 = \begin{cases} 1.05^2 & \text{on } [0, .64] \\ 1.05 \times .95 & \text{on } (.64, .8] \\ 1.05 \times .95 & \text{on } (.8, .96] \\ .95^2 & \text{on } (.96, 1] \end{cases}$$

A26

e.g.



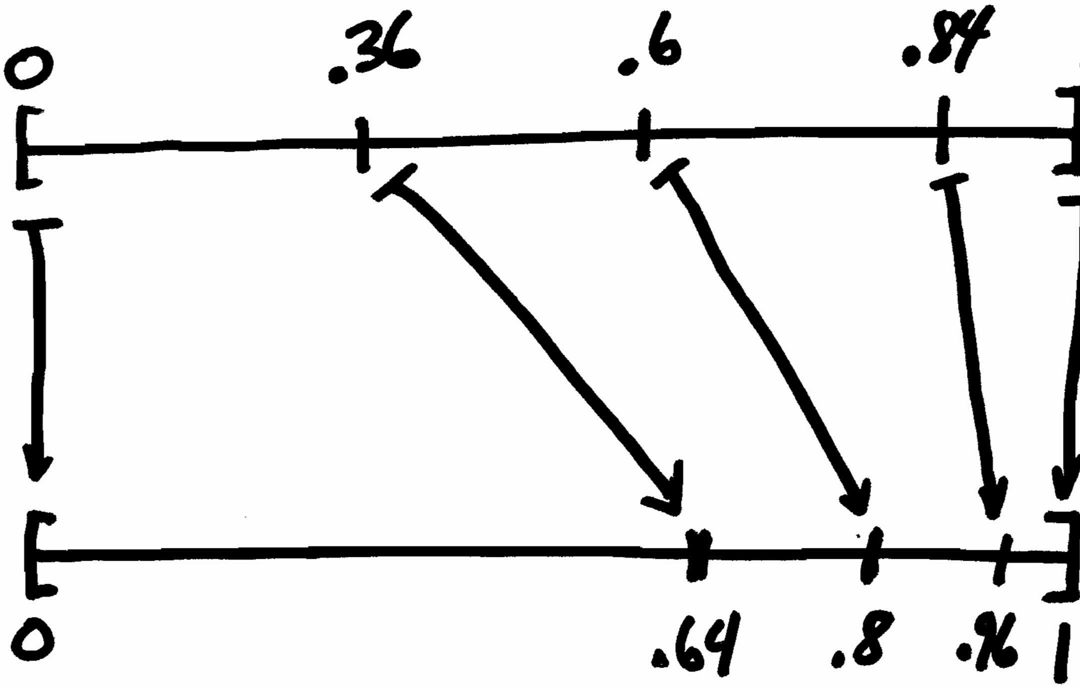
FEX risk-neutral

$$\tilde{\hat{E}}_0 = 1 : \Omega \longrightarrow \mathbb{R}$$

$$\tilde{\hat{E}}_1 = \begin{cases} 1.05 & \text{on } [0, .6] \\ .95 & \text{on } (.6, 1] \end{cases}$$

$$\tilde{\hat{E}}_2 = \begin{cases} 1.05^2 & \text{on } [0, .36] \\ 1.05 \times .95 & \text{on } (.36, .6] \\ 1.05 \times .95 & \text{on } (.6, .84] \\ .95^2 & \text{on } (.84, 1] \end{cases}$$

[redacted] A27



Q piecewise lin.
bij $\Omega \to \Omega$

$$\tilde{E}_t = E_t^Q \quad \forall t \in \{0, 1, 2\}$$

[redacted] A28

<u>Girsanov-1</u>: $|\alpha| < GP$

$$\tilde{W}_0 = 0$$

$$d\tilde{W}_t = dW_t + \alpha\, dt, \quad \forall t \in I^*$$

$\Longrightarrow \exists$ c.o.m. Q

$\ni: (\tilde{W}_t^Q)$ is $\approx$ a B.M.

"can compensate for changing drift by changing measure"

<u>Pf</u>: ...

$\widetilde{W}_t$ given by

$$\begin{tikzcd} & \sqrt{dt} + \alpha\, dt =: U \\ 0 \arrow[ur, ".5"] \arrow[dr, ".5"'] & \cdots\ GP \times BP \\ & -\sqrt{dt} + \alpha\, dt =: V \end{tikzcd}$$

$$V < 0 < U$$

Choose $p \ni: (1-p)V + pU = 0$

Choose c.o.m. $Q \ni:$

$\widetilde{W}_t^Q$ is given by

$$\begin{tikzcd} & U \\ 0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \cdots\ GP \times BP \\ & V \end{tikzcd}$$

$U = \sqrt{dt} + \alpha\, dt$, $V = -\sqrt{dt} + \alpha\, dt$

$d\tilde{W}_t^Q$ is iid w/ mean 0,

$$\text{var} = p U^2 + (1-p) V^2$$

$$= dt + \alpha^2 dt^2 + (p - (1-p))(2\alpha\, dt \sqrt{dt})$$

$$\approx dt$$

$\tilde{W}_t^Q$ is $\simeq$ B.M. QED

[redacted] / A31

$\forall$ int $N \geq 1$

$\mathcal{D}_N := \left\{ \left[0, \frac{1}{N}\right], \left(\frac{1}{N}, \frac{2}{N}\right], \ldots, \left(\frac{N-1}{N}, 1\right] \right\}$

$\forall t \in I$, $\boxed{\mathcal{F}_t} := \mathcal{D}_{2^t} = \mathcal{F}(W_0, \ldots, W_t)$

$\forall t \in I$, W_t is $\mathcal{F}_t$-msbl

X is $\mathcal{F}_t$-msbl iff

$\exists$ fn $f : \mathbb{R}^{t+1} \longrightarrow \mathbb{R}$

$\ni: X = f(W_0, W_1, \ldots, W_t)$

(X_t) is $\boxed{\text{adapted}}$ if

$\forall t \in I$, X_t is $\mathcal{F}_t$-msbl

[redacted] A32

$(S_t), (B_t)$ adapted SPs

$$E := \left((S_t), (B_t) \right)$$

"economy"

"primitive tradables"

A portfolio in E is a pair $(\phi_t), (\psi_t)$ of adapted SPs $\ni$:

$$d(\phi_t S_t + \psi_t B_t) = \phi_t \, dS_t + \psi_t \, dB_t$$

"self-financing"

A33

Fix (S_t), (B_t), $T \in I$

A $\boxed{\text{claim}}$ is an $\mathcal{F}_T$-msbl SRV

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claim = fn of $W_0, \dots, W_T$

---

complete w/ no arbitrage $\Longleftrightarrow$

$\forall$ claim C $\exists!$ portf. (ϕ_t), (ψ_t)

$$\ni: \; C = \phi_T S_T + \psi_T B_T$$

<u>Def</u>: $\boxed{\text{price}}$ of C

$$:= \phi_0 S_0 + \psi_0 B_0$$

Black-Scholes

$$\frac{dS_t}{S_t} = \sigma \, dW_t + \mu \, dt$$

$$\frac{dB_t}{B_t} = r \, dt$$

$$C = (S_T - K)^+$$

σ, r, K given

Goal: Price of C

A35

$$\tilde{W}_0 = 0$$

$$d\tilde{W}_t = dW_t + \frac{r-\mu}{\sigma}\,dt$$

$$\frac{dS_t}{S_t} = \sigma\, d\tilde{W}_t + r\,dt$$

$$\frac{dB_t}{B_t} = r\,dt$$

Choose c.o.m. Q

$\ni$: $\tilde{W}_t^Q$ is B.M.

$\exists$ portf. (ϕ_t), (ψ_t)

$\ni$: $C = \phi_T S_T + \psi_T B_T$

<u>Goal</u>: $\phi_0 S_0 + \psi_0 B_0$

$$V_t = \phi_t S_t + \psi_t B_t$$

$$dV_t = \phi_t \, dS_t + \psi_t \, dB_t$$

<u>Goal</u>: V_0

<u>Know</u>: $V_T = C = (S_T - K)^+$

A37

$\forall$ SRV Z, $\forall t \in I$,

$$\boxed{E_t^Q[Z]} := E\left[Z^Q \mid \tilde{W}_0^Q, \dots, \tilde{W}_t^Q\right]$$

$$E_t^Q[d\tilde{W}_t] = 0, \quad E_t^Q[dt] = dt$$

$$dS_t = \sigma S_t \, d\tilde{W}_t + r S_t \, dt$$

$$E_t^Q[dS_t] = r S_t^Q \, dt$$

$$dB_t = r B_t \, dt$$

$$E_t^Q[dB_t] = r B_t^Q \, dt$$

A38

$$V_t = \phi_t S_t + \psi_t B_t$$

$$dV_t = \phi_t \, dS_t + \psi_t \, dB_t$$

$$E_t^Q[dV_t] = \phi_t r S_t^Q dt + \psi_t r B_t^Q dt$$

$$= r V_t^Q dt$$

$$E[dV_t^Q] = r \cdot E[V_t^Q] dt$$

difference quotient

$$\frac{d}{dt} E[V_t^Q] = r \cdot E[V_t^Q]$$

$$E[V_t^Q] \approx V_0 e^{rt}$$

$\boxed{\text{A39}}$

Cost of option $= V_0 \approx$

$$e^{-RT} \cdot E[V_T^Q] =$$

$$e^{-RT} \cdot E[(S_T^Q - K)_+]$$

$$\frac{dS_t}{S_t} = \sigma\, d\tilde{W}_t + \mu\, dt$$

$$d(\ln S_t) = \sigma\, d\tilde{W}_t + \left(\mu - \frac{\sigma^2}{2}\right) dt$$

$$\ln S_T - \ln S_0 = \sigma \tilde{W}_T + \underbrace{\left(\mu - \frac{\sigma^2}{2}\right)}_{\nu} T$$

$$S_T = S_0\, e^{\sigma \tilde{W}_T + \nu T}$$

A40

$$\text{Cost} \approx e^{-RT} \cdot E\left[(S_T^Q - K)_+\right]$$

$$S_T^Q = S_0 e^{\sigma \tilde{W}_T^Q + \nu T}$$

$(\tilde{W}_t^Q)$ approx BM $\therefore$

approx PDF for $\tilde{W}_T^Q$ is $h^{\sqrt{T}}(w)$

$$\text{Cost} \approx e^{-RT} \int_{-\infty}^{\infty} \left[\quad \right] h^{\sqrt{T}}(w)\, dw$$

$$(S_0 e^{\sigma w + \nu T} - K)_+$$