VARIATIONS ON PRACTICE TEST 1

1-1. Let C be the part of the graph of $y = \ln(\cos x)$ between $x = 0$ and $x = \pi/4$. Find the length of C.

1-2. In xyz-space, let C be the curve with parametric equations $x = 2t$, $y = t^2$ and $z = t^3/3$, $0 \le t \le 1$. Find the length of C.

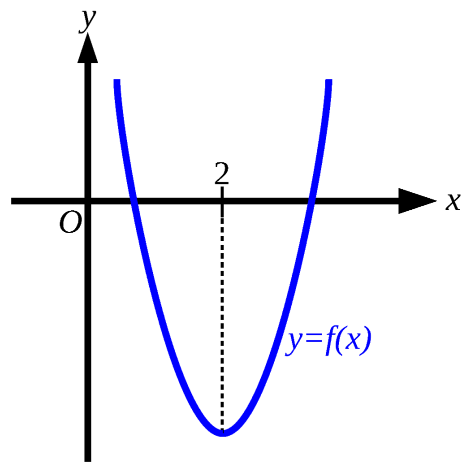
2-1. Give an equation of the line tangent to the graph of $y = 5x + \sin x$ at $x = \pi$.

3-1. If V is a 3-dimensional subspace of $\mathbb{R}^7$ and W is a 5-dimensional subspace of $\mathbb{R}^7$, what are the possible dimensions of $V \cap W$?

4-1. Let k be the number of real solutions of the equation $7 - x^5 - x = 0$ in the interval $[0, 1]$, and let n be the number of real solutions that are not in $[0, 1]$. Which of the following is true?

(A) $k = 0$ and $n = 1$
(B) $k = 1$ and $n = 0$
(C) $k = n = 1$
(D) $k > 1$
(E) $n > 1$

Date: Printout date: November 18, 2015.



5-1. Suppose b is a real number and $f(x) = 4x^2 + bx + 9$ defines a function on the real line, part of which is graphed above. Compute $f(5)$.

6-1. For what values of b does the curve $4x^2 + (y - b)^2 = 1$ have exactly one intersection point with $y = 2x$?

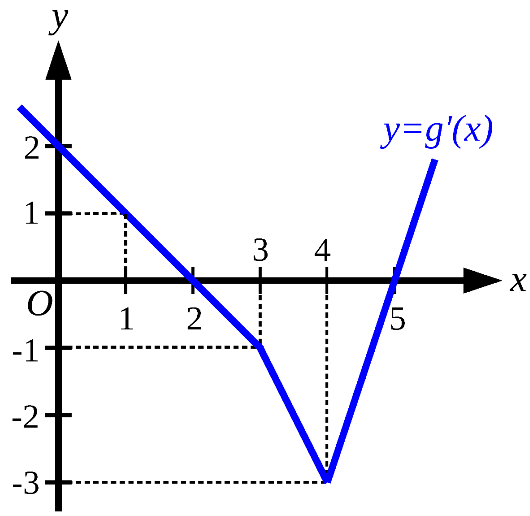
7-1. Compute $\displaystyle\int_{-3}^{3} e^{|x+1|}\,dx$.

8-1. Let R be a rectangle whose vertices are (x, y), $(-x, y)$, $(-x, 0)$ and $(x, 0)$. Assume that $0 < x < 3$, that $0 < y < 3$ and that $x^2 + y^2 = 9$. What is the maximum possible area inside such a rectangle R?

9-1. Define

$$\begin{aligned} J &:= \int_1^2 \sqrt{256 - x^4}\,dx \\ K &:= \int_1^2 \sqrt{256 + x^4}\,dx \\ L &:= \int_1^2 \sqrt{256 - x^8}\,dx \end{aligned}$$

Order 16, J, K, L from smallest to largest.



10-1. Let g be a function whose derivative g' is continuous and has the graph shown above. On $0 < x < 5$, what are the maximal open intervals of concavity for $g(x)$?

11-1. Approximate $[3.59]\left[(10)^{5/2}\right]$.

12-1. Let A be a 5×5 matrix such that the entries in each row add up to 10. Let $B := 6A^3 + 4A^2 + 7A$. True or False: The entries any row of B will add up to 6470.

13-1. We have available 75 square feet of material, and wish to use it to form the sides and bottom of an open-topped rectangular box. What is the maximum volume of the box?

14-1. What is the hundreds digit in the standard decimal expansion of the number 7^{26}?

15-1. True or False: Let f be a continuous real-valued function defined on the open interval $(-2, 3)$. Then f is bounded.

15-2. True or False: Let f be a continuous real-valued function defined on the closed interval $[-2, 3]$. There exists $c \in (-2, 3)$ such that f is differentiable at c and such that $5 \cdot [f'(c)] = [f(3)] - [f(-2)]$.

15-3. True or False: Let f be a continuous real-valued function defined on the closed interval $[-2,3]$. Assume that f is differentiable at 0 and that $f'(0)=0$. Then f has a local extremum at 0.

15-4. True or False: Let f be a continuous real-valued function defined on the closed interval $[-2,3]$. Assume that all of the following are true:

- f is twice-differentiable at 0,
- $f'(0)=0$ and
- $f''(0)\neq 0$.

Then f has a local extremum at 0.

16-1. What is the volume of the solid formed by revolving, about the x-axis, the region in the first quadrant of the xy-plane bounded by: the coordinate axes and the graph of the equation $y=\sqrt{\dfrac{x}{1+x^4}}$?

16-2. What is the volume of the solid formed by revolving, about the y-axis, the region in the first quadrant of the xy-plane bounded by: the coordinate axes and the graph of the equation $y=\dfrac{x^2}{(1+x^4)^{3/2}}$?

17-1. How many real roots does the polynomial x^5-5x+3 have?

18-1. Let V be the real vector space of all real homogeneous polynomials in x and y of degree 7 (together with the zero polynomial). Let W be the real vector space of all real polynomials in x of degree ≤ 3 (together with the zero polynomial). If T is a linear transformation from V <u>onto</u> W, what is the dimension of the subspace $\{\mathbf{v}\in V\,|\,T(\mathbf{v})=\mathbf{0}\}$ of V?

18-2. Let V be the real vector space of all real polynomials in x and y of degree ≤ 7 (together with the zero polynomial). Let W be the real vector space of all real polynomials in x of degree ≤ 3 (together with the zero polynomial). If T is a linear transformation from V <u>onto</u> W, what is the dimension of the subspace $\{\mathbf{v}\in V\,|\,T(\mathbf{v})=\mathbf{0}\}$ of V?

19-1. True or False: Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that, for all $x\in\mathbb{R}$, we have $-x^2\leq f(x)\leq x^2$. Then, for all $x\in\mathbb{R}$, we have $-2x\leq f'(x)\leq 2x$.

19-2. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that, for all $x \in \mathbb{R}$, we have $-x^2 \leq f(x) \leq x^2$. Then $f'(0) = 0$.

19-3. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(1) = 5$ and $f'(3) = 9$. Then $\exists c \in (1, 3)$ such that $f'(c) = 7$.

19-4. True or False: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then there exists $c \in \mathbb{R}$ such that f' is continuous at c.

20-1. Let f be the function defined on the real line by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational;} \\ 2x, & \text{if } x \text{ is irrational.} \end{cases}$$

Compute the set of points of discontinuity of f.

21-1. Let $p := 7919$, which is a prime number. Let $Q := \{p, 2p, 3p, \ldots\}$ be the set of multiples of p. Let $K := \{0, 1, \ldots, p\}$ denote the set of integers from 0 to p. For all $k \in K$, let C_k^p be the binomial coefficient "p choose k". Let $S := \{k \in K \,|\, C_1^p, \ldots, C_k^p \in Q\}$. So, for example, because $C_1^p = p \in Q$ and $C_2^p = [(p-1)/2]p = 3959p \in Q$, we get $2 \in S$. Compute the maximum element of S.

22-1. Let $C(\mathbb{R})$ be the collection of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Then $C(\mathbb{R})$ is a real vector space with vector addition defined by

$$\forall f, g \in C(\mathbb{R}),\ \forall x \in \mathbb{R}, \qquad (f+g)(x) \quad = \quad [f(x)] \ + \ [g(x)],$$

and with scalar multiplication defined by

$$\forall f \in C(\mathbb{R}),\ \forall r, x \in \mathbb{R}, \qquad (rf)(x) \quad = \quad r \cdot [f(x)].$$

Let S denote the set of $f \in C(\mathbb{R})$ such that all of the following hold:

- f is twice differentiable,
- for all $x \in \mathbb{R}$, $f(x + 2\pi) = f(x)$.
- $f'' = -f$.

True or False: S is a subspace of $C(\mathbb{R})$.

23-1. True or False: There exists a real number b such that the line $y = 10x$ tangent to the curve $y = bx^2 + 10x + 1$ at some point in the xy-plane.

24-1. Let h be the function defined by $h(x) = \int_0^{x^2} e^{(x+t)^2}\,dt$, for all real numbers x. Compute $h'(1)$.

25-1. Let $\{a_n\}_{n=1}^{\infty}$ be defined recursively by $a_1 = 7$ and

$$\text{for all integers } n \geq 1, \qquad a_{n+1} = \left(\frac{n}{n+3}\right) a_n.$$

Compute a_{25}.

26-1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 2x^2 - 4xy + y^4$. Find all the absolute extreme values of f, and where they occur.

27-1. Find the dimension of the solution space, in $\mathbb{R}^4$, of

$$\begin{array}{rcrcrcrcr} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & x & - & y & & & = & 2 \\ -\ w & + & 7x & - & y & - & 9z & = & -7. \end{array}$$

27-2. Find the dimension of the solution space, in $\mathbb{R}^4$, of

$$\begin{array}{rcrcrcrcr} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & 2x & - & y & & & = & 2 \\ -\ w & + & 7x & - & y & - & 9z & = & -7. \end{array}$$

27-3. Find the solution space, in $\mathbb{R}^4$, of

$$\begin{array}{rcrcrcrcl} 3w & + & 4x & - & 2y & - & 3z & = & 1 \\ 2w & + & x & - & y & & & = & 2 \\ -\ w & + & 7x & - & y & - & 9z & = & 5. \end{array}$$

28-1. Let T be a graph with 378 vertices. Assume T is a tree, which is a connected graph with no cycles. How many edges does T have?

29-1. For all positive functions f and g of the real variable x, let $\sim$ be a relation defined by

$$f \sim g \qquad \text{if and only if} \qquad \lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = 1.$$

True or False: Let f, g, ϕ, ψ be positive functions of x. Assume that $f \sim g$ and that $\phi \sim \psi$. Then $f + \phi \sim g + \psi$.

30-1. Let S and T be sets and assume that there exists a function $f : S \to T$ such that f is onto T. True or False: There must exist a function $g : T \to S$ such that g is one-to-one.

30-2. Let S and T be sets. Assume that there does NOT exist a function $f : S \to T$ such that f is one-to-one. True or False: There must exist a function $g : T \to S$ such that g is one-to-one.

31-1. True or False: There exists a solution $y : \mathbb{R} \to \mathbb{R}$ to the differential equation $y' = x^4 + 2x^2y^2 + y^4$ with the property that, for every $x \in \mathbb{R}$, we have $-1000 < y(x) < 1000$.

32-1. True or False: Let G be a group. Assume, for all $a, b \in G$, for all integers $n \geq 1$, that $(ab)^n = a^n b^n$. Then G is Abelian.

33-1. True or False: Let p and q be prime numbers, and let n be an integer. Assume that $p \neq q$. Then there exist integers k and ℓ such that $\dfrac{n}{p^2q} = \dfrac{k}{p^2} + \dfrac{\ell}{q}$.

33-2. True or False: Let p and q be prime numbers, and let n be an integer. Assume that $p \neq q$. Then there exist integers r, s, t, u such that $0 \leq s < p$ and $0 \leq t < p$ and $0 \leq u < q$ and $\dfrac{n}{p^2q} = r + \dfrac{s}{p} + \dfrac{t}{p^2} + \dfrac{u}{q}$.

33-3. True or False: Let $\mathbb{R}[x]$ denote the ring of polynomials, with real coefficients, in the indeterminate x. Let $p, q \in \mathbb{R}[x]$ be irreducible polynomials, and let $f \in \mathbb{R}[x]$. Assume that $p \neq q$. Then there exist $r, s, t, u \in \mathbb{R}[x]$ such that $\deg[s] < \deg[p]$ and $\deg[t] < \deg[p]$ and $\deg[u] < \deg[q]$ and $\dfrac{f}{p^2q} = r + \dfrac{s}{p} + \dfrac{t}{p^2} + \dfrac{u}{q}$.

34-1. Define $N : \mathbb{R}^2 \to [0,\infty)$ by $N(x,y) = [x^4 + y^4]^{1/4}$. (This is sometimes called the L^4-norm on $\mathbb{R}^2$.) Let $C := (1,2) \in \mathbb{R}^2$ and let $D := (3,5) \in \mathbb{R}^2$. Let

$$\begin{aligned} S &:= \{A \in \mathbb{R}^2 \,|\, N(A-C) = 1\} \\ T &:= \{B \in \mathbb{R}^2 \,|\, N(B-D) = 2\} \end{aligned}$$

(These are two L^4-spheres in $\mathbb{R}^2$.) Minimize $N(A-B)$ subject to the constraints $A \in S$ and $B \in T$. (That is, compute how close the one L^4-sphere gets to the other.)

42-1. Let $p : \mathbb{R} \to \mathbb{R}$ be defined by $p(x) = [e^{-x^2/2}]/[\sqrt{2\pi}\,]$. Let X and Y be independent random variables. Assume that X and Y are both standard normal, *i.e.*, that both X and Y have probability density function p. Compute the probability that $X < 9Y$.

46-1. TRUE OR FALSE: For any cyclic group G, for any homomorphism $f : G \to G$, there exists an integer n such that, for all $x \in G$, we have $f(x) = x^n$.

46-2. TRUE OR FALSE: For any Abelian group G, for any homomorphism $f : G \to G$, there exists an integer n such that, for all $x \in G$, we have $f(x) = x^n$.

49-1. Up to isomorphism, how many additive Abelian groups are there of order 12?

49-2. Up to isomorphism, how many additive Abelian groups G of order 12 have the property that, for all $x \in G$, $x+x+x+x+x+x = 0$?

49-3. Up to isomorphism, how many additive Abelian groups are there of order 24?

49-4. Up to isomorphism, how many additive Abelian groups G of order 24 have the property that, for all $x \in G$, $x+x+x+x+x = 0$?

49-5. Up to isomorphism, how many additive Abelian groups G of order 24 have the property that, for all $x \in G$, $x+x+x+x = 0$?

59-1. Let f be an analytic function of a complex variable $z = x + iy$ given by

$$f(z) = (3x + 5y) + i \cdot (g(x, y)),$$

where $g(x, y)$ is a real-valued function of the real variables x and y. If $g(0, 0) = 1$, then $g(7, 3) =$