SOLUTIONS TO PRACTICE TEST 2

$$\begin{aligned} y' + xy &= x \\ y(0) &= -1 \end{aligned}$$

44. If y is a real-valued function defined on the real line satisfying the initial value problem above, then $\lim\limits_{x \to -\infty} [y(x)] =$

(A) 0
(B) 1
(C) -1
(D) ∞
(E) $-\infty$

Solution: Following the notation given in the problem, y and $y(x)$ are used interchangeably. Also, y' and $y'(x)$ are used interchangeably. For all $x \in \mathbb{R}$, $y'(x) = x(1-y)$, and so $[y(x) = 1] \Rightarrow [y'(x) = 0]$. So, by Picard-Lindelöf, exactly one of the following three possibilities holds:

$$[\,\forall x \in \mathbb{R}, y(x) > 1\,] \text{ or } [\,\forall x \in \mathbb{R}, y(x) = 1\,] \text{ or } [\,\forall x \in \mathbb{R}, y(x) < 1\,].$$

So, as $y(0) = -1$, we get $\forall x \in \mathbb{R}, y(x) < 1$. Then, for all $x \in \mathbb{R}$,

$$\frac{d}{dx}[\ln(1-y)] \quad = \quad \frac{-y'}{1-y} \quad = \quad -x \quad = \quad \frac{d}{dx}\left[-\frac{x^2}{2}\right].$$

Choose $C \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$, $\ln(1-(y(x))) = -[x^2/2] + C$. Let $K := e^C$. Then, for all $x \in \mathbb{R}$, we have $1 - (y(x)) = Ke^{-x^2/2}$, and so $y(x) = 1 - Ke^{-x^2/2}$. Then, because $\lim\limits_{x \to -\infty}\left[e^{-x^2/2}\right] = 0$, we conclude that $\lim\limits_{x \to -\infty}[y(x)] = 1 - K \cdot 0 = 1$. Answer: (B) $\square$

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54. Choose a real number x uniformly at random in the interval $[0,3]$. Choose a real number y independently of x, and uniformly at random in the interval $[0,4]$. Find the probability that $x < y$.

(A) 1/2
(B) 7/12
(C) 5/8
(D) 2/3
(E) 3/4

Solution: Viewing this as a problem in measure theory, the answer is

$$\frac{\text{area of } \{\ (x,y) \in [0,3] \times [0,4] \mid x < y\ \}}{\text{area of } [0,3] \times [0,4]},$$

Let $T := \{(x,y) \in [0,3] \times [0,4] \,|\, x > y\}$. Then the answer is

$$\frac{12 - (\text{area of } T)}{12}.$$

Since T is a right isosceles triangle of leg length 3, it follows that the area of T is $(1/2)(3)(3) = 9/2$. Then the answer is

$$\frac{12-(9/2)}{12} = \frac{24-9}{24} = \frac{15}{24} = \frac{5}{8}.$$

Answer: (C) □

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61. A tank initially contains a salt solution of 3 grams of salt dissolved in 100 liters of water. A salt solution containing 0.02 grams of salt per liter of water is sprayed into the tank at a rate of 4 liters per minute. The sprayed solution is continually mixed with the salt solution in the tank, and the mixture flows out of the tank at a rate of 4 liters per minute. If the mixing is instantaneous, how many grams of salt are in the tank after 100 minutes have elapsed?

(A) 2
(B) $2 - e^{-2}$
(C) $2 + e^{-2}$
(D) $2 - e^{-4}$
(E) $2 + e^{-4}$

Solution: For all $t \in \mathbb{R}$, let $s(t)$ denote the number grams of salt in the tank at the t minute mark. Then $s(0) = 3$.

We use s and $s(t)$ interchangeably. We also use s' and $s'(t)$ interchangeably. The solution sprayed into the tank adds $(0.02)4 = 2/25$ grams of salt per minute. There are always 100 liters of liquid in the tank, containing s grams of salt. So the density of salt in the tank is $s/100$ grams per liter. The flow of water out of the tank therefore subtracts $4(s/100) = s/25$ grams of salt per minute. Then, for all $t \in \mathbb{R}$, we have $s'(t) = (2/25) - (s/25) = (2-s)/25$, and so $[s(t) = 2] \Rightarrow [s'(t) = 0]$. So, by Picard-Lindelöf, exactly one of the following three possibilities holds:

$$[\,\forall t \in \mathbb{R}, s(t) > 2\,] \text{ or } [\,\forall t \in \mathbb{R}, s(t) = 2\,] \text{ or } [\,\forall t \in \mathbb{R}, s(t) < 2\,].$$

So, as $s(0) = 3$, we get $\forall t \in \mathbb{R}, s(t) > 2$. Then, for all $t \in \mathbb{R}$,

$$\frac{d}{dt}\,[\ln(s-2)] \quad = \quad \frac{s'}{s-2} \quad = \quad \frac{-1}{25} \quad = \quad \frac{d}{dt}\left[-\frac{t}{25}\right].$$

Choose $C \in \mathbb{R}$ such that, for all $t \in \mathbb{R}$, $\ln((s(t) - 2)) = -[t/25] + C$. Let $K := e^C$. Then, for all $t \in \mathbb{R}$, we have $(s(t)) - 2 = Ke^{-t/25}$, and so $s(t) = 2 + Ke^{-t/25}$. Then $3 = s(0) = 2 + Ke^0 = 2 + K$, so $K = 1$. Then $s(100) = 2 + Ke^{-100/25} = 2 + 1 \cdot e^{-4} = 2 + e^{-4}$. Answer: (E) $\square$

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65. Let g be a differentiable function of two real variables, and let f be the function of a complex variable z defined by

$$f(z) = e^x(\sin y) + i \cdot (g(x,y)),$$

where x and y are the real and imaginary parts of z, respectively. If f is an analytic function on the complex plane, then $(g(4,2)) - (g(0,1)) =$

Solution: Define $Z : \mathbb{R}^2 \to \mathbb{C}$ by $Z(x,y) = x + iy$. Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x,y) = e^x(\sin y)$. Then $f \circ Z = h + ig$.

According to the Cauchy-Riemann equations, a counterclockwise 90° rotation of $(\partial_1 h, \partial_1 g)$ gives $(\partial_2 h, \partial_2 g)$. That is,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_1 h \\ \partial_1 g \end{bmatrix} = \begin{bmatrix} \partial_2 h \\ \partial_2 g \end{bmatrix}.$$

That is, $-\partial_1 g = \partial_2 h$ and $\partial_1 h = \partial_2 g$.

For all $x, y \in \mathbb{R}$, we have $h(x,y) = e^x(\sin y)$. Computing partial derivatives, for all $x, y \in \mathbb{R}$, we get $(\partial_1 h)(x,y) = e^x(\sin y)$ and $(\partial_2 h)(x,y) = e^x(\cos y)$.

Then, for all $x, y \in \mathbb{R}$, we have

$$-(\partial_1 g)(x,y) \;=\; e^x(\cos y) \qquad \text{and} \qquad (\partial_2 g)(x,y) \;=\; e^x(\sin y).$$

Multiplying the first equation by -1, and substituting $y :\to 2$, we see, for all $x \in \mathbb{R}$, that $(\partial_1 g)(x,2) = -e^x(\cos 2)$. So, integrating this equation from $x = 1$ to $x = 3$, we see that

$$[g(3,2)] \quad - \quad [g(1,2)] \qquad = \qquad \int_1^3 (-e^x(\cos 2))\,dx.$$

So, as $\displaystyle\int_1^3 (-e^x(\cos 2))\,dx = -\left(\int_1^3 e^x\,dx\right)(\cos 2) + = -(e^3 - e^1)(\cos 2)$, we get

$$[g(3,2)] \quad - \quad [g(1,2)] \qquad = \qquad -(e^3 - e^1)(\cos 2).$$

That is, $(g(3,2)) - (g(1,2)) = (e - e^3)(\cos 2)$. Answer: (E) $\square$
