SOLUTIONS OF VARIATIONS, PRACTICE TEST 4

52-1. Consider the following system of linear equations over the real numbers, where x, y and z are variables and b is a real constant.

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0 \\ 2x & + & 4y & + & 3z & = & 0 \\ x & + & 3y & + & bz & = & 0 \end{array}$$

Which of the following statements are true?

I. There exists a value of b for which the system has no solution.
II. There exists a value of b for which the system has exactly one solution.
III. There exists a value of b for which the system has more than one solution.

(A) II only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II and III

Solution: Let

$$M := \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & b \end{bmatrix} \quad \text{and} \quad v := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{0} := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then the given system can be written, in matrix form, as $Mv = \mathbf{0}$.

Since, for every $b \in \mathbb{R}$, $x = y = z = 0$ is a solution to $Mv = \mathbf{0}$, it follows that I is false. For every $b \in \mathbb{R}$,

the solution $x = y = z = 0$ is the **only** solution of $Mv = \mathbf{0}$

iff

$$\det M = 0.$$

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For every $b \in \mathbb{R}$, by expanding $\det M$ along the third row of M, we get

$$\begin{aligned} \det M &= 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + b \cdot \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \\ &= 1 \cdot 2 - 3 \cdot 1 + b \cdot 0 \;=\; -1. \end{aligned}$$

Therefore, for all $b \in \mathbb{R}$, the equation $Mv = \mathbf{0}$ has exactly one solution. So II is true and III is false. Answer: (A) □

52-2. Consider the following system of linear equations over the real numbers, where x, y and z are variables and b is a real constant.

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0 \\ 2x & + & 4y & + & 3z & = & 0 \\ 3x & + & 6y & + & bz & = & 0 \end{array}$$

Which of the following statements are true?

I. There exists a value of b for which the system has no solution.
II. There exists a value of b for which the system has exactly one solution.
III. There exists a value of b for which the system has more than one solution.

(A) II only
(B) I and II only
(C) I and III only
(D) II and III only
(E) III only

Solution: Let

$$M := \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & b \end{bmatrix} \quad \text{and} \quad v := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{0} := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then the given system can be written, in matrix form, as $Mv = \mathbf{0}$.

Since, for every $b \in \mathbb{R}$, $x = y = z = 0$ is a solution to $Mv = \mathbf{0}$, it follows that I is false. For every $b \in \mathbb{R}$,

the solution $x = y = z = 0$ is the **only** solution of $Mv = \mathbf{0}$

iff

$$\det M = 0.$$

For every $b \in \mathbb{R}$, by expanding $\det M$ along the third row of M, we get

$$\begin{aligned} \det M &= 3 \cdot \det \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} - 6 \cdot \det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + b \cdot \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \\ &= 3 \cdot 2 - 6 \cdot 1 + b \cdot 0 \;=\; 0. \end{aligned}$$

Therefore, for all $b \in \mathbb{R}$, the equation $Mv = \mathbf{0}$ has more than one solution. So II is false and III is true. Answer: (E) $\square$

53-1. In the complex plane, let C be the circle $|z+2| = 2$ with negative (clockwise) orientation. Compute $\displaystyle\int_C \frac{dz}{(z-1)(z+3)^2}$.

Solution: Define $g : \mathbb{C}\backslash\{1\} \to \mathbb{C}$ by $g(z) = \dfrac{1}{z-1}$. We wish to compute $\displaystyle\int_C \frac{g(z)}{(z+3)^2}\, dz$.

Let $c_0 := g(-3)$, $c_1 := g'(-3)$, $c_2 := \dfrac{g''(-3)}{2!}$, ... be the Taylor coefficients of g at -3. As g is holomorphic on $\mathbb{C}\backslash\{1\}$, it follows that g is holomorphic at -3. So choose a neighborhood N in $\mathbb{C}$ of -3 such that, for all $z \in N$, $g(z) = \displaystyle\sum_{k=0}^{\infty} c_k(z+3)^k$. Let C' be a circle in the complex plane, centered at -3, contained in N, with negative orientation. By Cauchy's Theorem, $\displaystyle\int_C \frac{g(z)}{(z+3)^2}\, dz = \int_{C'} \frac{g(z)}{(z+3)^2}\, dz$. Then $\displaystyle\int_C \frac{g(z)}{(z+3)^2}\, dz = \int_{C'} \left[\sum_{k=0}^{\infty} c_k(z+3)^{k-2}\right] dz$. Splitting off the first two terms, we conclude that

$$\begin{aligned} \int_C \frac{g(z)}{(z+3)^2}\, dz &= \left(\int_{C'} \frac{c_0}{(z+3)^2}\, dz\right) + \left(\int_{C'} \frac{c_1}{z+3}\, dz\right) + \\ &\qquad \left(\int_{C'} \left[\sum_{k=2}^{\infty} c_k(z+3)^{k-2}\right] dz\right). \end{aligned}$$

By Cauchy's Theorem, $\displaystyle\int_{C'} \left[\sum_{k=2}^{\infty} c_k(z+3)^{k-2}\right] dz = 0$; and note that this sum starts with $k = 2$, not $k = 0$. Then

$$\int_C \frac{g(z)}{(z+3)^2}\, dz \;=\; \left(\int_{C'} \frac{c_0}{(z+3)^2}\, dz\right) \;+\; \left(\int_{C'} \frac{c_1}{z+3}\, dz\right).$$

Parametrization of C' and computation (remembering the negative orientation of C') gives

$$\int_{C'} \frac{1}{(z+3)^2}\, dz = 0 \qquad \text{and} \qquad \int_{C'} \frac{1}{z+3}\, dz = -2\pi i.$$

We have $c_1 = g'(-3) = \dfrac{-1}{(-3-1)^2} = -\dfrac{1}{16}$. Then

$$\int_{C'} \frac{c_0}{(z+3)^2}\, dz = 0 \qquad \text{and} \qquad \int_{C'} \frac{c_1}{z+3}\, dz = -2\pi i c_1 = \frac{\pi i}{8}.$$

Then

$$\begin{aligned} \int_C \frac{g(z)}{(z+3)^2}\, dz &= \left(\int_{C'} \frac{c_0}{(z+3)^2}\, dz\right) + \left(\int_{C'} \frac{c_1}{z+3}\, dz\right) \\ &= 0 + \frac{\pi i}{8} = \frac{\pi i}{8}. \qquad \square \end{aligned}$$

Alternate solution: The integrand is holomorphic inside C, except at $z = -3$. The winding number of C around $z = -3$ is equal to -1. The Taylor expansion of $\dfrac{1}{z-4}$ about $z = 0$ begins $\dfrac{-1}{4} + \dfrac{-1}{16} z + \cdots$. Changing z to $z+3$, we see that the Taylor expansion of $\dfrac{1}{z-1}$ about $z = -3$ begins $\dfrac{-1}{4} + \dfrac{-1}{16}(z+3) + \cdots$. Then the residue of $\dfrac{1}{(z-1)(z+3)^2}$ at $z = -3$ is $\dfrac{-1}{16}$. So, by the Residue Theorem, the integral is

$$[-1]\left[\frac{-1}{16}\right][2\pi i] \quad = \quad \frac{\pi i}{8}. \qquad \square$$

53-2. In the complex plane, let C be the circle $|z| = 4$ with negative (clockwise) orientation. Compute $\displaystyle\int_C \frac{dz}{(z-1)(z+3)^2}$.

Solution: Define $f : \mathbb{C}\backslash\{-3, 1\} \to \mathbb{C}$ by $f(z) = \dfrac{1}{(z-1)(z+3)^2}$. We wish to compute $\displaystyle\int_C f(z)\, dz$.

Let C' be a small circle in $\mathbb{C}$ about $z = 1$ on which and inside of which the Taylor expansion of $\dfrac{1}{(z+3)^2}$ converges absolutely. Let C'' be a small circle in $\mathbb{C}$ about $z = -3$ on which and inside of which the Taylor expansion of $\dfrac{1}{z-1}$ converges absolutely. Give C' and C'' negative orientations. By Cauchy's Theorem,

$$\int_C f(z)\,dz \quad = \quad \left[\int_{C'} f(z)\,dz\right] \quad + \quad \left[\int_{C''} f(z)\,dz\right].$$

The Taylor expansion of $\dfrac{1}{(z+3)^2}$ about $z=1$ begins

$$\frac{1}{(z+3)^2} = \frac{1}{16} + \cdots .$$

Then, on C', we have

$$f(z) = \frac{1}{16(z-1)} + \cdots ,$$

where the remainder, denoted $\cdots$, is holomorphic, and so, by Cauchy, integrates to 0 around C'. Remembering that C' is oriented negatively,

$$\int_{C'} f(z)\,dz \;=\; \int_{C'} \frac{dz}{16(z-1)}\,dz \;=\; \frac{-2\pi i}{16} \;=\; -\frac{\pi i}{8}.$$

The Taylor expansion of $\dfrac{1}{z-4}$ about $z=0$ begins $\dfrac{-1}{4} + \dfrac{-1}{16}z + \cdots$. Changing z to $z+3$, we see that the Taylor expansion of $\dfrac{1}{z-1}$ about $z=-3$ begins $\dfrac{-1}{4} + \dfrac{-1}{16}(z+3) + \cdots$. Then, on C'', we have

$$f(z) = \frac{-1}{4(z+3)^2} + \frac{-1}{16(z+3)} + \cdots ,$$

where the remainder, denoted $\cdots$, is holomorphic, and so, by Cauchy, integrates to 0 around C''. Remembering that C'' is oriented negatively,

$$\begin{aligned}\int_{C''} f(z)\,dz \;&=\; \left[\int_{C''} \frac{-dz}{4(z+3)^2}\,dz\right] + \left[\int_{C''} \frac{-dz}{16(z+3)}\,dz\right] \\ &=\; 0 + \left[\frac{2\pi i}{16}\right] \;=\; \frac{\pi i}{8}.\end{aligned}$$

Then

$$\int_C f(z)\,dz \quad = \quad \left[-\frac{\pi i}{8}\right] \quad + \quad \left[\frac{\pi i}{8}\right] = 0. \qquad \square$$

Alternate solution: Let $f(z) = \dfrac{1}{(z-1)(z+3)^2}$ be the integrand, which is holomorphic inside C, except at $z=1$ and $z=-3$.

The winding number of C around $z = 1$ is equal to -1. The residue of $f(z)$ at $z = 1$ is $\dfrac{1}{(1+3)^2} = \dfrac{1}{16}$.

The winding number of C around $z = -3$ is equal to -1. The Taylor expansion of $\dfrac{1}{z-4}$ about $z = 0$ begins $\dfrac{-1}{4} + \dfrac{-1}{16}z + \cdots$. Changing z to $z + 3$, the Taylor expansion of $\dfrac{1}{z-1}$ about $z = -3$ begins $\dfrac{-1}{4} + \dfrac{-1}{16}(z+3) + \cdots$. Then the residue of $f(z)$ at $z = -3$ is $\dfrac{-1}{16}$.

So, by the Residue Theorem, the integral is

$$[-1]\left[\frac{1}{16}\right][2\pi i] \quad + \quad [-1]\left[\frac{-1}{16}\right][2\pi i] \quad = \quad 0. \qquad \square$$

Alternate solution: Let C_0 be the circle $|z| = 4$ with positive (counterclockwise) orientation; it is the same circle as C, but positively oriented. We will change the variable z to $\dfrac{16}{w}$ in the integral $\displaystyle\int_C \frac{dz}{(z-1)(z+3)^2}$. Then C changes to C_0 and dz changes to $\dfrac{-16\,dw}{w^2}$. Then

$$\begin{aligned}
\int_C \frac{dz}{(z-1)(z+3)^2} &= \int_{C_0} \frac{-16\,dw}{w^2((16/w)-1)((16/w)+3)^2} \\
&= \int_{C_0} \frac{-16w\,dw}{w((16/w)-1)w^2((16/w)+3)^2} \\
&= \int_{C_0} \frac{-16w\,dw}{(16-w)(16+3w)^2}.
\end{aligned}$$

The integrand $\dfrac{-16w\,dw}{(16-w)(16+3w)^2}$ is holomorphic except at 16 and $-16/3$ which are both outside of C. Thus, by Cauchy's Theorem, $\displaystyle\int_{C_0} \frac{-16w\,dw}{(16-w)(16+3w)^2} = 0$. Then $\displaystyle\int_C \frac{dz}{(z-1)(z+3)^2} = 0$. $\square$

NOTE: The idea of this last solution is that the original integrand (in z) has poles inside the circle, but, if we add a point at infinity, obtaining the Riemann sphere, then the poles are all on one side of the circle. On a sphere, the northern hemisphere is indisinguishable from the southern hemisphere, and the circles C and C_0 can be thought of as the equator. Geometrically, the change of variables (from z to w) simply interchanges the two hemispheres. This trick works in many situations, but we need to issue a ...

WARNING: If we try to use this trick to compute $\displaystyle\int_C \frac{dz}{z}$, we do not get 0, and, in fact, $\displaystyle\int_C \frac{dz}{z} = 2\pi i$. Here's ...

THE POINT: Recall that dz changes to $\dfrac{-16\,dw}{w^2}$. Some care is required to be sure that the transformed integrand (in w) does not have a pole at $w = 0$, coming from the pole (of order two) at $w = 0$ in $\dfrac{-16}{w^2}$.

In other words, if we simply transform the original integrand (in z), ignoring the differential dz, then, in the resulting expression (in w), no pole would appear. However, the differential dz itself transforms to a new differential with a pole of order two at $w = 0$. If we're lucky, then that "resulting expression (in w)" will have a zero of order (at least) two at $w = 0$. If so, then, by Cauchy, the transformed integral is 0, and, consequently, so is the original integral.

54-1. Assume that, in a certain two-dimensional world, the wind velocity at any point (x, y) is $(-11x+10y, -10x+14y)$. A small particle is simply pushed by the wind. Its position at any time t is given by $(f(t), g(t))$. Assume that its velocity at time t is

$$(\quad -11[f(t)] + 10[g(t)] \quad , \quad -10[f(t)] + 14[g(t)] \quad).$$

Because its velocity at time t is also given by $(f'(t), g'(t))$, its motion will satisfy the equations:

$$f'(t) = -11[f(t)] + 10[g(t)], \qquad\qquad g'(t) = -10[f(t)] + 14[g(t)].$$

Assume that the initial position of the particle is $(f(0), g(0)) = (0, 1)$. We stand at the origin and watch the particle. Along what slope line will we look, asymptotically, as $t \to \infty$? That is, compute $\displaystyle\lim_{t\to\infty} \frac{g(t)}{f(t)}$.

Solution: Let $M := \begin{bmatrix} -11 & 10 \\ -10 & 14 \end{bmatrix}$, and let $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Define the function $p : \mathbb{R} \to \mathbb{R}^{2\times 1}$ by $p(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$. The characteristic polynomial of M is $\det(M - \lambda I) = \lambda^2 - 3\lambda - 54 = (\lambda - 9)(\lambda + 6)$, so the eigenvalues of M are 9 and -6. We have $M - 9I = \begin{bmatrix} -20 & 10 \\ -10 & 5 \end{bmatrix}$. The

9-eigenspace of M is the kernel of $M - 9I$, which is spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We have $M + 6I = \begin{bmatrix} -5 & 10 \\ -10 & 20 \end{bmatrix}$. The (-6)-eigenspace of M is the kernel of $M + 6I$, which is spanned by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Let

$$v := \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad w := \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Then $Mv = 9v$ and $Mw = -6w$. Then, for all $t \in \mathbb{R}$, we have

$$e^{tM}v = e^{9t}v \qquad \text{and} \qquad e^{tM}w = e^{-6t}w.$$

At any time t, we have $p'(t) = M \cdot [p(t)]$. Let $u := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The initial position of the particle is $p(0) = u$. Then the position of the particle, at any time t, is given by $p(t) = e^{tM}u$. We calculate $v - (1/2)w = (3/2)u$, so $u = (2/3)v - (1/3)w$. Then, at any time t, we have

$$p(t) = (2/3)e^{9t}v - (1/3)e^{-6t}w,$$

so

$$\begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = p(t) = \begin{bmatrix} (2/3)e^{9t} - (2/3)e^{-6t} \\ (4/3)e^{9t} - (1/3)e^{-6t} \end{bmatrix},$$

so

$$\frac{g(t)}{f(t)} = \frac{(4/3)e^{9t} - (1/3)e^{-6t}}{(2/3)e^{9t} - (2/3)e^{-6t}} = \frac{(4/3) - (1/3)e^{-15t}}{(2/3) - (2/3)e^{-15t}}.$$

Then

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = \frac{4/3}{2/3} = 2. \quad \square$$

54-2. Assume that, in a certain two-dimensional world, the wind velocity at any point (x, y) is $(-y, x)$. A small particle is simply pushed by the wind. Its position at any time t is given by $(f(t), g(t))$. Assume that its velocity at time t is $(-[g(t)], f(t))$. Because its velocity at time t is also given by $(f'(t), g'(t))$, its motion will satisfy the equations:

$$f'(t) = -[g(t)], \qquad\qquad g'(t) = f(t).$$

Assume that the initial position of the particle is $(f(0), g(0)) = (2, 0)$. Find its position $(f(t), g(t))$ at any time t.

Solution: Let $M := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and let $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Define the function $p : \mathbb{R} \to \mathbb{R}^{2\times 1}$ by $p(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$. The characteristic polynomial of M is $\det(M-\lambda I) = \lambda^2+1 = (\lambda-i)(\lambda+i)$. Letting i denote a square root of -1 in the complex numbers, the eigenvalues of M are i and $-i$. We have $M - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$. The i-eigenspace of M is the kernel of $M-iI$, which is spanned by $\begin{bmatrix} i \\ 1 \end{bmatrix}$. We have $M + iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$. The $(-i)$-eigenspace of M is the kernel of $M+iI$, which is spanned by $\begin{bmatrix} 1 \\ i \end{bmatrix}$. Let $v := \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $w := \begin{bmatrix} 1 \\ i \end{bmatrix}$. Then $Mv = iv$ and $Mw = -iw$. Then, for all $t \in \mathbb{R}$, we have: $\qquad e^{tM}v = e^{it}v \qquad$ and $\qquad e^{tM}w = e^{-it}w.$

At any time t, we have $p'(t) = M \cdot [p(t)]$. Let $u := \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. The initial position of the particle is $p(0) = u$. Then the position of the particle, at any time t, is given by $p(t) = e^{tM}u$. We calculate $iv - w = -u$, so $u = -iv + w$. Then, at any time t, we have

$$p(t) = -ie^{it}v + e^{-it}w,$$

so

$$\begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = p(t) = \begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix},$$

so $f(t) = e^{it} + e^{-it} = 2\cos t$ and $g(t) = -ie^{it} + ie^{-it} = 2\sin t$, so

$$(\, f(t)\, ,\, g(t)\,) \quad = \quad (\, 2\cos t\, ,\, 2\sin t\,). \quad \square$$

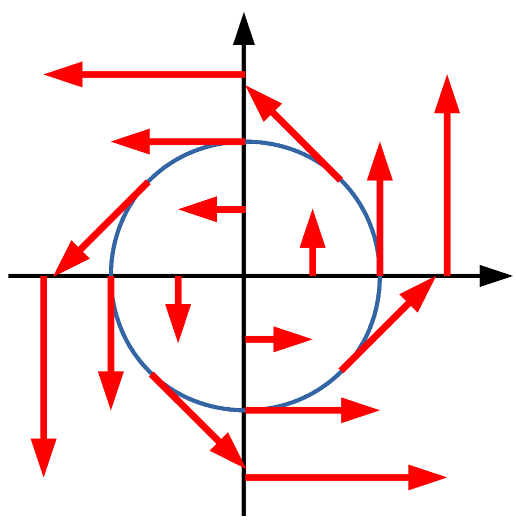
Alternate solution: At any point (x, y), the velocity of the wind is $(-y, x)$, and this is not hard to visualize, as follows: Pick a finite collection of points (x_i, y_i) in the plane, and, for each of these, draw an arrow that

- starts at (or "is footed at") the point (x_i, y_i),
- has run $-y_i$,
- has rise x_i, and, therefore,
- ends at $(x_i - y_i, y_i + x_i)$.

We will draw this picture, using the following 16 points:

$$\begin{array}{llll} (1,0), & (0,1), & (-1,0), & (0,-1), \\ (3,0), & (0,3), & (-3,0), & (0,-3), \\ (2,0), & (0,2), & (-2,0), & (0,-2), \\ (\sqrt{2},\sqrt{2}), & (-\sqrt{2},\sqrt{2}), & (-\sqrt{2},-\sqrt{2}), & (\sqrt{2},-\sqrt{2}). \end{array}$$

Note that the last eight points are on the circle $x^2 + y^2 = 4$, which is represented in blue in the picture below. The last arrow starts at $(\sqrt{2}, -\sqrt{2})$, runs $-(-\sqrt{2}) = \sqrt{2}$, and rises $\sqrt{2}$. It therefore ends at $(\sqrt{2}+\sqrt{2}, -\sqrt{2}+\sqrt{2}) = (2\sqrt{2}, 0)$. Thus the last arrow starts on the blue circle $x^2 + y^2 = 4$ and ends on the x-axis, slightly to the left of $(3, 0)$. This arrow, together with the other 15, results in the 16 red arrows appearing in the picture below.



Note that each red arrow is perpendicular to

the line from its footpoint (*i.e.*, starting point) to the origin.

Moreover, any radius from a point on a circle to the circle's center is perpendicular to the circle's tangent at that same point. From these observations we see that, if a particle, at a certain time t, is on

a circle centered at the origin,

then it must be travelling tangent to that circle, and so the instantaneous rate of change in

the distance from the particle to the origin is equal to zero. That is, the distance from the particle to the origin has derivative 0 at any time. That is, the distance from the particle to the origin is constant. The particle we wish to study in this question starts at $(2, 0)$, which is on the blue circle, and so it must travel around that circle. Moreover, every arrow footed on the blue circle will have length 2, and so our particle travels with speed 2 at all times. The distance around the circle is 4π, so the particle revolves once around the origin (*i.e.*, covers 2π radians) every 2π units of time. Thus, between time 0 and a time t, it has traveled through t radians. It also travels counterclockwise. Therefore, by trigonometry, we see that its position at any time t is given by $(f(t), g(t)) = (2\cos t, 2\sin t)$. □

55-1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. True or False: If $f'(0) = 0$, then $f(x)$ has a local extremum at $x = 0$.

Solution: False. Counterexample: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Then $f'(0) = 0$, but f is increasing on $\mathbb{R}$, so f has no local extrema. □

55-2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. True or False: If $f(x)$ has a local extremum at $x = 0$, then $f'(0) = 0$.

Solution: True. This is Fermat's Theorem (a.k.a. the Interior Extremum Theorem). □

55-3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. True or False: If $f'(x)$ has a local extremum at $x = 0$, then $f(x)$ has a point of inflection at $x = 0$.

Solution: True. Proof: Assume that $f'(x)$ has a local extremum at $x = 0$. We wish to show that $f(x)$ has a point of inflection at $x = 0$.

We will assume that $f'(x)$ has a local maximum at $x = 0$; the proof for local minimum is similar. Choose $\delta > 0$ such that $f'(x)$ is increasing on $-\delta < x < 0$ and such that $f'(x)$ is decreasing on $0 < x < \delta$. Then $f(x)$ is concave up on $-\delta < x < 0$ and concave down on $0 < x < \delta$. Thus $f(x)$ has a point of inflection at $x = 0$. □

55-4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. True or False: If $f(x)$ has a point of inflection at $x = 0$, then $f'(x)$ has a local extremum at $x = 0$.

Solution: True. Proof: Assume that $f(x)$ changes concavity at $x = 0$. We wish to show that $f'(x)$ has a local extremum at $x = 0$.

We will assume that $f(x)$ changes from concave up to concave down at $x = 0$, when x moves from left to right; the proof for

changing from concave down to concave up

is similar. Choose $\delta > 0$ such that $f(x)$ is concave up on $-\delta < x < 0$ and concave down on $0 < x < \delta$. It then follows that $f'(x)$ is increasing on $-\delta < x < 0$ and decreasing on $0 < x < \delta$. Then $f'(x)$ has a local maximum at $x = 0$. Then $f'(x)$ has a local extremum at $x = 0$. $\square$

55-5. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. True or False: If $f''(0) = 0$, then $f(x)$ has a point of inflection at $x = 0$.

Solution: False. Counterexample: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^4$. Then $f''(0) = 0$. On the other hand, f is concave up on $\mathbb{R}$, so f has no points of inflection. $\square$

55-6. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. True or False: If $f(x)$ has a point of inflection at $x = 0$, then $f''(0) = 0$.

Solution: True. Proof: By Problem 55-2, $f'(x)$ has a local extremum at $x = 0$. Then, by Fermat's Theorem (a.k.a. the Interior Extremum Theorem), $f''(0) = 0$. $\square$

56-1. True or false: For any metric d on $\mathbb{R}$, there is a norm $\| \bullet \|$ on $\mathbb{R}$ such that, for all $x, y \in \mathbb{R}$, $d(x,y) = \|x - y\|$.

Solution: False. Counterexample: Let $| \bullet |$ denote absolute value on $\mathbb{R}$ and define a metric d on $\mathbb{R}$ by $d(x,y) = \min\{|x-y|, 1\}$. Let $\| \bullet \|$ be a norm on on $\mathbb{R}$, and assume: for all $x, y \in \mathbb{R}$, $d(x,y) = \|x - y\|$. We aim for a contradiction.

Let $a := \|1\|$. Then $a > 0$. Let $x := 2/a$ and let $y := 0$. Then, by assumption, $d(x,y) = \|x - y\|$. Since $x > 0$, we get $|x| = x = 2/a$. Then $|x| \cdot \|1\| = xa = 2$. We have $d(x,y) = d(x,0) = \min\{|x|, 1\} \leq 1$. Also, we have $\|x - y\| = \|x - 0\| = \|x\| = |x| \cdot \|1\| = 2$. Then $d(x,y) \leq 1 < 2 = \|x - y\|$. Contradiction.

NOTE: Every norm on $\mathbb{R}$ is a positive multiple of absolute value, so the collection of norms on $\mathbb{R}$ is very restricted. (For $n \geq 2$, norms on $\mathbb{R}^n$ are much more plentiful.) On the other hand, there are lots of metrics on $\mathbb{R}$, and there are many ways to manipulate one metric to get another. For example, if δ is a metric on a set S, then we can form another metric δ' on S by defining $\delta'(x,y) = \min\{\delta(x,y), 1\}$. In fact, if we define $f : [0, \infty) \to [0, \infty)$ by $f(t) = \min\{t, 1\}$, then f is semi-increasing and semi-concave down and, moreover, we have $f(0) = 0$. It turns out that because of these three properties of f, we can prove that $f \circ \delta$ is a metric. Since $\delta' = f \circ \delta$, we get that δ' is a metric. If you are aware of all of these facts, then it's easy to find a metric on $\mathbb{R}$ that doesn't come from a norm. $\square$

56-2. True or false: For every norm $\| \bullet \|$ on $\mathbb{R}$, there is an inner product $\langle \bullet, \bullet \rangle$ on $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, we have $\|x\|^2 = \langle x, x \rangle$.

Solution: True. Proof: Let $\| \bullet \|$ be a norm on $\mathbb{R}$. We wish to show that there is an inner product $\langle \bullet, \bullet, \rangle$ on $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, we have $\|x\|^2 = \langle x, x \rangle$.

Let $a := \|1\|$. Then, for all $x \in \mathbb{R}$, we have $\|x\| = |x| \cdot \|1\| = a \cdot |x|$. Define an inner product $\langle \bullet, \bullet \rangle$ on $\mathbb{R}$ by $\langle x, y \rangle = a^2 xy$. Given $x \in \mathbb{R}$. We wish to prove that $\|x\|^2 = \langle x, x \rangle$.

We have $|x|^2 = x^2$, so $\|x\|^2 = (a \cdot |x|)^2 = a^2 x^2 = \langle x, x \rangle$. $\square$

NOTE: Every norm on $\mathbb{R}$ is a positive multiple of absolute value, so the collection of norms on $\mathbb{R}$ is very restricted. Every inner product on $\mathbb{R}$ is a positive multiple of multiplication, so the collection of inner products on $\mathbb{R}^2$ is similarly restricted. So, since the absolute value norm comes from the multiplication inner product, it follows that every norm comes from an inner product.

56-3. True or false: For every norm $\| \bullet \|$ on $\mathbb{R}^2$, there is an inner product $\langle \bullet, \bullet \rangle$ on $\mathbb{R}^2$ such that, for all $v \in \mathbb{R}^2$, we have $\|v\|^2 = \langle v, v \rangle$.

Solution: False. Counterexample: Let $| \bullet |$ denote the absolute value on $\mathbb{R}$. Define a norm $\| \bullet \|$ on $\mathbb{R}^2$ by $\|(x, y)\| = |x| + |y|$. Let $\langle \bullet, \bullet \rangle$ be a metric on $\mathbb{R}^2$. Assume, for all $v \in \mathbb{R}^2$, that $\|v\|^2 = \langle v, v \rangle$. We aim for a contradiction.

Let $p := (1, 0)$ and $q := (0, 1)$. We have

$$\langle p, p \rangle = \|p\|^2 = [|1| + |0|]^2 = 1$$

and

$$\langle q, q \rangle = \|q\|^2 = [|0| + |1|]^2 = 1$$

and

$$\langle p + q, p + q \rangle = \|p + q\|^2 = [|1| + |1|]^2 = 4.$$

Then

$$4 = \langle p + q, p + q \rangle = \langle p, p \rangle + 2\langle p, q \rangle + \langle q, q \rangle = 1 + 2\langle p, q \rangle + 1,$$

so $2 = 2\langle p, q \rangle$, so $1 = \langle p, q \rangle$. Then

$$\langle p - q, p - q \rangle = \langle p, p \rangle - 2\langle p, q \rangle + \langle q, q \rangle = 1 - 2 \cdot 1 + 1 = 0,$$

so $\|p - q\|^2 = \langle p - q, p - q \rangle = 0$. On the other hand, we calculate $\|p - q\|^2 = [|1| + |-1|]^2 = 4$. Contradiction. $\square$

NOTE: A norm is always determined by its unit level set. Thus questions about norms can be rephrased in geometric terms.

For a norm coming from an inner product on $\mathbb{R}^2$, the unit level set of the norm will be an ellipse; it will have no corners. It's easy to make a norm on $\mathbb{R}^2$ whose unit level set has corners. One example is $\|(x, y)\| = |x| + |y|$; the unit level set of this norm is a diamond with corners at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Once you realize that this norm, for geometric reasons, cannot possibly come from an inner product, you know that the answer is false.

The argument above is simply an effort to find an algebraic argument to confirm that geometric reasoning. By polarization, when a norm does come from an inner product, that inner product is determined from the norm. One may therefore calculate quantities like $\langle p, q\rangle$ by using the polarization formula. If you accumulate enough calculated inner product values, you can form the matrix of $\langle \bullet, \bullet \rangle$, and see that it's not positive definite, giving the contradiction.

57-1. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}[x]$ the ring of polynomials in x with coefficients in $\mathbb{R}$. Which of the following subsets of $\mathbb{R}[x]$ is a subring of $\mathbb{R}[x]$?

I. All polynomials in $\mathbb{R}[x]$ whose coefficient of x^2 is zero
II. All polynomials in $\mathbb{R}[x]$ all of whose terms have even degree, including the zero polynomial.
III. All polynomials in $\mathbb{R}[x]$ whose coefficients are nonnegative real numbers.

Solution: Let W_I, W_{II} and W_{III} be the three subsets of $\mathbb{R}[x]$ described in I, II and III, respectively. Let $U := \{1, x, x^2, x^3, x^4, x^5, x^6, \ldots\}$ be the set of monomials in x. Then $\mathbb{R}[x]$ is the $\mathbb{R}$-span of U.

Since $x \in W_I$, and since $x \cdot x = x^2 \notin W_I$, it follows that W_I is not closed under multiplication, and so is not a subring of $\mathbb{R}[x]$.

Let $U_{II} := \{1, x^2, x^4, x^6, \ldots\}$. Then W_{II} is the $\mathbb{R}$-span of U_{II}. Since U_{II} is closed under multiplication, so is W_{II}. Since W_{II} is closed under finite $\mathbb{R}$-linear combinations, it follows that W_{II} is closed under subtraction. Then W_{II} is a subring of $\mathbb{R}[x]$.

Since $x, 2x \in W_{III}$, but $x - 2x = -x \notin W_{III}$, it follows that W_{III} is not closed under subtraction, and so is not a subring of $\mathbb{R}[x]$. $\square$

58-1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and injective. Let U be an open subset of $\mathbb{R}$. True or false: $f(U)$ is necessarily an open subset of $\mathbb{R}$.

Solution: True. Proof: Any continuous injective function $\mathbb{R} \to \mathbb{R}$ is either increasing or decreasing. We will assume that f is increasing; the proof for decreasing f is similar.

Let $\mathcal{V}$ be a set of bounded open intervals such that $\cup\mathcal{V} = U$. Then $f(U) = \cup\{f(V) \,|\, V \in \mathcal{V}\}$, so it suffices to prove, for all $V \in \mathcal{V}$, that $f(V)$ is an open subset of $\mathbb{R}$. Given $V \in \mathcal{V}$. We wish to show that $f(V)$ is open.

Let $a := \inf V$ and let $b := \sup V$. Then $V = (a, b)$. So, since f is increasing $f(V) = (f(a), f(b))$. Then $f(V)$ is a bounded open interval, and, in particular, is open. $\square$

58-2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let U be an open subset of $\mathbb{R}$. True or false: $f(U)$ is necessarily an open subset of $\mathbb{R}$.

Solution: False. Counterexample: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$, and let $U := (-1, 1)$. Then $f(U) = [0, 1)$, so $f(U)$ is not open. $\square$

58-3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let U be an open subset of $\mathbb{R}$. True or false: $f^{-1}(U)$ is necessarily an open subset of $\mathbb{R}$.

Solution: True, because the preimage of an open set under a continuous function is always open. $\square$

58-4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let B be a bounded subset of $\mathbb{R}$. True or false: $f(B)$ is necessarily a bounded subset of $\mathbb{R}$.

Solution: True. Proof: Since B is bounded, choose a compact inteval C in $\mathbb{R}$ such that $B \subseteq C$. Then $f(B) \subseteq f(C)$. The image of a compact set under a continuous map is compact, so $f(C)$ is compact. Then $f(C)$ is bounded. So, since $f(B) \subseteq f(C)$, we see that $f(B)$ is bounded. $\square$

58-5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let B be a bounded subset of $\mathbb{R}$. True or false: $f^{-1}(B)$ is necessarily a bounded subset of $\mathbb{R}$.

Solution: False. Counterexample: Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(x^2+1)$, and let $B := (0, 1]$. For all $x \in \mathbb{R}$, we have $x^2+1 \geq 1$, so $f(x) \leq 1$. For all $x \in \mathbb{R}$, we have $x^2 + 1 > 0$, so $f(x) > 0$. Then, for all $x \in \mathbb{R}$, we have $0 < f(x) \leq 1$. Then $f(\mathbb{R}) \subseteq (0, 1] = B$. Then $\mathbb{R} \subseteq f^{-1}(B)$. Thus $f^{-1}(B)$ is bounded. $\square$

NOTE: The function f defined above is not injective, so leaves open the question of whether an injective counterexample exists. It turns out that it's not possible to find an injective rational counterexample. However, the agebraic function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ is injective and continuous and satisfies $f(\mathbb{R}) = (-1, 1)$. So, setting $B := (-1, 1)$, we have $f^{-1}(B) = \mathbb{R}$, and so $f^{-1}(B)$ is bounded.

58-6. Let $f : (0, 1) \to \mathbb{R}$ be continuous. Let B be a bounded subset of $\mathbb{R}$. Assume that $B \subseteq (0, 1)$. True or false: $f(B)$ is necessarily a bounded subset of $\mathbb{R}$.

Solution: False: Counterexample: Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = 1/x$ and let $B := (0, 1)$. For every integer $n > 1$, we have $1/n \in B$, so $n = f(1/n) \in f(B)$. Thus $\{2, 3, 4, \ldots\} \subseteq f(B)$. So, since $\{2, 3, 4, \ldots\}$ is unbounded, it follows that $f(B)$ is unbounded. $\square$
