CALCULUS
Functions and expressions

p. 8 (from §1.3):

A **function** is a rule that assigns to any element of one set (called the **domain**) exactly one element of another set (called the **target**).

(In this course, domain and target will be subsets of $\mathbb{R}$, unless otherwise specified.)

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e.g.: $f = \sqrt{(\bullet) - 3}$

f is the function that subtracts 3, then takes the square root.

If we input 19 into f,

then the output is $\sqrt{19 - 3} = 4$.

"**value**"

Notation to convey this: $f(19) = \sqrt{19 - 3} = 4$

$f(7) = \sqrt{7 - 3} = 2$

Let x be a variable. $f(x) = \sqrt{x - 3}$

NOTE: 2 is an illegal input to f; we say:

"f is undefined at 2."

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e.g.: $f = \sqrt{(\bullet) - 3}$

Domain: $\{$all numbers greater than or equal to $3\} = [3, \infty)$

CONVENTION: Unless otherwise indicated, the domain is
the set of all legal inputs.
Unless otherwise indicated, the target is $\mathbb{R}$.

We indicate domain and target
using this notation: $f : [3, \infty) \to \mathbb{R}$

Let x be a variable. $f(x) = \sqrt{x - 3}$

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e.g.: $f = \sqrt{(\bullet) - 3}$ $\qquad f : [3, \infty) \to \mathbb{R}$

$$f(x) = \sqrt{x - 3}$$

NOTE: f is the function,
$f(x)$ is the corresponding expression of x.

$f : [3, \infty) \to \mathbb{R}$

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e.g.: $f = \sqrt{(\bullet) - 3}$ ⟵ a function $\quad f : [3, \infty) \to \mathbb{R}$

(function of x)

$f(x) = \sqrt{x - 3}$ ⟵ the corresponding expression of x

$f(t) = \sqrt{t - 3}$ ⟵ the corresponding expression of t

(function of t)

etc.

NOTE: f is the function,
$f(x)$ is the corresponding expression of x.

Many don't distinguish between them,
but I typically do (unless I'm being sloppy).

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to any element of one set (called the **domain**)
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e.g.: $f = \sqrt{(\bullet) - 3}$ ⟵ a function $\qquad f : [3, \infty) \to \mathbb{R}$

$f(x) = \sqrt{x - 3}$ ⟵ the corresponding expression of x

$f : [3, \infty) \to \mathbb{R}$

Let $g : [4, \infty) \to \mathbb{R}$ be the function
defined by $g = \sqrt{(\bullet) - 3}$.

i.e. let $g(x) = \sqrt{x - 3}, \quad x \geq 4$.

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e.g.: $f = \sqrt{(\bullet) - 3}$ ← a function

$f(x) = \sqrt{x - 3}$ ← the corresponding expression of x

$f : [3, \infty) \to \mathbb{R}$

Let $g : [4, \infty) \to \mathbb{R}$ be the function defined by $g = \sqrt{(\bullet) - 3}$.

i.e. let $g(x) = \sqrt{x - 3}$, $\quad x \geq 4$.

WARNING: $f \neq g$ $\qquad$ $f(3) = 0$, but g is undefined at 3.

g is the "restriction" of f to $[4, \infty)$, written: $g = f|[4, \infty)$.

Definition: The **image** of a function is its set of outputs.

Notation: $\forall$function v, $\boxed{\text{dom}[v]} :=$ (the domain of v)

$\boxed{\text{im}[v]} :=$ (the image of v)

WARNING: Some use **range** to mean target.
Some use **range** to mean image.

I avoid using "range".

---

e.g.: $f = \sqrt{(\bullet) - 3}$ ← a function

$f(x) = \sqrt{x - 3}$ ← the corresponding expression of x

$\text{dom}[f] = [3, \infty)$

$f : [3, \infty) \to \mathbb{R}$

$\text{im}[f] = [0, \infty)$

$\text{dom}[g] = [4, \infty)$

Let $g : [4, \infty) \to \mathbb{R}$ be the function defined by $g = \sqrt{(\bullet) - 3}$.

i.e. let $g(x) = \sqrt{x - 3}$, $x \geq 4$.

$\text{im}[g] = [1, \infty)$

WARNING: $f \neq g$

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e.g.: $f = \sqrt{(\bullet) - 3}$ ⟵ a function

$f(x) = \sqrt{x - 3}$ ⟵ the corresponding expression of x

$f : [3, \infty) \to \mathbb{R}$

Let $h : [3, \infty) \to [-10, \infty)$ be the function defined by $h = \sqrt{(\bullet) - 3}$.

i.e. let $h(x) = \sqrt{x - 3}$.

NOTE: $f = h$ Changing the target doesn't change the fn, but the target must contain the image. . .

~~Let $w : [3, \infty) \to [10, \infty)$ be def'd by $w(x) = \sqrt{x - 3}$~~

~~MAKES NO SENSE.~~

Fahrenheit is related to Celsius by

$$F = (9/5)C + 32$$

American temperature is related to real temperature by

$$A = (9/5)R + 32$$

These formulas

$$F = (9/5)C + 32$$

$$A = (9/5)R + 32$$

Different variables, same function.

are different,

but are clearly closely related.

Let $h(x) = (9/5)x + 32$.

Dependent variables → $F = h(C)$, $A = h(R)$ ← Independent variables

Advantage of functions (over expressions):

No arbitrary choices of variable names.

$$h = (9/5) \bullet + 32$$

$h(x) = (9/5)x + 32$

h converts Celsius to Fahrenheit.

$h(x) = (9/5)x + 32$

$$h(x) = (9/5)x + 32$$

h converts Celsius to Fahrenheit.

$$k(x) = (5/9)(x - 32)$$

k converts Fahrenheit to Celsius.

$$k(h(x)) = x$$
$$h(k(x)) = x$$

h and k are "inverses".

More on this in a later topic.

e.g.: $f(x) = \frac{1}{x-5}$, $g(x) = \sqrt{x-3}$

$f : \mathbb{R}\backslash\{5\} \to \mathbb{R}$ $g : [3, \infty) \to \mathbb{R}$

$$\mathbb{R}\backslash\{5\} \quad \cap \quad [3, \infty) \quad = \quad [3, \infty)\backslash\{5\}$$

$$f + g : [3, \infty)\backslash\{5\} \to \mathbb{R}$$

$$(f+g)(x) = \frac{1}{x-5} + \sqrt{x-3}$$

The domain of the sum is
the intersection of the domains.

e.g.: $f(x) = \frac{1}{x-5}$, $g(x) = \sqrt{x-3}$

$f : \mathbb{R}\backslash\{5\} \to \mathbb{R}$ $g : [3,\infty) \to \mathbb{R}$

$$\mathbb{R}\backslash\{5\} \quad \cap \quad [3,\infty) \quad = \quad [3,\infty)\backslash\{5\}$$

$$f - g : [3,\infty)\backslash\{5\} \to \mathbb{R}$$

$$(f-g)(x) = \frac{1}{x-5} - \sqrt{x-3}$$

The domain of the difference is
the intersection of the domains.

e.g.: $f(x) = \dfrac{1}{x-5}$, $\quad g(x) = \sqrt{x-3}$

$f : \mathbb{R}\backslash\{5\} \to \mathbb{R}$ $\quad g : [3,\infty) \to \mathbb{R}$

$$\mathbb{R}\backslash\{5\} \quad \cap \quad [3,\infty) \quad = \quad [3,\infty)\backslash\{5\}$$

$$fg : [3,\infty)\backslash\{5\} \to \mathbb{R}$$

$$(fg)(x) = \left[\frac{1}{x-5}\right]\left[\sqrt{x-3}\right]$$

The domain of the product is
the intersection of the domains.

Next: Scalar multiplication

e.g.: $f(x) = \dfrac{1}{x-5}$, $\quad g(x) = \sqrt{x-3}$

$f : \mathbb{R}\backslash\{5\} \to \mathbb{R} \qquad g : [3,\infty) \to \mathbb{R}$

Let $Z := \{x \in [3,\infty) \,|\, g(x) = 0\} = \{3\}$

$$\mathbb{R}\backslash\{5\} \cap [3,\infty) = [3,\infty)\backslash\{5\}$$

$$(\ \mathbb{R}\backslash\{5\} \cap [3,\infty)\ ) \backslash Z = (3,\infty)\backslash\{5\}$$

$$f/g : (3,\infty)\backslash\{5\} \to \mathbb{R}$$

$$(f/g)(x) = \frac{1}{x-5} \Big/ \sqrt{x-3}$$

The domain of the quotient is the intersection of the domains minus the zero set of the denominator.

OPERATIONS ON FUNCTIONS: Scalar multiplication

"**scalar**" means $\underbrace{\text{real}}$ number, *e.g.*, 12, 7, $-\pi$, *etc.*, NOT $\sqrt{-1}$.

e.g.: $f(x) = \dfrac{1}{x-5}$

Let's multiply f by a scalar, say, by 7.

$f : \mathbb{R}\backslash\{5\} \to \mathbb{R}$

$(7f)(x) = \dfrac{7}{x-5}$

$7f : \mathbb{R}\backslash\{5\} \to \mathbb{R}$

Scalar multiplication does not change the domain.

The **linear operations** are:
addition and SCALAR multiplication.

Problem: Starting with $p(x) = x^2$, $q(x) = x$ and $r(x) = 1$, and using only linear operations, what functions can we create?

e.g.: $7q+(-8)r+3p+2q=3p+9q-8r$

is a linear combination of p, q and r
with coefficients 3, 9 and -8.

$$\begin{aligned}(7q+(-8)r+3p+2q)(x) &= 7x+(-8)1+3x^2+2x \\ &= 3x^2+9x-8\end{aligned}$$

is a linear combination of x^2, x and 1
with coefficients 3, 9 and -8.

A **linear combination** is
a sum of scalar multiples.

The **linear operations** are:
addition and SCALAR multiplication.

Problem: Starting with $p(x)=x^2$, $q(x)=x$ and $r(x)=1$,
and using only linear operations,
what functions can we create?

A linear combination of x^2, x and 1 is called
a **quadratic polynomial in** x.
More on this in a later topic.

$3x^2 + 9x - 8$
is a linear combination of x^2, x and 1
with coefficients 3, 9 and -8.

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Problem: Starting with $p(x) = x^2$, $q(x) = x$ and $r(x) = 1$,
and using only linear operations,
what functions can we create?

e.g.: $f(x) = x^3$

Eval. f at 2: $f(2) = 2^3 = 8$

$[x^3]_{x:\to 2} = 2^3 = 8$

e.g.: $g(x) = \sqrt[3]{x}$

Eval. g at 2: $g(2) = \sqrt[3]{2}$

$[\sqrt[3]{x}]_{x:\to 2} = \sqrt[3]{2}$

Same for multiplication . . .

Evaluation distributes over addition:

$$(f + g)(2) = (f(2)) + (g(2))$$

$$\left[x^3 + \sqrt[3]{x}\right]_{x:\to 2} = \left[x^3\right]_{x:\to 2} + \left[\sqrt[3]{x}\right]_{x:\to 2}$$

Evaluation is **additive**.

A **linear combination** is
a sum of scalar multiples.

The **linear operations** are:
addition and SCALAR multiplication.

OPERATIONS ON FUNCTIONS: Evaluation

e.g.: $f(x) = x^3$

Eval. f at 2: $f(2) = 2^3 = 8$

$[x^3]_{x:\to 2} = 2^3 = 8$

e.g.: $g(x) = \sqrt[3]{x}$

Eval. g at 2: $g(2) = \sqrt[3]{2}$

$[\sqrt[3]{x}]_{x:\to 2} = \sqrt[3]{2}$

Next: scalar multiplication . . .

Evaluation distributes over multiplication:

$$(fg)(2) = (f(2))(g(2))$$

$$\left[\left(x^3\right)\left(\sqrt[3]{x}\right)\right]_{x:\to 2} = \left(\left[x^3\right]_{x:\to 2}\right)\left(\left[\sqrt[3]{x}\right]_{x:\to 2}\right)$$

Evaluation is **multiplicative**.

A **linear combination** is
a sum of scalar multiples.

The **linear operations** are:
addition and SCALAR multiplication.

OPERATIONS ON FUNCTIONS: Evaluation

e.g.: $f(x) = x^3$

Eval. f at 2: $f(2) = 2^3 = 8$

$$[x^3]_{x:\to 2} = 2^3 = 8$$

Evaluation commutes with scalar multiplication:

$$(5f)(2) = 5(f(2))$$

$$\left[5x^3\right]_{x:\to 2} = 5\left[x^3\right]_{x:\to 2}$$

Commutes refers to traveling.
The scalar travels. . .

$$\left[5x^3\right]_{x:\to 2}$$

A **linear combination** is
a sum of scalar multiples.

The **linear operations** are:
addition and SCALAR multiplication.

OPERATIONS ON FUNCTIONS: Evaluation

An operation on functions is **linear** if
both it is additive
and it commutes with scalar multiplication.

e.g.: Evaluation is linear.

It is ALSO multiplicative.

Evaluation commutes with scalar multiplication:

$$(5f)(2) = 5(f(2))$$

$$\left[5x^3\right]_{x:\to 2} = 5\left[x^3\right]_{x:\to 2}$$

A **linear combination** is
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The **linear operations** are:
addition and SCALAR multiplication.

An operation on functions is **linear** if
both it is additive
and it commutes with scalar multiplication.

e.g.: Evaluation is linear.

It is ALSO multiplicative.

Most of the operations we study in this course
will be linear, but not multiplicative.

Next: Difference evaluation . . .

A **linear combination** is
a sum of scalar multiples.

The **linear operations** are:
addition and SCALAR multiplication.

OPERATIONS ON FUNCTIONS: Difference evaluation

e.g.: $f(x) = x^3$

Diff. in f from 2 to 1:

$$f|_2^1 = 1^3 - 2^3 = -7$$

$$[x^3]_{x:\to 2}^{x:\to 1} = 1^3 - 2^3 = -7$$

Difference evaluation is linear, but not multiplicative.

Most of the operations we study in this course will be linear, but not multiplicative.

A **linear combination** is a sum of scalar multiples.

The **linear operations** are: addition and SCALAR multiplication.

OPERATIONS ON FUNCTIONS: Difference evaluation

e.g.: $f(x) = x^3$

Diff. in f from 2 to 1:

$$f|_2^1 = 1^3 - 2^3 = -7$$

$$[x^3]_{x:\to 2}^{x:\to 1} = 1^3 - 2^3 = -7$$

Difference evaluation is linear, but not multiplicative.

$$\left[5s^3 + 7s - 4\right]_{s:\to 6}^{s:\to 8} = 5\left(\left[s^3\right]_{s:\to 6}^{s:\to 8}\right) + 7\left([s]_{s:\to 6}^{s:\to 8}\right) - 4\left([1]_{s:\to 6}^{s:\to 8}\right)$$
$$= 5\left(8^3 - 6^3\right) + 7(8-6) - 4(1-1) = \cdots$$

A **linear combination** is
a sum of scalar multiples.

The **linear operations** are:
addition and SCALAR multiplication.

OPERATIONS ON FUNCTIONS: Difference evaluation

e.g.: $f(x) = x^3$

Diff. in f from 2 to 1:

$$f|_2^1 = 1^3 - 2^3 = -7$$

$$[x^3]_{x:\to 2}^{x:\to 1} = 1^3 - 2^3 = -7$$

Difference evaluation is linear, but not multiplicative.

$$\left[5s^3 + 7s - 4\right]_{s:\to 6}^{s:\to 8} = 5\left(\left[s^3\right]_{s:\to 6}^{s:\to 8}\right) + 7\left([s]_{s:\to 6}^{s:\to 8}\right) - 4\left([1]_{s:\to 6}^{s:\to 8}\right)$$

$$= 5\left(8^3 - 6^3\right) + 7(8-6) - 4(1-1) = \cdots$$

$$[(t+3)(4t+7)]_{t:\to 2}^{t:\to 8} \quad \neq \quad \left([t+3]_{t:\to 2}^{t:\to 8}\right)\left([4t+7]_{t:\to 2}^{t:\to 8}\right)$$

---

cocycle identity: $[f(x)]_{x:\to a}^{x:\to c} = \left([f(x)]_{x:\to a}^{x:\to b}\right) + \left([f(x)]_{x:\to b}^{x:\to c}\right)$

STOP

Exercise: Check that

$$[x^3]_{x:\to 5}^{x:\to 9} = \left([x^3]_{x:\to 5}^{x:\to 6}\right) + \left([x^3]_{x:\to 6}^{x:\to 9}\right)$$