CALCULUS
Limits of power functions

The domain of $\bullet^r$, x^r, t^r, *etc.*

Any $r \in \mathbb{Q}\backslash\{0\}$ can be written $r = \pm p/q$,

with $p, q > 0$ integers, not both even.

$$\begin{aligned} 8/28 &= 4/14 = 2/7 \\ 12/32 &= 6/16 = 3/8 \\ 24/56 = 12/28 &= 6/14 = 3/7 \end{aligned}$$

For all $p, q > 0$ integers, p odd, q even,

$x^{p/q} := \sqrt[q]{x^p} = [\sqrt[q]{x}\,]^p$ has domain $x \in [0, \infty)$

and $x^{-p/q} := 1/[x^{p/q}]$ has domain $x \in (0, \infty)$.

$x^{3/8} := \sqrt[8]{x^3} = [\sqrt[8]{x}\,]^3$ has domain $x \in [0, \infty)$

and $x^{-3/8} := 1/[x^{3/8}]$ has domain $x \in (0, \infty)$.

$x^{3/7} := \sqrt[7]{x^3} = [\sqrt[7]{x}\,]^3$ has domain $x \in \mathbb{R}$

and $x^{-3/7} := 1/[x^{3/7}]$ has domain $x \in \mathbb{R}\backslash\{0\}$.

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x^0 has domain $x \in \mathbb{R}\backslash\{0\}$. $\qquad x^0 \underset{x \neq 0}{=} 1$

$x^\pi := \lim\limits_{\mathbb{Q} \ni q \to \pi} x^q$ For all $r \in \mathbb{R}\backslash\mathbb{Q}$, $x^r := \lim\limits_{\mathbb{Q} \ni q \to r} x^q$

Spp *e.g.*: $x^\pi = \lim x^3, x^{3.1}, x^{3.14}, x^{3.141}, \ldots$

The domain of $\bullet^r$, x^r, t^r, *etc.*

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For all $r \in \mathbb{R}\backslash\mathbb{Q}$, $x^r := \lim\limits_{\mathbb{Q} \ni q \to r} x^q$

dom: $x \in [0, \infty)$, if $r > 0$,

dom: $x \in (0, \infty)$, if $r < 0$.

The domain of $\bullet^r$, x^r, t^r, *etc.*

For all $p, q > 0$ integers, p odd, q even,

$x^{p/q} := \sqrt[q]{x^p} = [\sqrt[q]{x}]^p$ has domain $x \in [0, \infty)$

continuous at $x = a$, $\forall a \in (0, \infty)$

continuous from the right at $x = 0$

and $x^{-p/q} := 1/[x^{p/q}]$ has domain $x \in (0, \infty)$.

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dom: $x \in [0, \infty)$, if $r > 0$,

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Spp

The domain of $\bullet^r$, x^r, t^r, *etc.*

For all $p, q > 0$ integers, p odd, q even,

$$x^{p/q} := \sqrt[q]{x^p} = [\sqrt[q]{x}\,]^p \text{ has domain } x \in [0, \infty)$$

- continuous at $x = a$, $\forall a \in (0, \infty)$
- continuous from the right at $x = 0$

and $x^{-p/q} := 1/[x^{p/q}]$ has domain $x \in (0, \infty)$.

- continuous at $x = a$, $\forall a \in (0, \infty)$

For all $p, q > 0$ integers, q odd,

$$x^{p/q} := \sqrt[q]{x^p} = [\sqrt[q]{x}\,]^p \text{ has domain } x \in \mathbb{R}$$

- continuous at $x = a$, $\forall a \in \mathbb{R}$

and $x^{-p/q} := 1/[x^{p/q}]$ has domain $x \in \mathbb{R}\backslash\{0\}$.

- continuous at $x = a$, $\forall a \in \mathbb{R}\backslash\{0\}$

x^0 has domain $x \in \mathbb{R}\backslash\{0\}$.

- continuous at $x = a$, $\forall a \in \mathbb{R}\backslash\{0\}$

For all $r \in \mathbb{R}\backslash\mathbb{Q}$, $x^r := \lim\limits_{\mathbb{Q} \ni q \to r} x^q$

dom: $x \in [0, \infty)$, if $r > 0$,
dom: $x \in (0, \infty)$, if $r < 0$.

- continuous at $x = a$, $\forall a > 0$
- continuous from the right at $x = 0$, if $r > 0$

Spp

The domain of $\bullet^r$, x^r, t^r, *etc.*

For all $p, q > 0$ integers, p odd, q even,

$$x^{p/q} := \sqrt[q]{x^p} = [\sqrt[q]{x}\,]^p \text{ has domain } x \in [0, \infty)$$

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For all $r \in \mathbb{R}\backslash\mathbb{Q}$, $x^r := \lim_{\mathbb{Q} \ni q \to r} x^q$

dom: $x \in [0, \infty)$, if $r > 0$,

dom: $x \in (0, \infty)$, if $r < 0$.

Spp

$\forall a \in \mathbb{R}$, $\bullet^a$ is contin. on its domain.

Limits at ∞ of power functions x^r

$$x^0 \underset{x \neq 0}{=} 1$$

$$\lim_{x \to \infty} x^0 = 1$$

$$\forall r > 0, \lim_{x \to \infty} x^r = \infty$$

$$\lim_{x \to \infty} x^3 = \infty \qquad \text{“}\infty^3 = \infty\text{”}$$

$$\lim_{x \to \infty} x^{2/9} = \infty \qquad \text{“}\infty^2 = \infty\text{”} \qquad \text{“}\sqrt[9]{\infty} = \infty\text{”}$$

$$\lim_{x \to \infty} x^{\sqrt{2}} = \infty$$

Spp

Limits at ∞ of power functions x^r ☺

$$\lim_{x\to\infty} x^0 = 1$$

$$\forall r > 0, \lim_{x\to\infty} x^r = \infty$$

$$\forall r < 0, \lim_{x\to\infty} x^r = 0$$

"$1/\infty = 0$"

$$\frac{1}{x^r} = x^{-r}$$

$$\forall r > 0, \lim_{x\to\infty} \frac{1}{x^r} = 0$$

$$\forall r > 0, \lim_{x\to\infty} x^{-r} = 0$$

$$r :\to 3, 2/9, \sqrt{2}$$

$$\lim_{x\to\infty} x^{-3} = 0$$

$$\lim_{x\to\infty} x^{-2/9} = 0$$

$$\lim_{x\to\infty} x^{-\sqrt{2}} = 0$$

Spp

Limits at $-\infty$ of power functions x^r

$\lim_{x \to -\infty} x^0 = 1$ $\quad x^0 \underset{x \neq 0}{=} 1$

$\forall r \notin \mathbb{Q}$, $\quad x^r$ is not defined on $x \in (-\infty, 0)$,

so $\lim_{x \to -\infty} x^r$ DNE.

$\forall r \in \left\{ \frac{\text{odd}}{\text{even}} \right\}$,

x^r is not defined on $x \in (-\infty, 0)$,

so $\lim_{x \to -\infty} x^r$ DNE.

Spp

Limits at $-\infty$ of power functions x^r

$$\lim_{x\to-\infty} x^0 = 1$$

$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\},\ \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall r \notin \mathbb{Q},\ \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall \text{positive } r \in \left\{\frac{\text{even}}{\text{odd}}\right\},$$

$$\lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\},$$

$$\lim_{x\to-\infty} x^r \text{ DNE}$$

Limits at $-\infty$ of power functions x^r

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$$\forall \text{positive } r \in \left\{\frac{\text{even}}{\text{odd}}\right\}, \lim_{x\to-\infty} x^r = \infty$$

"$(-\infty)^2 = \infty$"

"$\sqrt[3]{\infty} = \infty$"

$$\lim_{x\to-\infty} x^{2/3} = \lim_{x\to-\infty} \sqrt[3]{x^2} = \infty$$

Spp

Limits at $-\infty$ of power functions x^r

$$\lim_{x\to-\infty} x^0 = 1$$

$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\}, \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall r \notin \mathbb{Q}, \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall \text{positive } r \in \left\{\frac{\text{even}}{\text{odd}}\right\}, \lim_{x\to-\infty} x^r = \infty$$

"$1/\infty = 0$"

so $$\lim_{x\to-\infty} \frac{1}{x^r} = 0,$$

i.e., $$\lim_{x\to-\infty} x^{-r} = 0.$$

$$\forall \text{negative } r \in \left\{\frac{\text{even}}{\text{odd}}\right\}, \lim_{x\to-\infty} x^r = 0.$$

$$\lim_{x\to-\infty} x^{-2/3} = 0$$

Spp

Limits at $-\infty$ of power functions x^r

$$\lim_{x \to -\infty} x^0 = 1$$

$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\}, \lim_{x \to -\infty} x^r \text{ DNE}$$

$$\forall r \notin \mathbb{Q}, \lim_{x \to -\infty} x^r \text{ DNE}$$

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Spp

Limits at $-\infty$ of power functions x^r

$$\lim_{x\to-\infty} x^0 = 1$$

$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\}, \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall r \notin \mathbb{Q}, \lim_{x\to-\infty} x^r \text{ DNE}$$

"$(-\infty)^3 = -\infty$"

$$\forall \text{positive } r \in \left\{\frac{\text{odd}}{\text{odd}}\right\}, \lim_{x\to-\infty} x^r = -\infty$$

"$\sqrt[5]{-\infty} = -\infty$"

$$\lim_{x\to-\infty} x^{3/5} = \lim_{x\to-\infty} \sqrt[5]{x^3} = -\infty$$

$$\forall \text{positive } r \in \left\{\frac{\text{even}}{\text{odd}}\right\}, \lim_{x\to-\infty} x^r = \infty$$

$$\forall \text{negative } r \in \left\{\frac{\text{even}}{\text{odd}}\right\}, \lim_{x\to-\infty} x^r = 0$$

Limits at $-\infty$ of power functions x^r

$$\lim_{x \to -\infty} x^0 = 1$$

$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\}, \lim_{x \to -\infty} x^r \text{ DNE}$$

$$\forall r \notin \mathbb{Q}, \lim_{x \to -\infty} x^r \text{ DNE}$$

$$\forall \text{positive } r \in \left\{\frac{\text{odd}}{\text{odd}}\right\}, \lim_{x \to -\infty} x^r = -\infty$$

"$1/(-\infty) = 0$"

so $\displaystyle\lim_{x \to -\infty} \frac{1}{x^r} = 0,$

$$\forall \text{positive } r \in \left\{\frac{\text{even}}{\text{odd}}\right\}, \lim_{x \to -\infty} x^r = \infty$$

$$\forall \text{negative } r \in \left\{\frac{\text{even}}{\text{odd}}\right\}, \lim_{x \to -\infty} x^r = 0$$

Spp

Limits at $-\infty$ of power functions x^r

$$\lim_{x\to-\infty} x^0 = 1$$

$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\}, \quad \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall r \notin \mathbb{Q}, \quad \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall \text{positive } r \in \left\{\frac{\text{odd}}{\text{odd}}\right\}, \quad \lim_{x\to-\infty} x^r = -\infty$$

$$\text{so } \lim_{x\to-\infty} \frac{1}{x^r} = 0,$$

$$\text{i.e., } \lim_{x\to-\infty} x^{-r} = 0.$$

$$\forall \text{negative } r \in \left\{\frac{\text{odd}}{\text{odd}}\right\}, \quad \lim_{x\to-\infty} x^r = 0$$

$$\lim_{x\to-\infty} x^{-3/5} = 0$$

Spp

Limits at $-\infty$ of power functions x^r ☺

$$\lim_{x \to -\infty} x^0 = 1$$

$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\}, \lim_{x \to -\infty} x^r \text{ DNE}$$

$$\forall r \notin \mathbb{Q}, \lim_{x \to -\infty} x^r \text{ DNE}$$

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Limits at $-\infty$ of power functions x^r ☺

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$$\forall r \in \left\{\frac{\text{odd}}{\text{even}}\right\}, \quad \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall r \notin \mathbb{Q}, \quad \lim_{x\to-\infty} x^r \text{ DNE}$$

$$\forall \text{positive } r \in \left\{\frac{\text{even}}{\text{odd}}\right\}, \quad \lim_{x\to-\infty} x^r = \infty$$

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SKILL
limits of powers
at numbers,
& at $\pm\infty$

Spp

Exercise: Find the numbers at which the fn f f def'd below is discontinuous. At which of them is f continuous from the right, from the left, or neither? Sketch the gph of f.

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

SKILL
find types of continuity

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continuous at all numbers
except *possibly* 0 and 1,
which we proceed to investigate. . .

Exercise: Find the numbers at which the fn f f def'd below is discontinuous. At which of them is f continuous from the right, from the left, or neither? Sketch the gph of f.

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

SKILL
find types of continuity

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = [-x]_{x :\to 0} = 0$$

$$f(0) = [e^x]_{x :\to 0} = e^0 = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^x \;=\; (= f(0))$$

f is continuous from the right at 0,
but not from the left

Exercise: Find the numbers at which the fn f f def'd below is discontinuous. At which of them is f continuous from the right, from the left, or neither? Sketch the gph of f.

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

SKILL
find types of continuity

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} e^x$$

$$\| $$

$$f(1) = [e^x]_{x :\to 1} = e^1 = e$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3 - x) = [3 - x]_{x :\to 1} = 3 - 1$$

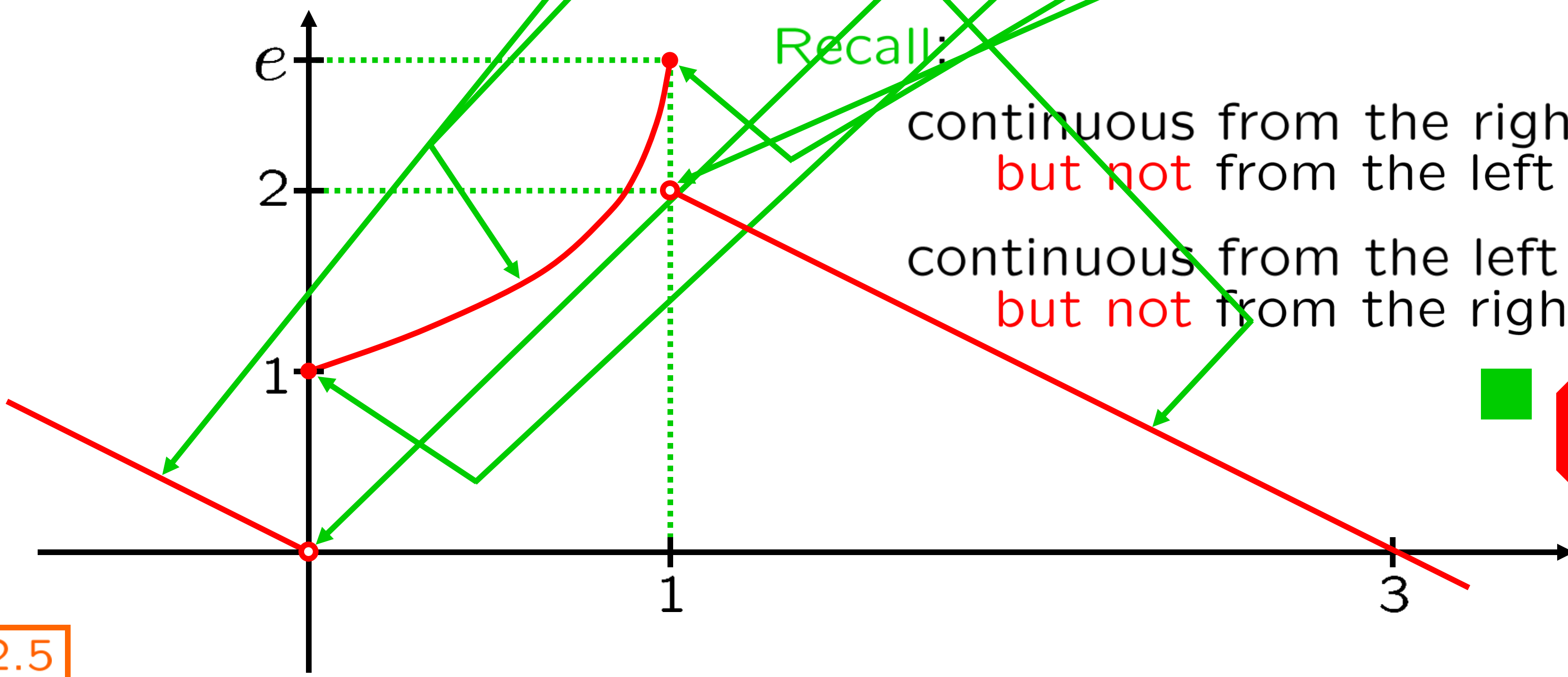
$$= 2$$

f is continuous from the left at 1,
but not from the right ■

Exercise: Find the numbers at which the fn f def'd below is discontinuous. At which of them is f continuous from the right, from the left, or neither? Sketch the gph of f.

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

SKILL
find types of continuity



Recall:

continuous from the right at 0, but not from the left

continuous from the left at 1, but not from the right

STOP