# CALCULUS
## Problems involving horizontal asymptotes

Exercise: For the function $g$ whose graph appears below, state the following.

(a) $\lim_{x \to \infty} g(x)$ 3

(b) $\lim_{x \to -\infty} g(x)$ 1

(c) $\lim_{x \uparrow -2} g(x)$ $-3$

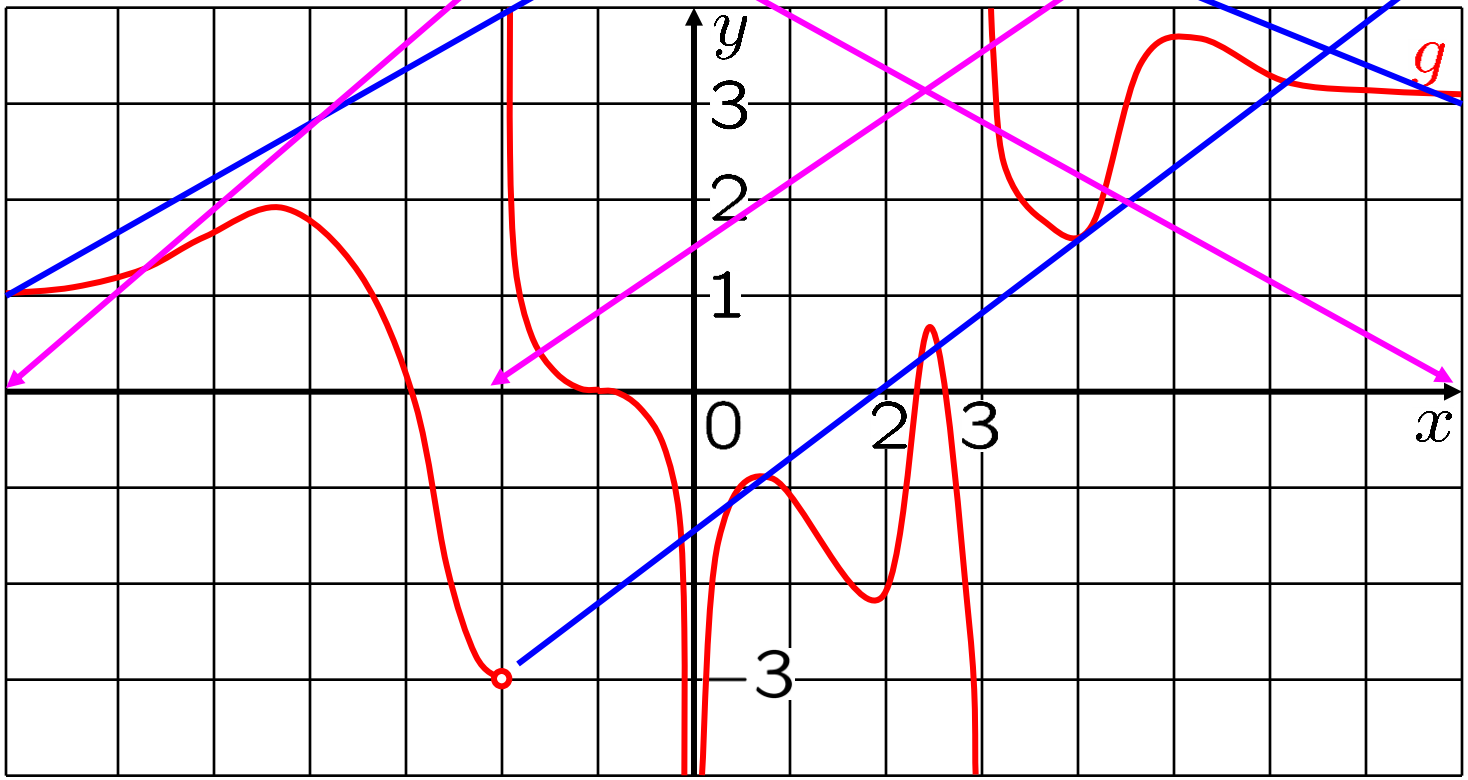
(d) $\lim_{x \uparrow 3} g(x)$

(e) $\lim_{x \downarrow 3} g(x)$

(f) $\lim_{x \to 3} g(x)$

(g) $\lim_{x \to 0} g(x)$

(h) The equations of the asymptotes




SKILL
lim from gph



Spp

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(e) $\lim_{x\downarrow 3} g(x)$ $\infty$

(f) $\lim_{x\to 3} g(x)$ DNE

(g) $\lim_{x\to 0} g(x)$

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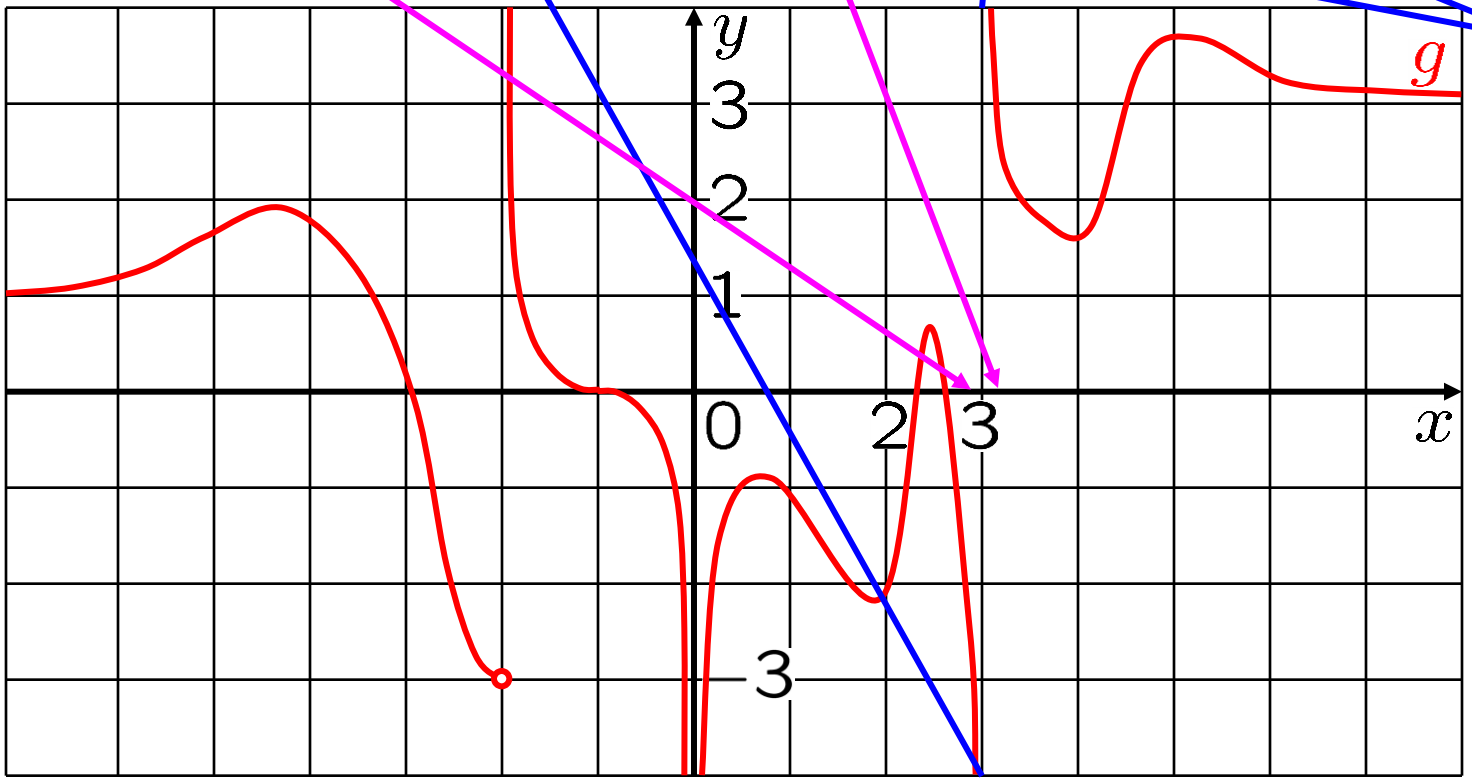


SKILL
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Spp

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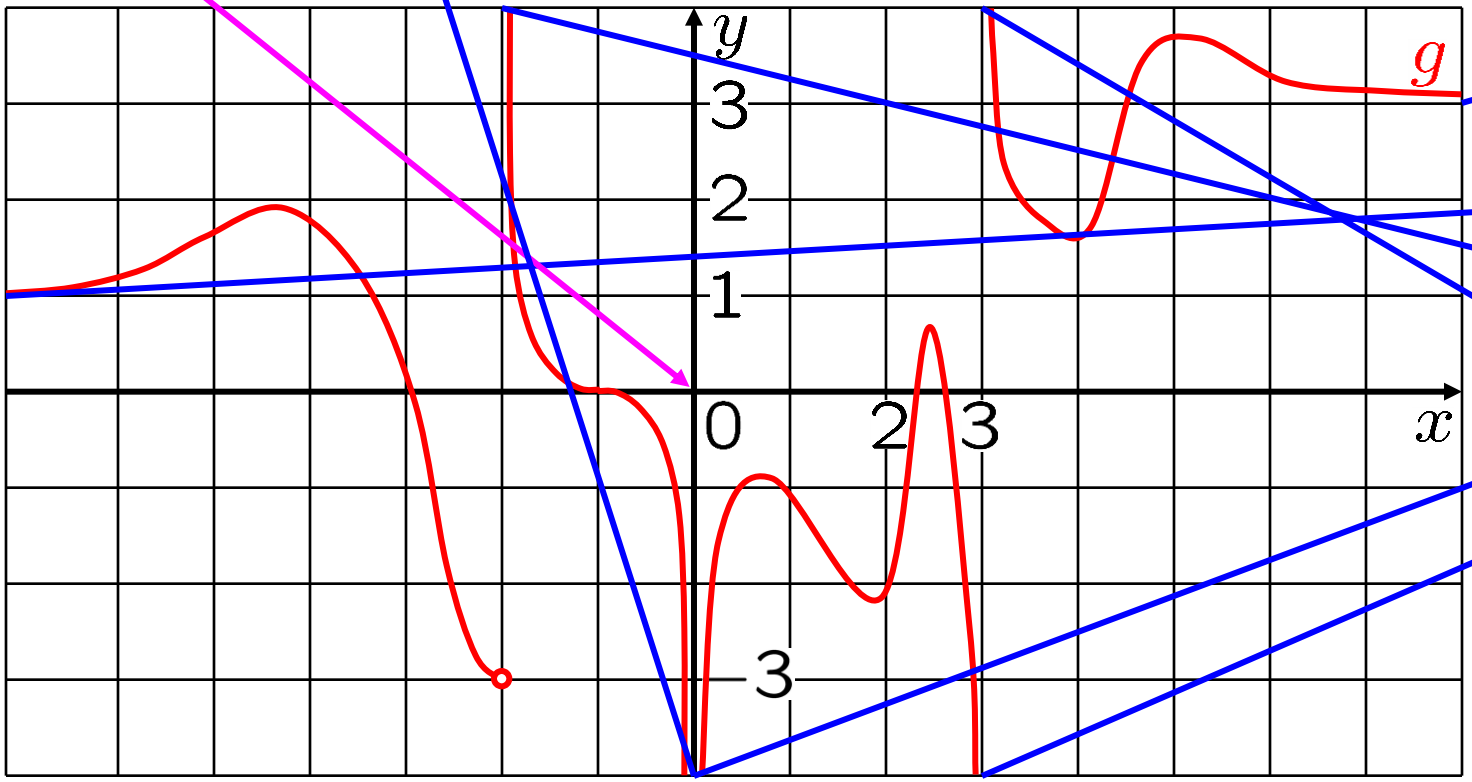


horizontal:
$y = 3$
$y = 1$

vertical:
$x = -2$
$x = 0$
$x = 3$

SKILL
lim from gph



Spp

Exercise:Compute $\lim_{x \to \infty} \frac{5x + 7}{4x - 8}$.

SKILL
limit rat'l fn

$$\| \frac{5}{4} \blacksquare$$

Spp

Exercise: Compute $\lim_{x\to\infty} \dfrac{8x-6}{\sqrt{4x^2+1}}$.

SKILL
limit rat'l sqrt

$$\| $$

$$\lim_{x\to\infty} \frac{8x}{2x} = 4 \blacksquare$$

Exercise: Compute $\lim_{x\to-\infty} \dfrac{8x-6}{\sqrt{4x^2+1}}$.

$$\| $$

$$\lim_{x\to-\infty} \frac{8x}{-2x} = -4 \blacksquare$$



Spp

Exercise:Compute $\lim_{x\to-\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL

limit rat'l sqrt

$\infty$ =



"$\infty+\infty=\infty$"



Spp

Exercise:Compute $\lim_{x \to \infty} \left( -2x + \sqrt{4x^2 + 7x + 6} \right)$.

SKILL

limit rat'l sqrt



"$-\infty + \infty$"

is indeterminate



Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL

limit rat'l sqrt

$\not=$

$\lim_{x\to\infty}(-2x+2x)$

$\|$

$0$

---

$\sqrt{4x^2+7x+6} \underset{x\to\infty}{\sim} \sqrt{4x^2}$

$\|x\geq 0$

$2x$

---

$f\sim g$

$p\sim q$

WARNING: $\not\Downarrow$

$f+p\sim g+q$



Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL
limit rat'l sqrt

$\exists C>0$ s.t. $\forall x\geq C,\quad g(x)\geq 0$ and $q(x)\geq 0$

$g,q\geq 0$ near $\infty$

$$\begin{array}{c} f\sim g \\ p\sim q \end{array}$$

$$\Downarrow$$

$$f+p\sim g+q$$

---

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Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL
limit rat'l sqrt

$$g, q \geq 0 \text{ near } \infty$$

$$f \sim g$$
$$p \sim q$$

$$\Downarrow$$

$$f + p \sim g + q$$

*e.g.*: $2x, \sqrt{4x^2} \geq 0$, for $x$ near $\infty$

$x \geq 0$

$$2x \underset{x\to\infty}{\simeq} 2x$$

$$\sqrt{4x^2+7x+6} \underset{x\to\infty}{\simeq} \sqrt{4x^2}$$

Spp so $2x+\sqrt{4x^2+7x+6} \underset{x\to\infty}{\simeq} 2x+\sqrt{4x^2}$

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL
limit rat'l sqrt

$$g, q \geq 0 \text{ near } \infty$$
$$f \sim g$$
$$p \sim q$$
$$\Downarrow$$
$$f+p \sim g+q$$

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so $2x+\sqrt{4x^2+7x+6} \underset{x\to\infty}{\simeq} 2x+\sqrt{4x^2}$

Spp

Exercise: Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL
limit rat'l sqrt

---

$\exists C>0$ s.t. $\forall x\geq C,\quad g(x)\leq 0$ and $q(x)\leq 0$

$g, q \leq 0$ near $\infty$

$f\sim g$
$p\sim q$

$\Downarrow$

$f+p\sim g+q$

---

*e.g.*: $2x, \sqrt{4x^2}\geq 0$, for $x$ near $\infty$

$$2x \underset{x\to\infty}{\simeq} 2x$$

$$\sqrt{4x^2+7x+6} \underset{x\to\infty}{\simeq} \sqrt{4x^2}$$

so $2x+\sqrt{4x^2+7x+6} \underset{x\to\infty}{\simeq} 2x+\sqrt{4x^2}$

Spp

Problem: opposite signs.

Exercise: Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL
limit rat'l sqrt

New idea: Let's conjugate!

$$g, q \le 0 \text{ near } \infty$$

$$f \sim g$$
$$p \sim q$$

$$\Downarrow$$

$$f + p \sim g + q$$

*e.g.*: $-2x, -\sqrt{4x^2} \le 0$, for $x$ near $\infty$

$$-2x \underset{x\to\infty}{\simeq} -2x$$

$$-\sqrt{4x^2+7x+6} \underset{x\to\infty}{\simeq} -\sqrt{4x^2}$$

so $-2x-\sqrt{4x^2+7x+6} \underset{x\to\infty}{\simeq} -2x-\sqrt{4x^2}$

Exercise:Compute $\lim_{x\to\infty}\left(\boxed{-2x+\sqrt{4x^2+7x+6}}\right)$.

SKILL
limit rat'l sqrt

$$\frac{-2x+\sqrt{4x^2+7x+6}}{1}\,\frac{-2x-\sqrt{4x^2+7x+6}}{-2x-\sqrt{4x^2+7x+6}}$$

$$=$$

$$\frac{(2x)^2-\left(\sqrt{4x^2+7x+6}\right)^2}{-2x-\sqrt{4x^2+7x+6}}$$

$$=$$

$$\frac{(4x^2)-(4x^2+7x+6)}{-2x-\sqrt{4x^2+7x+6}}$$

Spp

$-2x-\sqrt{4x^2+7x+6}\underset{x\to\infty}{\simeq}-2x-\sqrt{4x^2}$

Exercise:Compute $\lim_{x\to\infty}\left(\boxed{-2x+\sqrt{4x^2+7x+6}}\right)$.

SKILL

limit rat'l sqrt

$$=\!\!=$$

$$\frac{(\cancel{4x^2})-(\cancel{4x^2}+7x+6)}{-2x-\sqrt{4x^2+7x+6}}$$

$$\|$$

$$\frac{-7x-6}{-2x-\sqrt{4x^2+7x+6}}$$

$$\frac{(4x^2)-(4x^2+7x+6)}{-2x-\sqrt{4x^2+7x+6}}$$

$$-2x-\sqrt{4x^2+7x+6}\underset{x\to\infty}{\simeq}-2x-\sqrt{4x^2}$$

Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$. 7/4

SKILL
limit rat'l sqrt

$$=\frac{(4x^2)-(4x^2+7x+6)}{-2x-\sqrt{4x^2+7x+6}}$$

$\wr\, x\to\infty$

$$=\frac{-7x-6}{-2x-\sqrt{4x^2+7x+6}}$$

$\|\, x>0$

$$=\frac{-7x}{-2x-2x}\overset{x\neq 0}{=}\frac{-7}{-2-2}\overset{x\to\infty}{\to}\frac{7}{4}$$

$\sqrt{4x^2}\underset{x\geq 0}{=}2x$

$-2x-\sqrt{4x^2+7x+6}\underset{x\to\infty}{\simeq}-2x-\sqrt{4x^2}$

Spp

Exercise: Compute $\lim_{x\to\infty}\left(\boxed{-2x+\sqrt{4x^2+7x+6}}\right)$.

SKILL
limit rat'l sqrt

7/4

$$=\frac{(\cancel{4x^2})-(\cancel{4x^2}+7x+6)}{-2x-\sqrt{4x^2+7x+6}}$$

$\wr\, x\to\infty$

$$\frac{-7x\not>6}{-2x-\sqrt{4x^2\not>7x+6}}$$

$\|\, x>0$

$$\frac{-7x}{-2x-2x}\overset{x\neq 0}{=}\frac{-7}{-2-2}\to\frac{7}{4}$$

not traditional in freshman calc
to teach asymptotic methods; instead. . .



Spp

Exercise:Compute $\lim_{x\to\infty}\left(\boxed{-2x+\sqrt{4x^2+7x+6}}\right)$.

SKILL

limit rat'l sqrt

$$=\!\!=$$

$$\frac{(\cancel{4x^2})-(\cancel{4x^2}+7x+6)}{-2x-\sqrt{4x^2+7x+6}}$$

$$=\!\!=$$

$$\frac{-7x-6}{-2x-\sqrt{4x^2+7x+6}}$$

Back up to here.

not traditional in freshman calc
to teach asymptotic methods; instead. . .



Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL

limit rat'l sqrt

$$=\!\!=$$

$$\frac{(\cancel{4x^2})-(\cancel{4x^2}+7x+6)}{-2x-\sqrt{4x^2+7x+6}}$$

$$=\!\!=$$

$$\frac{-7x-6}{-2x-\sqrt{4x^2+7x+6}}$$

$$=\!\!=$$

$$\frac{(-7x-6)/x}{\left(-2x-\sqrt{4x^2+7x+6}\right)/x}$$

Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL

limit rat'l sqrt

$$=$$

$$\frac{(-7x-6)/x}{\left(-2x-\sqrt{4x^2+7x+6}\right)/x}$$

$$\frac{(-7x-6)/x}{\left(-2x-\sqrt{4x^2+7x+6}\right)/x}$$



Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL

limit rat'l sqrt

$$=\!\!=$$

$$\frac{(-7x-6)/x}{\left(-2x-\sqrt{4x^2+7x+6}\right)/x}$$

$$\|x>0$$

$$\frac{-7-(6/x)}{-2-\left[\left(\sqrt{4x^2+7x+6}\right)/\sqrt{x^2}\right]}$$

$$\|$$

$$\frac{-7-(6/x)}{-2-\sqrt{4+(7/x)+(6/x^2)}}$$



Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL

limit rat'l sqrt

$$=$$

$$\frac{-7-(6/x)}{-2-\sqrt{4+(7/x)+(6/x^2)}}$$

$x \to \infty$

$$\frac{-7-(6/x)}{-2-\sqrt{4+(7/x)+(6/x^2)}}$$



Spp

Exercise:Compute $\lim_{x\to\infty}\left(-2x+\sqrt{4x^2+7x+6}\right)$.

SKILL

limit rat'l sqrt

$$= \frac{-7-(6/x)}{-2-\sqrt{4+(7/x)+(6/x^2)}}$$

($6/x \to 0$, $7/x \to 0$, $6/x^2 \to 0$)

$x \to \infty$

$$\frac{-7-(0)}{-2-\sqrt{4+(0)+(0)}} = \frac{-7}{-2-2} = \frac{7}{4}$$



Spp

Exercise:Compute $\lim_{x\to\infty} \dfrac{x^3-4x+7}{2+4x-8x^2}.$

SKILL
limit rat'l fn

$$\lim_{x\to\infty} \frac{x^3}{-8x^2}$$

$$\lim_{x\to\infty} \frac{x}{-8}$$

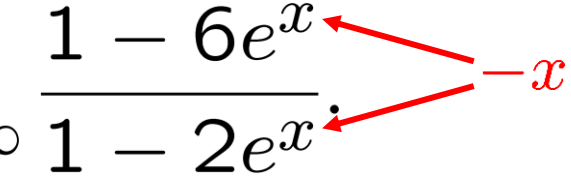
$$-\infty$$

■



Spp

Exercise:Compute $\lim_{x\to\infty} \dfrac{1-6e^x}{1-2e^x}$.




SKILL
limit at $\infty$

$1-6e^x \underset{x\to\infty}{\sim} -6e^x$

$1-2e^x \underset{x\to\infty}{\sim} -2e^x$

$$\lim_{x\to\infty} \frac{-6e^x}{-2e^x}$$

$$\lim_{x\to\infty} \frac{-6}{-2}$$

$$\lim_{x\to\infty} 3$$

$$3$$

■



Spp

Exercise:Compute $\lim_{x\to\infty} \dfrac{1-6e^{-x}}{1-2e^{-x}}$.

SKILL
limit at $\infty$

easier:||

$$\frac{\lim_{x\to\infty}(1-6e^{-x})}{\lim_{x\to\infty}(1-2e^{-x})} = \frac{1}{1} = 1 \blacksquare$$

WARNING: $1-6e^{-x} \not\sim 6e^{-x}$ (as $x\to\infty$)

$1-6e^{-x} \sim 1$ (as $x\to\infty$)

$1-2e^{-x} \sim 1$ (as $x\to\infty$)

$$\lim_{x\to\infty} \frac{1-6e^{-x}}{1-2e^{-x}} = \lim_{x\to\infty} \frac{1}{1} = 1$$



Spp

Exercise: Compute $\lim_{x\to\infty}(e^{-5x}\sin x)$.

SKILL
oscillatory limit

$e^{-5x}$: $+$, $\to 0$

$\sin x$: bdd between $-1$ and $1$

$= 0$

Upper envelope: $e^{-5x} \xrightarrow{x\to\infty} 0$

$\Big| \vee$

$e^{-5x}\sin x \xrightarrow[\text{Squeeze Th'm at } \infty]{x\to\infty} 0$

$\Big| \vee$

Lower envelope: $-e^{-5x} \xrightarrow{x\to\infty} 0$



Spp

Exercise: Compute $\lim_{x\to\infty}(e^{5x}\sin x)$.

$e^{5x}\to\infty$

bdd between $-1$ and $1$

SKILL
oscillatory limit

Turns out that $\lim_{x\to\infty}(e^{5x}\sin x)$ does not exist. ■

---

Upper envelope: $e^{5x} \xrightarrow{x\to\infty} \infty$

$\wr\vee$

$e^{5x}\sin x$ $\not\exists$

$\wr\vee$

Lower envelope: $-e^{5x} \xrightarrow{x\to\infty} -\infty$

NO info from the squeeze theorem

Exercise:Compute $\lim_{x\to\infty}(e^{5x}\sin x)$.

SKILL
oscillatory limit

Turns out that $\lim_{x\to\infty}(e^{5x}\sin x)$ does not exist. ■

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Exercise:Compute $\lim_{x\to-\infty}(e^{5x}\sin x)$.

$e^{5x}$ → 0

$\sin x$: bdd between $-1$ and $1$

Squeeze Th'm at $-\infty$ implies that

$$\lim_{x\to-\infty}(e^{5x}\sin x) = 0. \blacksquare$$



Spp

$$\lim_{x\to\infty}(1/\sqrt[3]{x}) = 0$$

Intuition: $x$ large $\Rightarrow$ $\underbrace{|(1/\sqrt[3]{x}) - 0|}_{1/\sqrt[3]{x}} \approx 0$

Exercise: How large do we have to take $x$ so that $1/\sqrt[3]{x} < 0.000001$?

SKILL
find input spec

$0.000001 = 10^{-6}$

$\sqrt[3]{x} > 10^6$

$x > (10^6)^3 = 10^{18}$