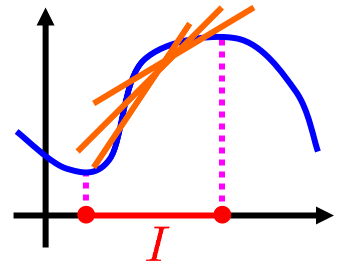
CALCULUS
Derivative tests and graphing

DEFINITION: Let I be an interval.

A function f is called **increasing on** I if

$f(s) < f(t)$ whenever $s, t \in I$ and $s < t$.

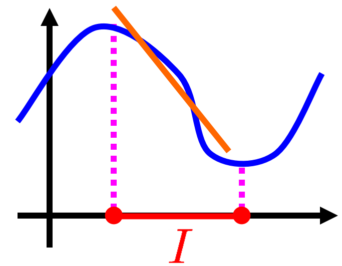
"secant lines run uphill" (slopes > 0)



A function f is called **decreasing on** I if

$f(s) > f(t)$ whenever $s, t \in I$ and $s < t$.

"secant lines run downhill" (slopes < 0)

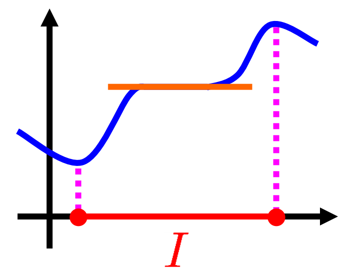


(semi-increasing)

A function f is called **nondecreasing on** I if

$f(s) \leq f(t)$ whenever $s, t \in I$ and $s \leq t$.

"secant lines don't run downhill" (slopes ≥ 0)

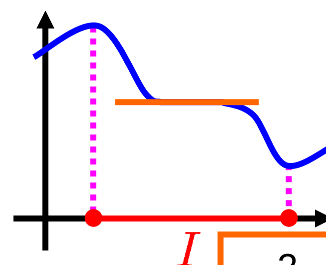


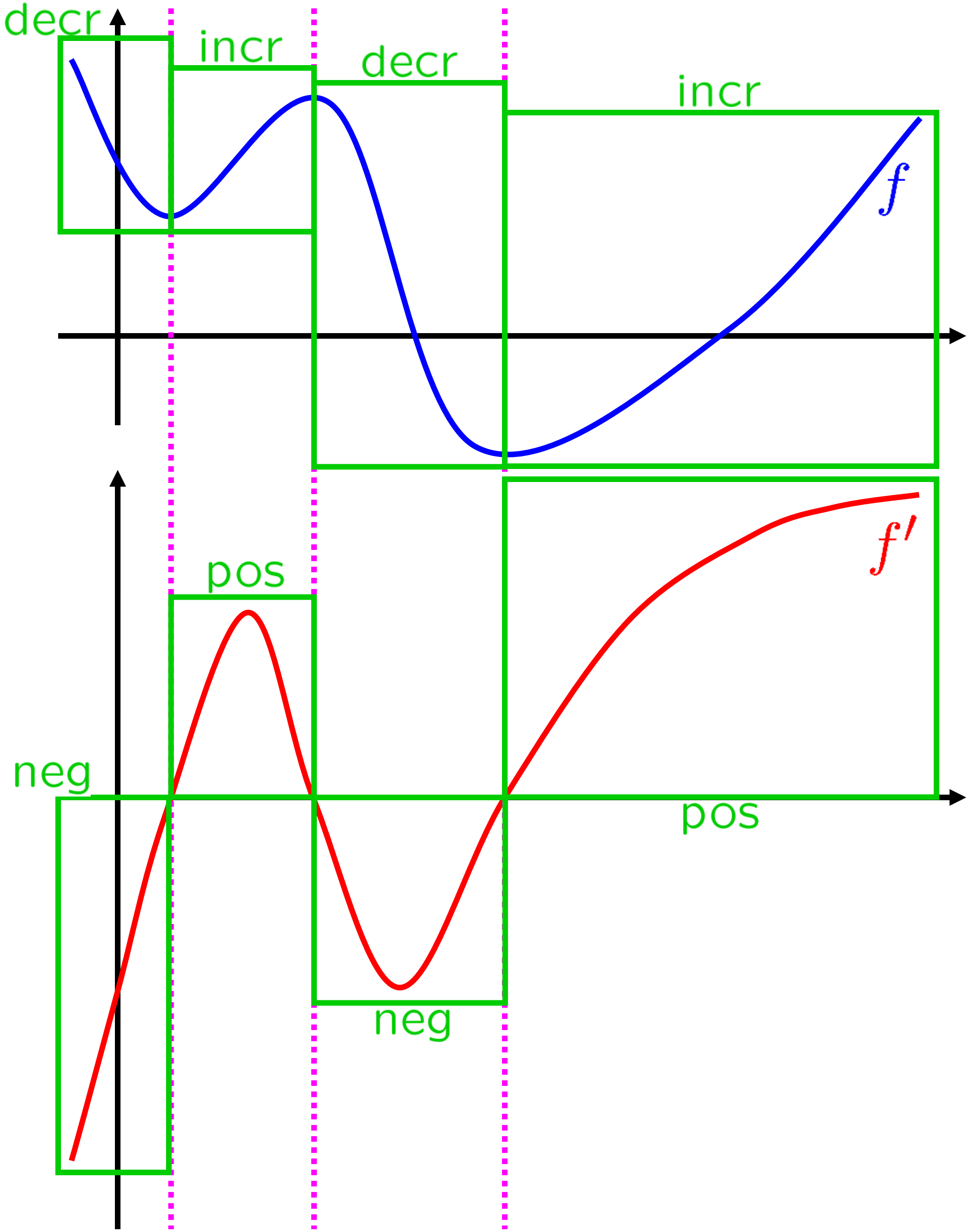
(semi-decreasing)

A function f is called **nonincreasing on** I if

$f(s) \geq f(t)$ whenever $s, t \in I$ and $s \leq t$.

"secant lines don't run uphill" (slopes ≤ 0)




decr
incr
decr
incr
f
pos
f'
neg
pos
neg

INCREASING TEST:

If $f'(x) > 0$, for all x in an interval I,
then f is increasing on I.

DECREASING TEST:

If $f'(x) < 0$, for all x in an interval I,
then f is decreasing on I.

INCREASING/DECREASING TEST:

If $f'(x) > 0$ (resp. < 0), for all x in an interval I,
then f is increasing (resp. decreasing) on I.

Spp

EXAMPLE: Find where the function

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 7$$

is increasing and where it is decreasing.

$$\begin{aligned} f'(x) &= 12x^3 + 12x^2 - 24x \\ &= 12x(x^2 + x - 2) \\ &= 12x(x+2)(x-1) \end{aligned}$$

f'	neg	0	pos	0	neg	0	pos
		-2		0		1	
f	decr on $(-\infty, -2)$		incr on $(-2, 0)$		decr on $(0, 1)$		incr on $(1, \infty)$

INCREASING/DECREASING TEST:

If $f'(x) > 0$ (resp. < 0), for all x in an interval I, then f is increasing (resp. decreasing) on I.

IMPROVED INCREASING/DECREASING TEST:

(Assuming f continuous on I.)

If $f'(x) > 0$ (resp. < 0), for all but finitely many x in an interval I, then f is increasing (resp. decreasing) on I.

Spp

EXAMPLE: Find where the function

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 7$$

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$$\begin{aligned} f'(x) &= 12x^3 + 12x^2 - 24x \\ &= 12x(x^2 + x - 2) \\ &= 12x(x+2)(x-1) \end{aligned}$$

NOT neg at -2 only one number

f'	neg	0	pos	0	neg	0	pos
		-2		0		1	
f	decr on $(-\infty, -2)$ and even on $(-\infty, -2]$		incr on $(-2, 0)$		decr on $(0, 1)$		incr on $(1, \infty)$

IMPROVED INCREASING/DECREASING TEST:

If $f'(x) > 0$ (resp. < 0), (Assuming f continuous on I.)

for all but finitely many x in an interval I,

then f is increasing (resp. decreasing) on I.

Spp

EXAMPLE: Find where the function

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 7$$

is increasing and where it is decreasing.

$$f'(x) = 12x^3 + 12x^2 - 24x$$
$$= 12x(x^2 + x - 2)$$
$$= 12x(x+2)(x-1)$$

NOT pos at $-2, 0$ only two numbers

f'	neg	0	pos	0	neg	0	pos
f	decr on $(-\infty, -2)$ and even on $(-\infty, -2]$	-2	incr on $(-2, 0)$ and even on $[-2, 0]$	0	decr on $(0, 1)$	1	incr on $(1, \infty)$

IMPROVED INCREASING/DECREASING TEST: (Assuming f continuous on I.)

If $f'(x) > 0$ (resp. < 0),

for all but finitely many x in an interval I,

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Spp

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f'	neg	0	pos	0	neg	0	pos
		-2		0		1	
f	decr on $(-\infty,-2)$		incr on $(-2,0)$		decr on $(0,1)$		incr on $(1,\infty)$
	and even on $(-\infty,-2]$		and even on $[-2,0]$		and even on $[0,1]$		and even on $[1,\infty)$

■

BIGGER INTERVALS BETTER ANSWER

IMPROVED INCREASING/DECREASING TEST:

(Assuming f continuous on I.)

If $f'(x) > 0$ (resp. < 0),

for all but finitely many x in an interval I,

then f is increasing (resp. decreasing) on I.

Spp

ANOTHER EXAMPLE: Find where the function

$$f(x) = -4x^5 + 15x^4 - 40x^2 + 3$$

is increasing and where it is decreasing.

$$f'(x) = -20x^4 + 60x^3 - 80x$$
$$= -20x(x^3 - 3x^2 + 4)$$
$$= -20x(x-2)^{\boxed{2}}(x+1)$$

NOT neg at $0, 2$ — only two numbers

2 is a root of of multiplicty two.

f'	neg	0	pos	0	neg	$0^{\boxed{2}}$	neg
		-1		0		2	
f	decr on $(-\infty, -1)$		incr on $(-1, 0)$		decr on $(0, 2)$		decr on $(2, \infty)$

and even on $(-\infty, -1]$

and even on $[-1, 0]$

and even on $[0, \infty)$ ■

IMPROVED INCREASING/DECREASING TEST:

(Assuming f continuous on I.)

If $f'(x) > 0$ (resp. < 0),

for all but finitely many x in an interval I,

then f is increasing (resp. decreasing) on I.

Spp

ANOTHER EXAMPLE: Find where the function

$$f(x) = 12x^5 - 15x^4 - 40x^3 - 9$$

is increasing and where it is decreasing.

$$f'(x) = 60x^4 - 60x^3 - 120x^2$$
$$= 60x^2(x^2 - x - 2)$$
$$= 60x^{\boxed{2}}(x-2)(x+1)$$

NOT neg at $-1, 0, 2$ only three numbers

0 is a root of of multiplicty two.

f'	pos	0	neg	$0^{\boxed{2}}$	neg	0	pos
		-1		0		2	
f	incr on $(-\infty, -1)$		decr on $(-1, 0)$		decr on $(0, 2)$		incr on $(2, \infty)$

and even on $(-\infty, -1]$

and even on $[-1, 2]$

and even on $[2, \infty)$ ■

IMPROVED INCREASING/DECREASING TEST:

(Assuming f continuous on I.)

If $f'(x) > 0$ (resp. < 0),

for all but finitely many x in an interval I,

then f is increasing (resp. decreasing) on I.

Spp

ANOTHER EXAMPLE: Find where the function

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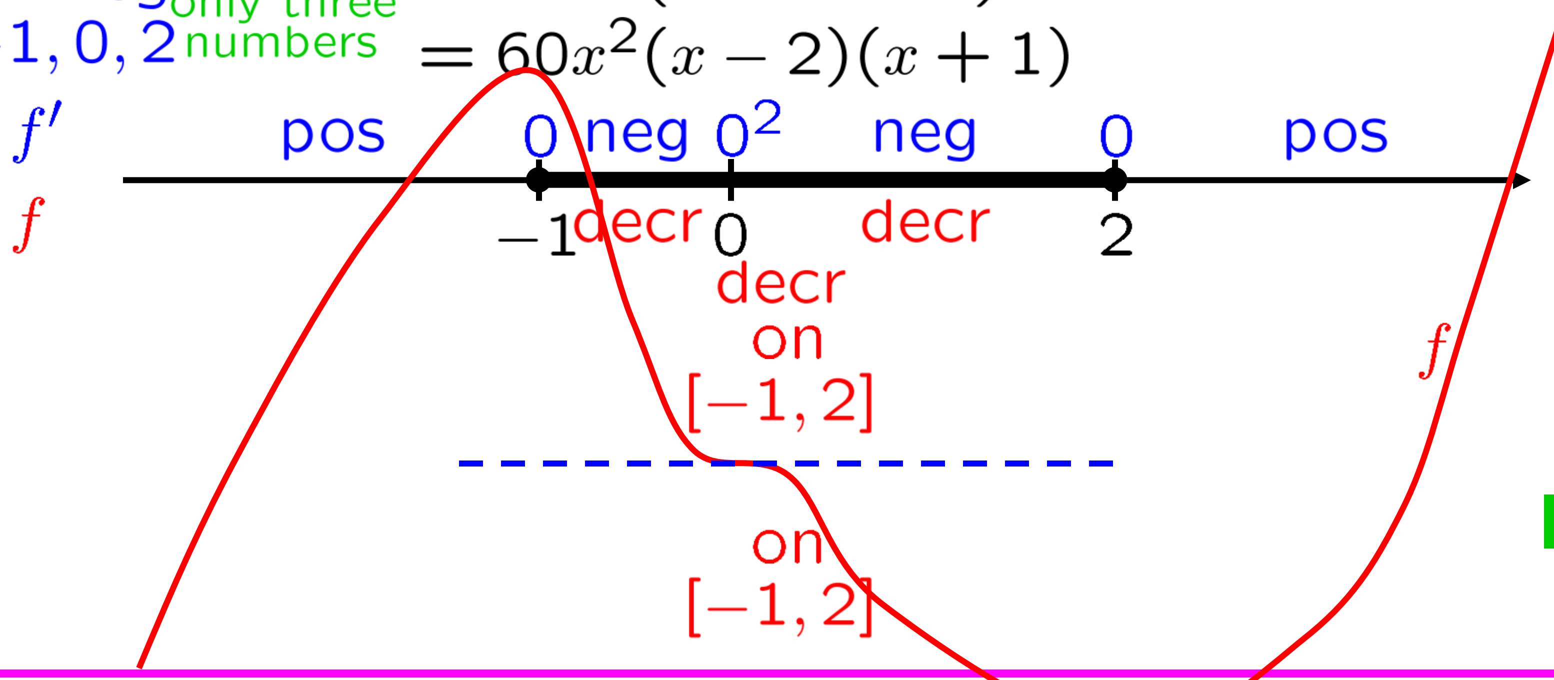
NOT neg at $-1, 0, 2$ only three numbers



IMPROVED INCREASING/DECREASING TEST:

(Assuming f continuous on I.)

If $f'(x) > 0$ (resp. < 0),

for all but finitely many x in an interval I,

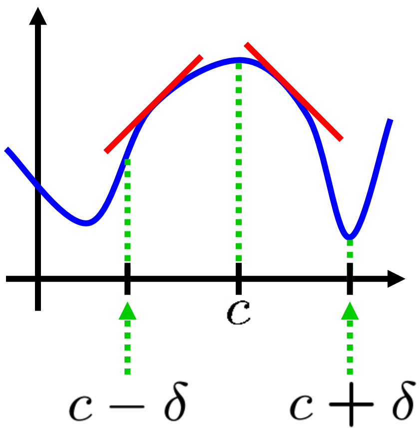
then f is increasing (resp. decreasing) on I.

Spp

cf. §5.2, p. 97–99 THE FIRST DERIVATIVE TEST:

Suppose c is a critical number of a continuous fn f.

(a) If f' changes from positive to negative at c,
then f has a local maximum at c.



Restatement:

If $\exists \delta > 0$ s.t. $f' > 0$ on $(c-\delta, c)$
and s.t. $f' < 0$ on $(c, c+\delta)$,
then f has a local maximum at c.

IMPROVED INCREASING/DECREASING TEST:

(Assuming f continuous on I.)

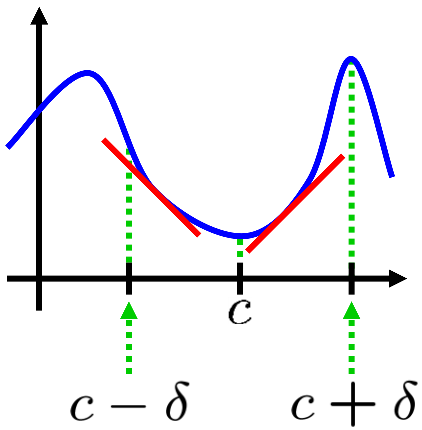
If $f'(x) > 0$ (resp. < 0),
for all but finitely many x in an interval I,
then f is increasing (resp. decreasing) on I.

cf. §5.2, p. 97–99 THE FIRST DERIVATIVE TEST:

Suppose c is a critical number of a continuous fn f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.



IMPROVED INCREASING/DECREASING TEST:

(Assuming f continuous on I.)

If $f'(x) > 0$ (resp. < 0),

for all but finitely many x in an interval I,

then f is increasing (resp. decreasing) on I.

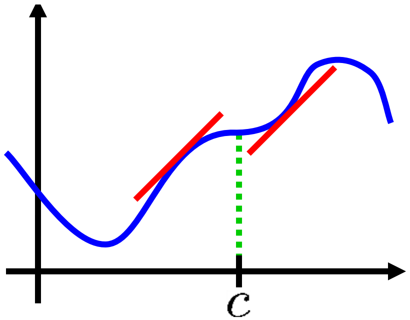
cf. §5.2, p. 97–99 THE FIRST DERIVATIVE TEST:

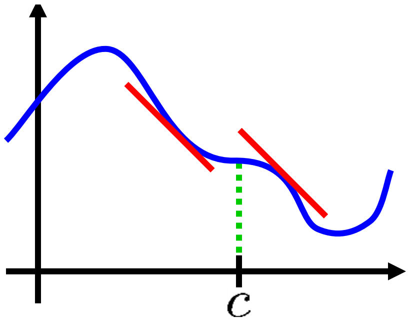
Suppose c is a critical number of a continuous fn f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c, then f does not have a local extremum at c.





IMPROVED INCREASING/DECREASING TEST:

(Assuming f continuous on I.)

If $f'(x) > 0$ (resp. < 0),

for all but finitely many x in an interval I,

then f is increasing (resp. decreasing) on I.

EXAMPLE: Find the local maxima and minima of $f(x) = 3x^4 + 4x^3 - 12x^2 + 7$.

$$\begin{aligned} f'(x) &= 12x^3 + 12x^2 - 24x \\ &= 12x(x^2 + x - 2) \\ &= 12x(x+2)(x-1) \end{aligned}$$

f'	neg	0	pos	0	neg	0	pos
f	decr on $(-\infty,-2)$ and even on $(-\infty,-2]$	-2	incr on $(-2,0)$ and even on $[-2,0]$	0	decr on $(0,1)$ and even on $[0,1]$	1	incr on $(1,\infty)$ and even on $[1,\infty)$

f has a local min at -2 with local min value $f(-2) = -25$.

f has a local max at 0 with local max value $f(0) = 7$.

f has a local min at 1 with local min value $f(1) = 2$. ■

SKILL
local extrema via 1st deriv test

ANOTHER EXAMPLE: Find the local extrema of

$$f(x) = -4x^5 + 15x^4 - 40x^2 + 3.$$

$$\begin{aligned} f'(x) &= -20x^4 + 60x^3 - 80x \\ &= -20x(x^3 - 3x^2 + 4) \\ &= -20x(x-2)^2(x+1) \end{aligned}$$

		-1		0		2	
f'	neg	0	pos	0	neg	0^2	neg
f	decr on $(-\infty,-1)$		incr on $(-1,0)$		decr on $(0,2)$		decr on $(2,\infty)$
	and even on $(-\infty,-1]$		and even on $[-1,0]$		and even on $[0,\infty)$		

SKILL
local extrema via 1st deriv test

f has a local min at -1 with local min value $f(-1) = -18$.

f has a local max at 0 with local max value $f(0) = 3$. ■

f does not have a local extremum at 2.

Spp

cf. §5.4, pp. 100–101, DEFINITION: Let I be an interval.

strictly (convex)

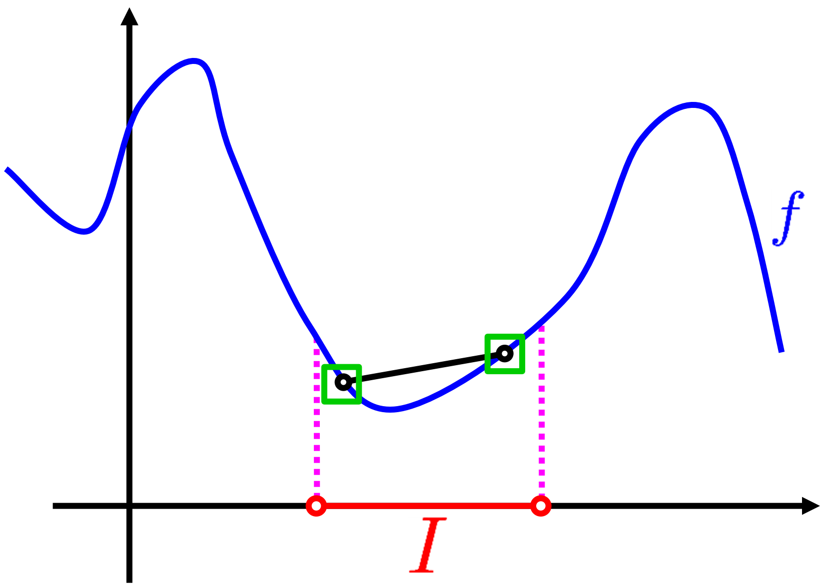
A function f is called **concave up on** I if

the (open) secant line segment from $(s, f(s))$ to $(u, f(u))$

lies above the graph of f,

whenever $s, u \in I$.

e.g.:



Typical to make the interval
as large as possible. . .

cf. §5.4, pp. 100–101, DEFINITION: Let I be an interval.

strictly

(convex)

A function f is called **concave up on** I if

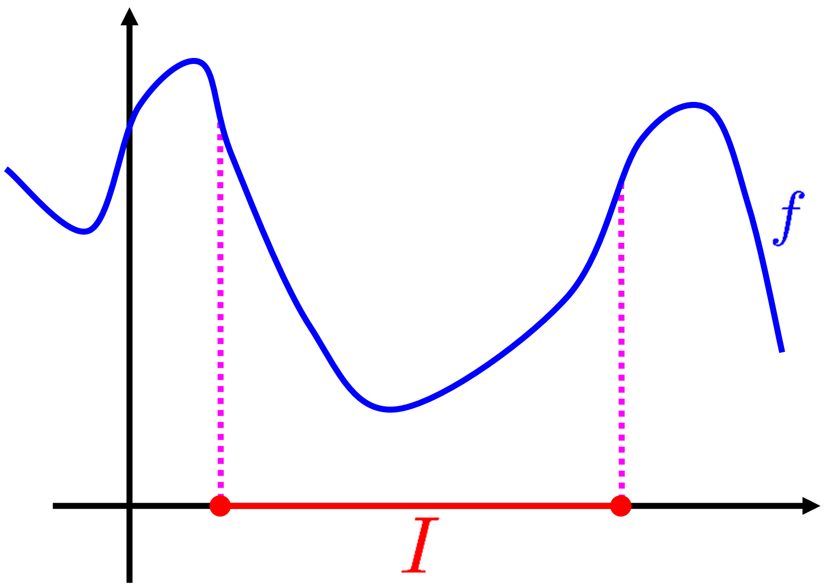
the secant line segment from $(s, f(s))$ to $(u, f(u))$

open

lies above the graph of f,

whenever $s, u \in I$.

e.g.:



Typical to make the interval as large as possible. . .

Next: Concave down. . .

cf. §5.4, pp. 100–101, DEFINITION: Let I be an interval.

strictly

(concave)

A function f is called **concave down on** I if

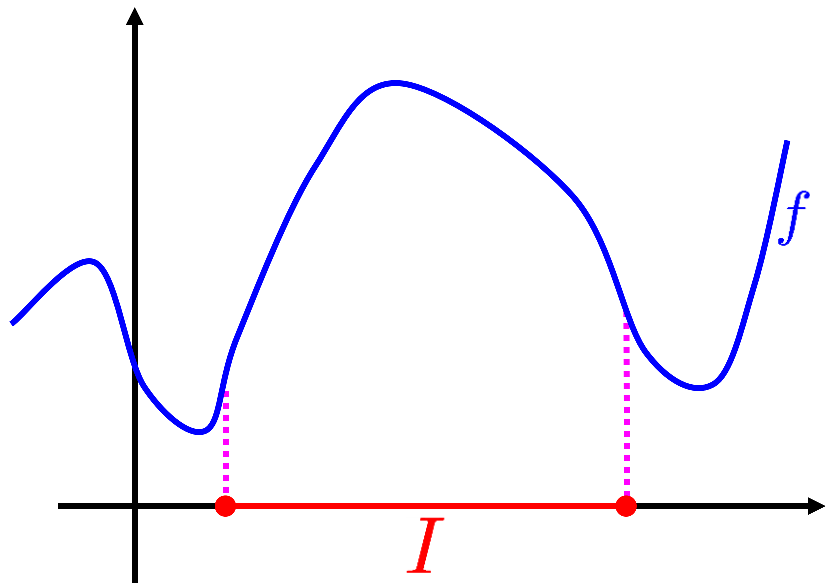
the secant line segment from $(s, f(s))$ to $(u, f(u))$

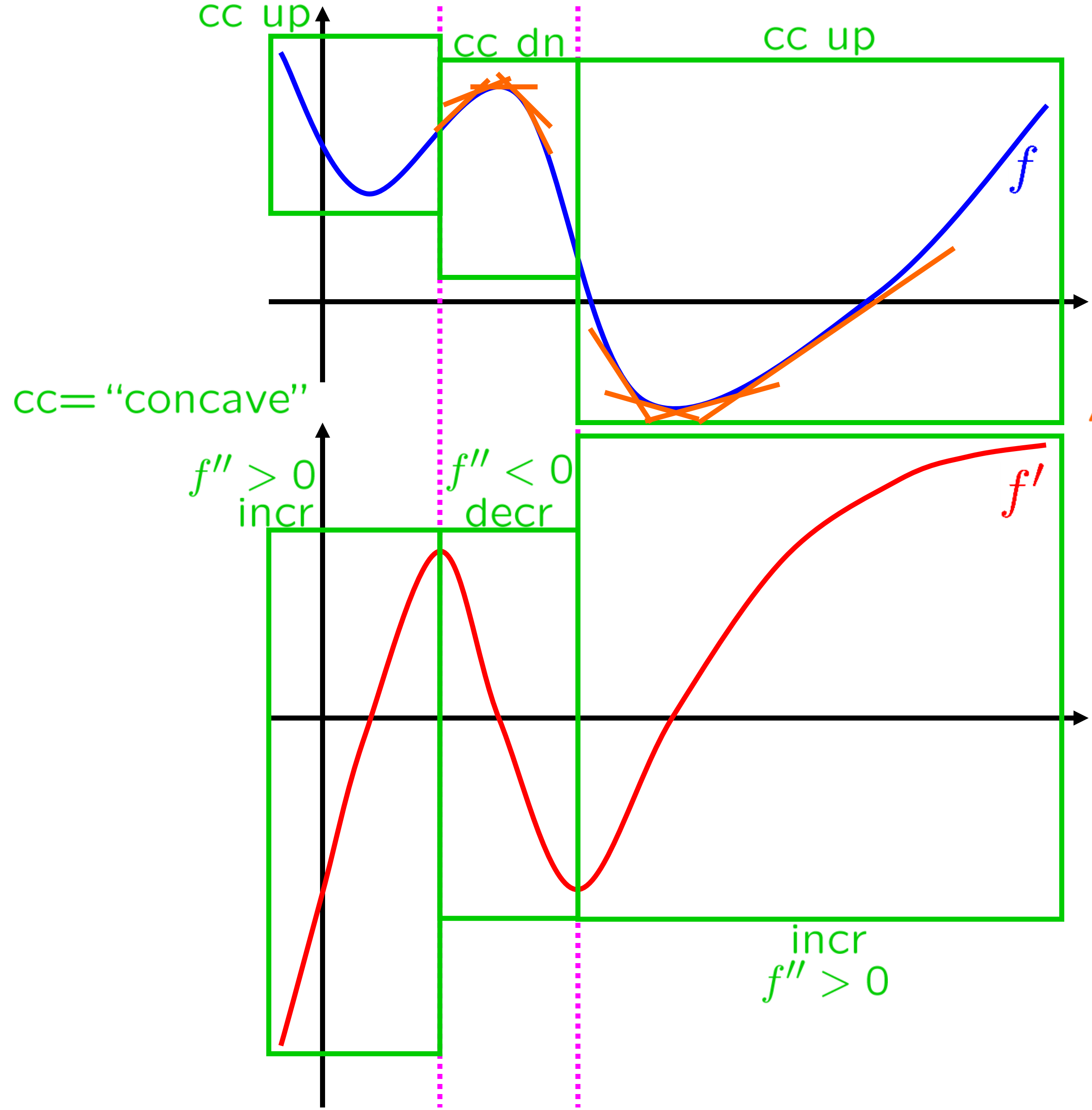
open

lies below the graph of f,

whenever $s, u \in I$.

e.g.:





TANGENT LINE SLOPES
negative
less negative
positive
more positive

TANGENT LINE SLOPES ARE INCREASING

CONCAVITY TEST:

Let I be an interval.

(a) If $f''(x) > 0$ for all $x \in I$,
then the graph of f is concave up on I.

(b) If $f''(x) < 0$ for all $x \in I$,
then the graph of f is concave down on I.

Definition: The **interior** of an interval is the open interval with the same endpoints.

e.g.: The interior of $[1, 2]$ is $(1, 2)$.
The interior of $(3, 4]$ is $(3, 4)$.
The interior of $(5, 6)$ is $(5, 6)$.
The interior of $[7, \infty)$ is $(7, \infty)$.
The interior of $(-\infty, 8]$ is $(-\infty, 8)$.

IMPROVED CONCAVITY TEST:

Let I be an interval. (Assuming f continuous on I and differentiable on the interior of I.)

(a) If $f''(x) > 0$ for all <u>but finitely many</u> $x \in I$,
then the graph of f is concave up on I.

(b) If $f''(x) < 0$ for all <u>but finitely many</u> $x \in I$,
then the graph of f is concave down on I.

EXAMPLE: Find the maximal intervals of concavity for

$$f(x) = -x^6 + 3x^5 + 20x^4$$

$$f'(x) = -6x^5 + 15x^4 + 80x^3$$

$$\begin{aligned} f''(x) &= -30x^4 + 60x^3 + 240x^2 \\ &= -30x^2(x^2 - 2x - 8) \\ &= -30x^2(x+2)(x-4) \end{aligned}$$

f''	neg	0	pos	$0^{\boxed{2}}$	pos	0	neg
		-2		0		4	

f cc dn on $(-\infty, -2]$	cc up on $[-2, 4]$	cc dn on $[4, \infty)$ ■

IMPROVED CONCAVITY TEST:

Let I be an interval. (Assuming f continuous on I and differentiable on the interior of I.)

(a) If $f''(x) > 0$ for all but finitely many $x \in I$, then the graph of f is concave up on I.

(b) If $f''(x) < 0$ for all but finitely many $x \in I$, then the graph of f is concave down on I.

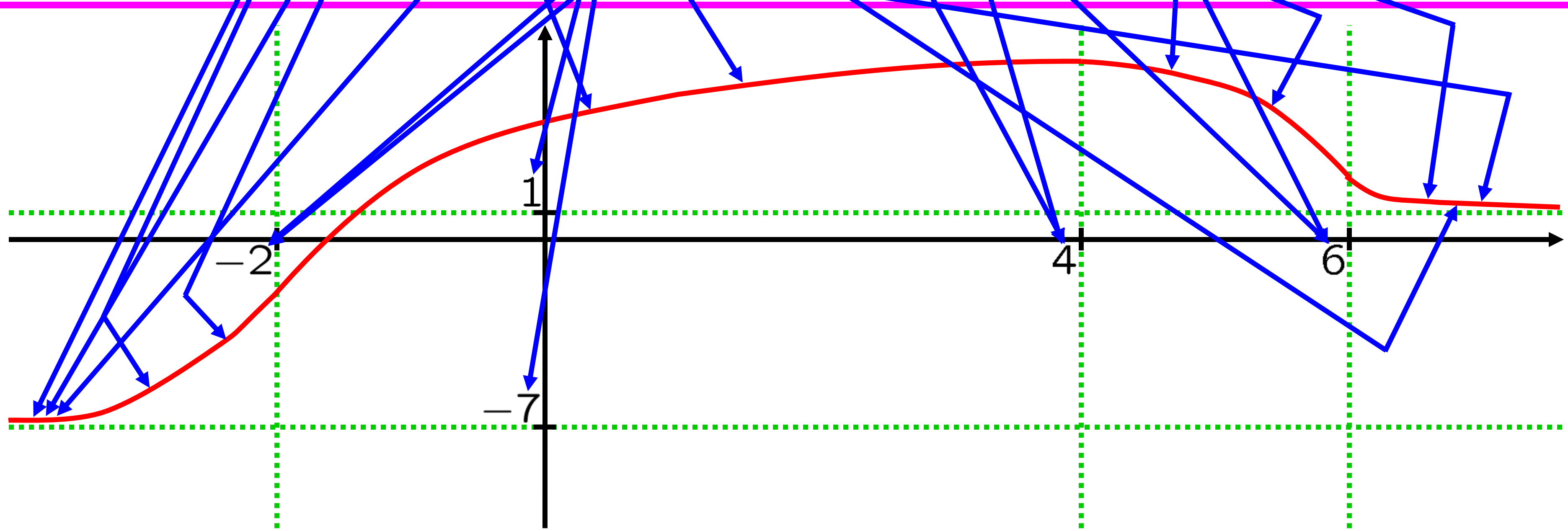
Spp

EXAMPLE: Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $x \in (-\infty, 4)$,
$f'(x) < 0$ on $x \in (4, \infty)$

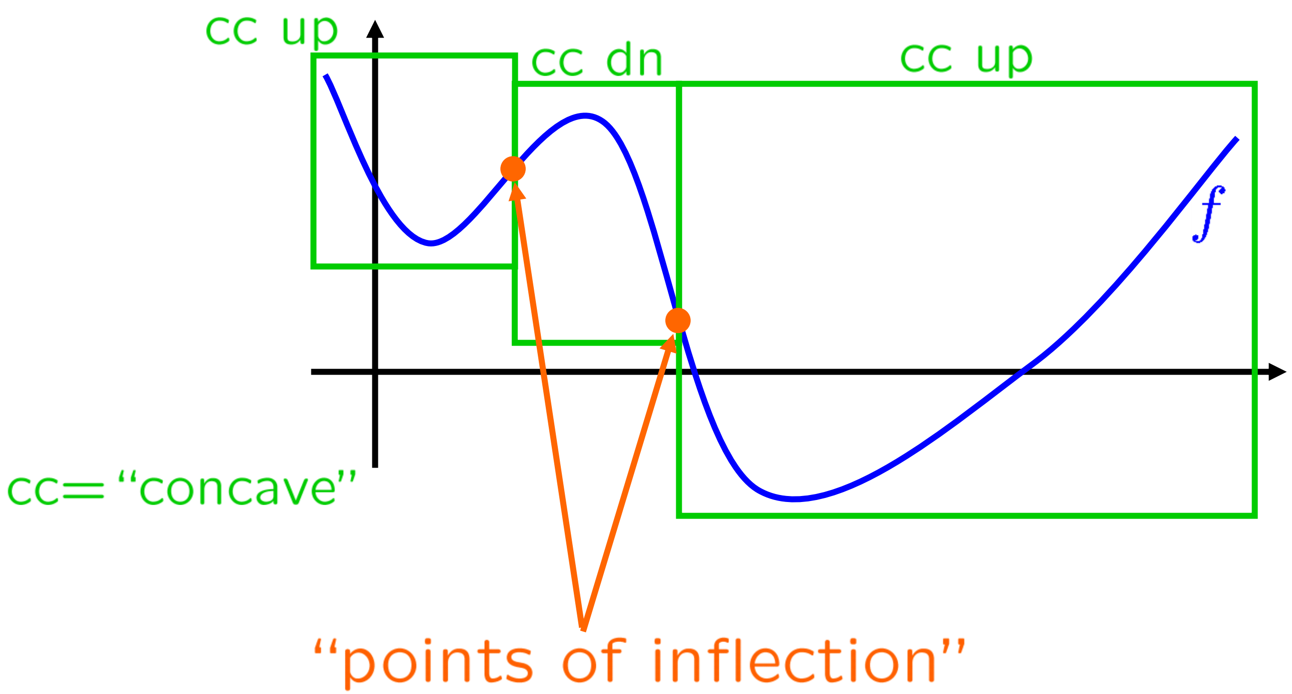
(ii) $f''(x) > 0$ on $x \in (-\infty, -2) \cup (6, \infty)$,
$f''(x) < 0$ on $x \in (-2, 6)$

(iii) $\lim_{x \to -\infty} f(x) = -7$,
$\lim_{x \to \infty} f(x) = 1$



SKILL
sketching, given data

many other examples



Next: def'n of point of inflection
a.k.a. inflection point . . .

cf. §5.4, p. 100–102 DEFINITION: "**point of inflection**"

"**flex point**"

A point (a, b) on a curve $y = f(x)$ is called **an inflection point** if all of the following hold:

f is continuous at a

and $\displaystyle\lim_{h \to 0} \frac{(f(a+h)) - (f(a))}{h}$ exists (possibly $\pm\infty$)

and the curve changes

from concave up to concave down

or

from concave down to concave up

at (a, b).

"$y = f(x)$ changes concavity at a" meaning:

$\exists \delta > 0$ s.t.

either

f is cc up on $(a-\delta, a)$ and f is cc dn on $(a, a+\delta)$

or

f is cc dn on $(a-\delta, a)$ and f is cc up on $(a, a+\delta)$.

cf. §5.4, p. 100–102 DEFINITION: "**point of inflection**"

A point (a, b) on a curve $y = f(x)$ is called **an inflection point** if all of the following hold:

f is continuous at a

and $\boxed{\lim_{h\to 0} \frac{(f(a+h)) - (f(a))}{h} \text{ exists (possibly } \pm\infty)}$

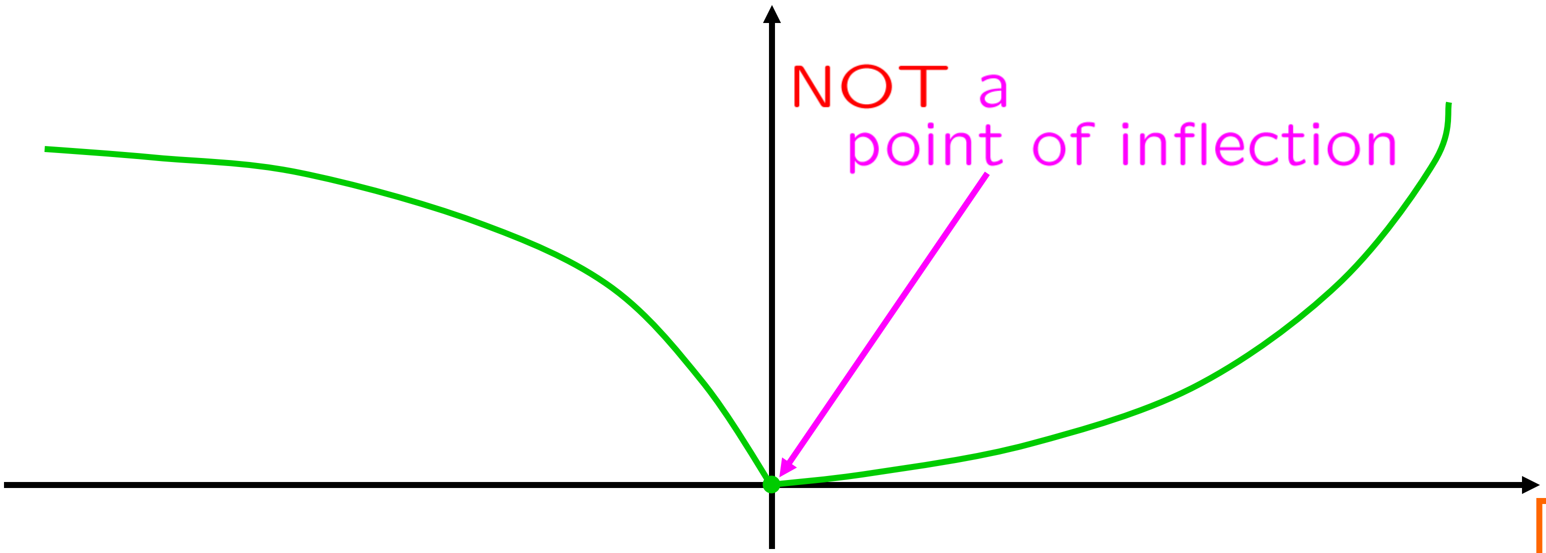
and the curve changes

from concave up to concave down

or

from concave down to concave up

at (a, b).



§5.4

cf. §5.4, p. 100–102 DEFINITION: "**point of inflection**"

A point (a, b) on a curve $y = f(x)$ is called **an inflection point** if all of the following hold:

f is continuous at a

and $\lim_{h \to 0} \frac{(f(a+h)) - (f(a))}{h}$ exists (possibly $\boxed{\pm\infty}$)

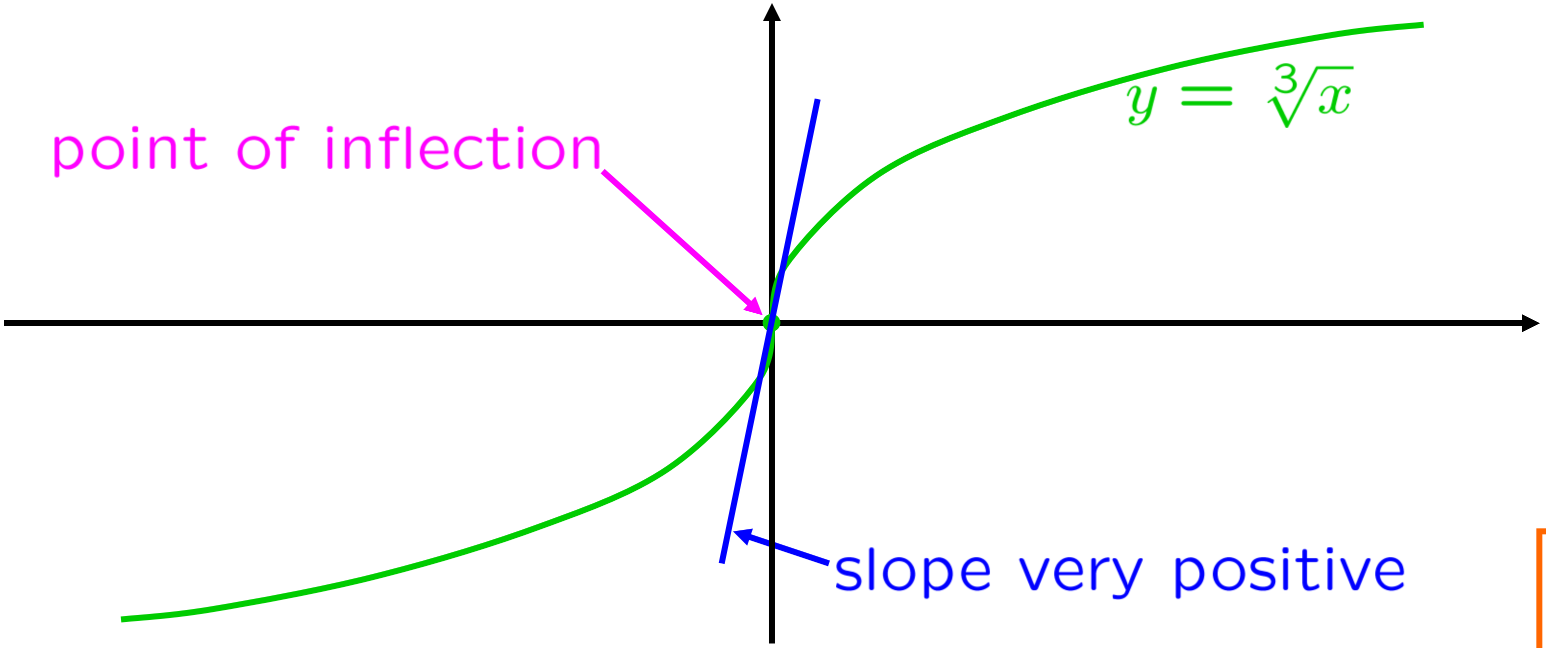
and the curve changes

from concave up to concave down

or

from concave down to concave up

at (a, b).



cf. §5.4, p. 100–102 DEFINITION: "**point of inflection**"

A point (a,b) on a curve $y=f(x)$ is called **an inflection point** if all of the following hold:

$\boxed{f \text{ is continuous at } a}$

and $\displaystyle\lim_{h\to 0}\frac{(f(a+h))-(f(a))}{h}$ exists (possibly $\pm\infty$)

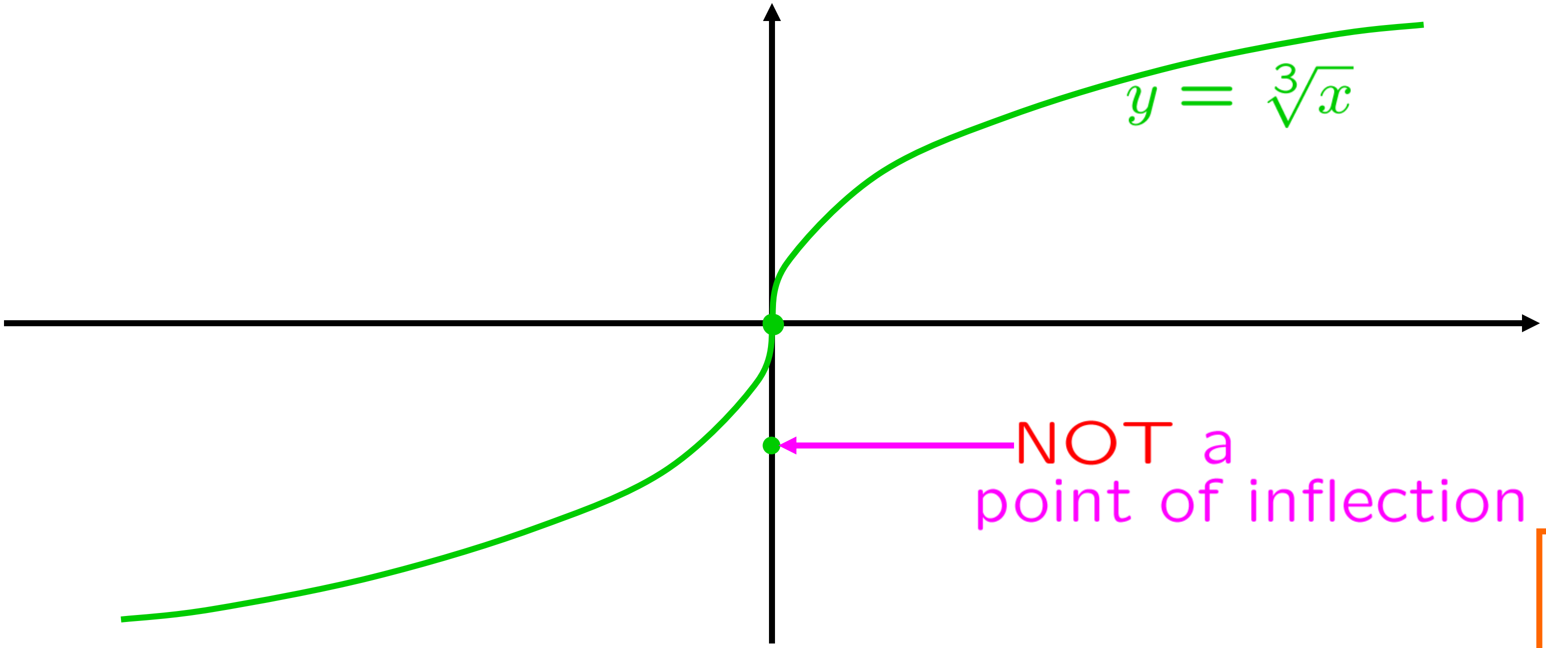
and the curve changes

from concave up to concave down

or

from concave down to concave up

at (a,b).



EXAMPLE: Describe the curve $y = x^4 + 4x^3$
in terms of concavity,
points of inflection,
and local maxima and minima.

SKILL
sketching

Sketch the curve.

$$y = x^4 + 4x^3 = x^3(x+4)$$
$$dy/dx = 4x^3 + 12x^2 = 4x^2(x+3)$$
$$d^2y/dx^2 = 12x^2 + 24x = 12x(x+2)$$

$4x^2(x+3)$	neg	0	pos		pos	0^2	pos
$12x(x+2)$	pos		pos	0	neg	0	pos
x		-3		-2		0	

$x^4 + 4x^3$
$\|$
$x^3(x+4)$
0 at $x = 0$
0 at $x = -4$

decr on $(-\infty, -3]$ | incr on $[-3, \infty)$ | level at 0

cc up on $(-\infty, -2]$ | cc dn on $[-2, 0]$ | cc up on $[0, \infty)$

EXAMPLE: Describe the curve $y = x^4 + 4x^3$

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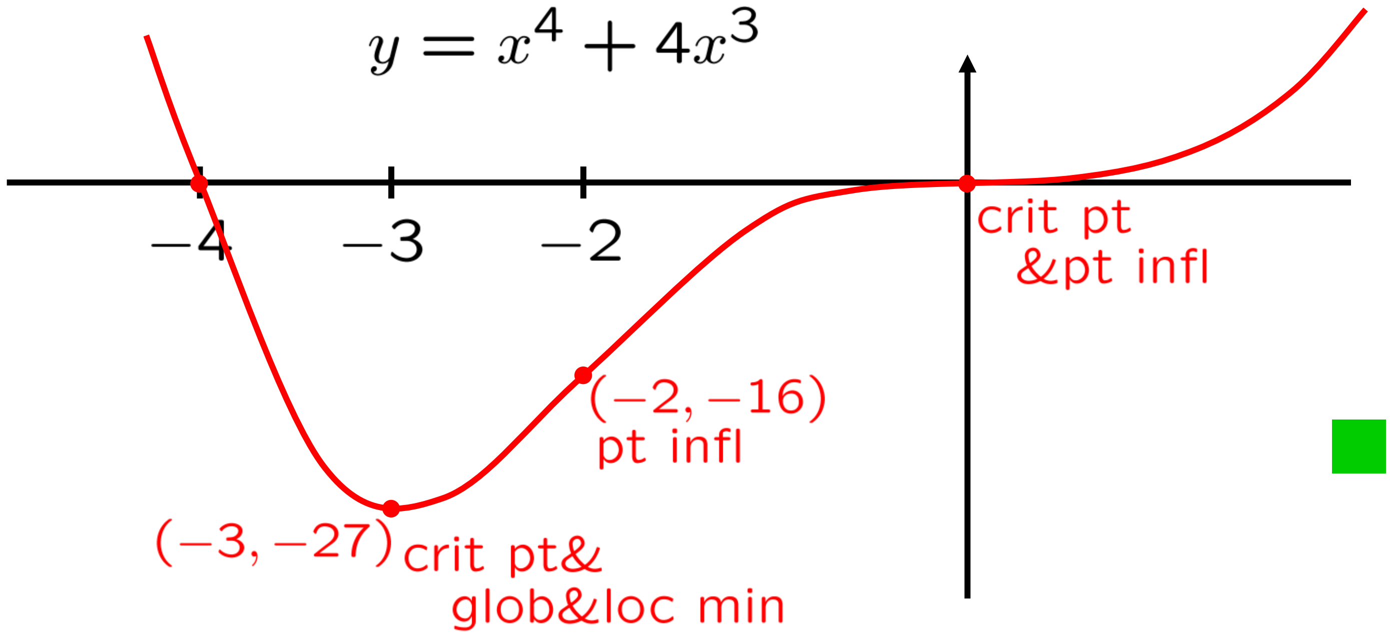
SKILL
sketching

Sketch the curve.



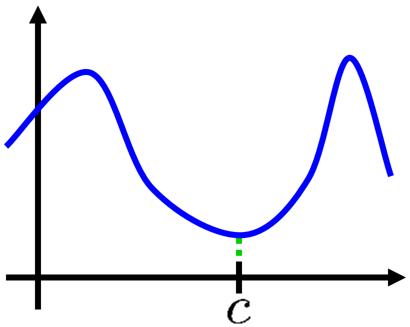
$x^4 + 4x^3$
$\|$
$x^3(x + 4)$
0 at $x = 0$
0 at $x = -4$

decr on $(-\infty, -3]$	incr on $[-3, \infty)$	level at 0
cc up on $(-\infty, -2]$	cc dn on $[-2, 0]$	cc up on $[0, \infty)$

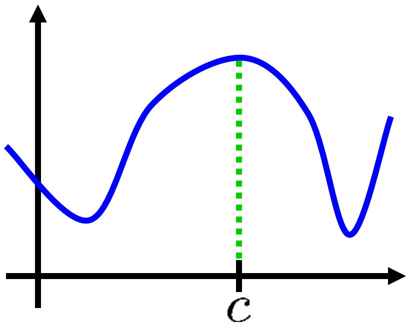
cf. §5.3, p. 99–100 THE SECOND DERIVATIVE TEST:

Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$,
then f has a local minimum at c.



(b) If $f'(c) = 0$ and $f''(c) < 0$,
then f has a local maximum at c.



Note: No information if $f'(c) = 0$ and $f''(c) = 0$.

SKILL
intervals, loc max/min

EXAMPLE: Let $f(x) = \dfrac{x^2}{x^2 - 5}$.

$\pm\sqrt{5} \notin \mathrm{dom}[f]$

(a) Find the maximal intervals on which f is incr or decr.

(b) Find the local extrema of f.

(c) Find the max intervals of concavity and inflection points.

$$f'(x) = \frac{(\cancel{x^2} - 5)(2x) - \cancel{(x^2)(2x)}}{(x^2-5)^2} = \frac{\overbrace{-10x}^{>0,\ \text{if } x<0;\ <0,\ \text{if } x>0}}{\underbrace{(x^2-5)^2}_{\text{always } \geq 0}}$$

(a): f incr on $(-\infty, -\sqrt{5})$, $(-\sqrt{5}, 0]$

f decr on $[0, \sqrt{5})$, $(\sqrt{5}, \infty)$

(b): f has loc max at 0 with value 0

$$f''(x) = \frac{(x^2-5)^{\cancel{2}}(-10) + (+10x)(2\cancel{(x^2-5)}(2x))}{(x^2-5)^{\cancel{4}\,3}}$$

$$= \frac{(-10x^2+50)+(40x^2)}{(x^2-5)^3} = \frac{\overbrace{30x^2+50}^{\text{always pos}}}{\boxed{(\boxed{x^2-5})^3}} \begin{cases} >0, & \text{if } x > \sqrt{5} \\ <0, & \text{in between} \\ >0, & \text{if } x < -\sqrt{5} \end{cases}$$

(c): f cc up on $(-\infty, -\sqrt{5})$ $\quad$ f cc up on $(\sqrt{5}, \infty)$

no infl. pts $\quad$ f cc dn on $(-\sqrt{5}, \sqrt{5})$

EXAMPLE: Let $f(x) = \dfrac{x^2}{x^2 - 5}$.

Find the local maxima and minima of f using the First Derivative Test.

$$f'(x) = \qquad = \frac{-10x}{(x^2-5)^2} \longleftarrow \text{always} \geq 0$$

$$f'(x) = \frac{-10x}{}$$

(a): f incr on $(-\infty, -\sqrt{5})$, $(-\sqrt{5}, 0]$

f decr on $[0, \sqrt{5})$, $(\sqrt{5}, \infty)$

(a): f incr on $(-\infty, -\sqrt{5})$, $(-\sqrt{5}, 0]$

f decr on $[0, \sqrt{5})$, $(\sqrt{5}, \infty)$

$f''(x) =$

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EXAMPLE: Let $f(x) = \dfrac{x^2}{x^2 - 5}$.

Find the local maxima and minima of f using the First Derivative Test.

SKILL
deriv tests

Find the local max and min values of f using the Second Derivative Test.

Which of the two methods do you prefer?

$$f'(x) = \frac{-10x}{(x^2-5)^2}$$ ← always ≥ 0

0 is the only critical number

(a): f incr on $(-\infty, -\sqrt{5})$, $(-\sqrt{5}, 0]$
f decr on $[0, \sqrt{5})$, $(\sqrt{5}, \infty)$

f has loc max at 0 with value 0

$$f''(x) = \frac{30x^2+50}{(x^2-5)^3}$$ ← always pos

$$f''(0) = \frac{50}{(-5)^3} < 0$$

f has loc max at 0 with value 0

EXAMPLE: Let $B(x) = 4x^{3/4} - 3x$.

(a) Find the max intervals of increase or decrease.

(b) Find local maximum and minimum values.

(c) Find the max intervals of concavity and the inflection points.

(d) Sketch the graph.

$$B'(x) = 4(3/4)x^{-1/4} - 3 = 3(x^{-1/4} - 1)$$

$$x > 1 \quad \Rightarrow \quad \left[x^{1/4} > 1 \right] \quad \times \quad x^{-1/4}$$

$$\Rightarrow \quad 1 > x^{-1/4} \qquad \Rightarrow \quad x^{-1/4} < \boxed{1}$$

$$\Rightarrow \quad 3(x^{-1/4} - 1) < 0$$

(a): B decr on $[1, \infty)$

$B' < 0$ on $(1, \infty)$

EXAMPLE: Let $B(x) = 4x^{3/4} - 3x$. domain of B is $[0, \infty)$.

(a) Find the max intervals of increase or decrease.

(b) Find local maximum and minimum values.

(c) Find the max intervals of concavity and the inflection points.

(d) Sketch the graph.

$$B'(x) = 4(3/4)x^{-1/4} - 3 = 3(x^{-1/4} - 1)$$

$$\boxed{\begin{aligned} x > 1 \quad &\Rightarrow \quad x^{1/4} > 1 \\ &\Rightarrow \quad 1 > x^{-1/4} \\ &\Rightarrow \quad 3(x^{-1/4} - 1) < 0 \end{aligned}}$$

(a): B decr on $[1, \infty)$

$B' < 0$ on $(1, \infty)$

EXAMPLE: Let $B(x) = 4x^{3/4} - 3x$. domain of B is $[0, \infty)$.

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(d) Sketch the graph.

$$B'(x) = 4(3/4)x^{-1/4} - 3 = 3(x^{-1/4} - 1)$$

$$0 < x < 1 \Rightarrow 0 < x^{1/4} < 1$$
$$\Rightarrow 1 < x^{-1/4}$$
$$\Rightarrow 3(x^{-1/4} - 1) > 0$$

(a): B incr on $[0, 1]$, B decr on $[1, \infty)$

$B' > 0$ on $(0, 1)$, $B' < 0$ on $(1, \infty)$

EXAMPLE: Let $B(x) = 4x^{3/4} - 3x$. domain of B is $[0, \infty)$.

(a) Find the max intervals of increase or decrease.

(b) Find local maximum and minimum values.

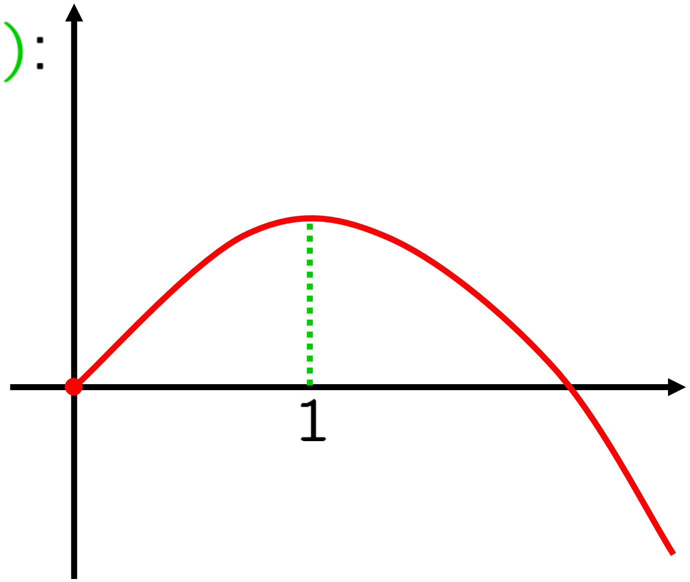
(c) Find the max intervals of concavity and the inflection points.

(d) Sketch the graph.

$$B'(x) = 4(3/4)x^{-1/4} - 3 = 3(x^{-1/4} - 1)$$

$$B''(x) = 3(-1/4)x^{-5/4} = -(3/4)x^{-5/4}$$

(d):



(a): B incr on $[0, 1]$, B decr on $[1, \infty)$

(b): B has loc max at 1,
max value $= B(1) = 1$

(c): B cc dn on $[0, \infty)$
$B'' < 0$ on $(0, \infty)$

SKILL
intervals, loc max/min

EXAMPLE: (a) Show that $e^x > 1$, for all $x > 0$. ☺

(b) Show that $e^x > 1 + x$, for all $x > 0$. ☺

(c) Show that $e^x > 1 + x + \frac{1}{2}x^2$, for all $x > 0$. ☺

(a) $(d/dx)(e^x - 1) = e^x > 0$

$e^x - 1$ is increasing in x.
$[e^x - 1]_{x:\to 0} = 0$

$e^x - 1 > 0, \; \forall x > 0$

(b) $(d/dx)(e^x - 1 - x) = e^x - 1 > 0, \; \forall x > 0$

$e^x - 1 - x$ is increasing on $x \geq 0$.
$[e^x - 1 - x]_{x:\to 0} = 0$

$e^x - 1 - x > 0, \; \forall x > 0$

(c) $(d/dx)(e^x - 1 - x - \frac{1}{2}x^2) = e^x - 1 - x > 0, \; \forall x > 0$

$e^x - 1 - x - \frac{1}{2}x^2$ is increasing on $x \geq 0$.
$[e^x - 1 - x - \frac{1}{2}x^2]_{x:\to 0} = 0$

$e^x - 1 - x - \frac{1}{2}x^2 > 0, \; \forall x > 0$

SKILL
inequalities
from incr/decr

EXAMPLE: Show that a cubic (*i.e.*, third degree) polynomial always has exactly one point of inflection. ☺

If its graph has three x-intercepts a, b and c, show that the x-coordinate of the inflection point is $(a+b+c)/3$.

$px^3 + qx^2 + rx + s$ changes cc at $x = -q/(3p)$

$$\frac{d}{dx}(px^3 + qx^2 + rx + s) = 3px^2 + 2qx + r$$

$$\frac{d^2}{dx^2}(px^3 + qx^2 + rx + s) = 6px + 2q$$

x-intercept: $-q/(3p)$

changes sign at $x = -q/(3p)$

$p \neq 0$

$8x + 3$ x-intercept: $-3/8$

changes sign at $x = -3/8$

$-5x + 7$ x-intercept: $7/5$

changes sign at $x = 7/5$

EXAMPLE: Show that a cubic (*i.e.*, third degree) polynomial always has exactly one point of inflection.

If its graph has three x-intercepts a, b and c, show that the x-coordinate of the inflection point is $(a+b+c)/3$.

The cubic is divisible by

$$(x-a)(x-b)(x-c)$$

and the quotient is a constant.

$$p(x-a)(x-b)(x-c) \text{ changes cc at } x=(a+b+c)/3$$

$$\frac{d}{dx}[p(x-a)(x-b)(x-c)]$$
$$= p[(x-a)(x-b)+(x-a)(x-c)+(x-b)(x-c)]$$

$$\frac{d^2}{dx^2}[p(x-a)(x-b)(x-c)]$$
$$= p[2(x-a)+2(x-b)+2(x-c)]$$

x-intercept: $(a+b+c)/3$

changes sign at $x=(a+b+c)/3$

$p \neq 0$

SKILL

1st deriv test

Whitman problems

§5.2, p. 98–99, #1-17

SKILL

2nd deriv test

Whitman problems

§5.3, p. 100, #1-18

SKILL

concavity, inflection pts

Whitman problems

§5.4, p. 101–102, #1-21