CALCULUS

The Integral Mean Value Theorem

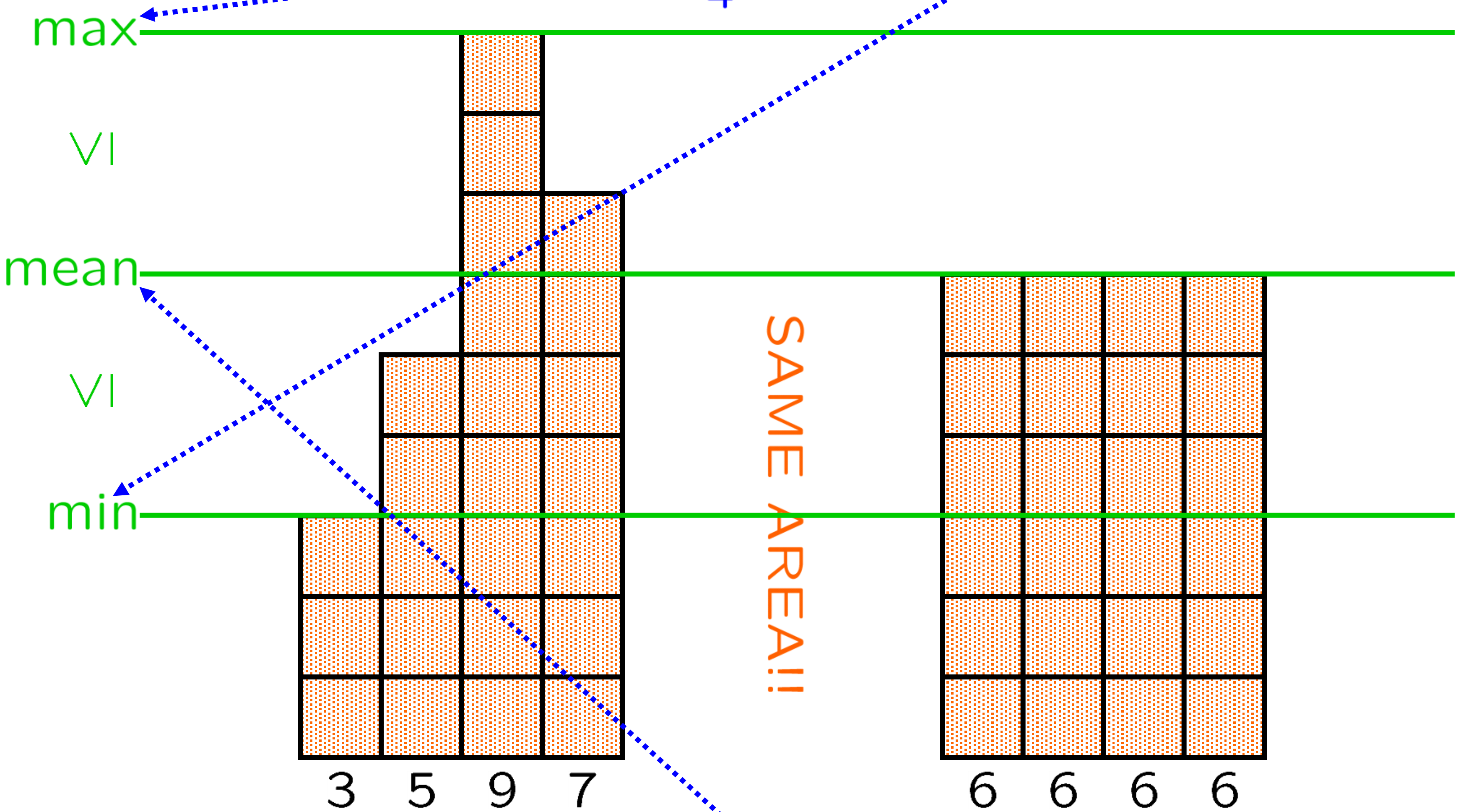
Calculate the mean (or average) of 3, 5, 9, 7.

mean doesn't appear

$3 + 5 + 9 + 7 = 24$

$\frac{24}{4} = 6$

$6 + 6 + 6 + 6 = 24$



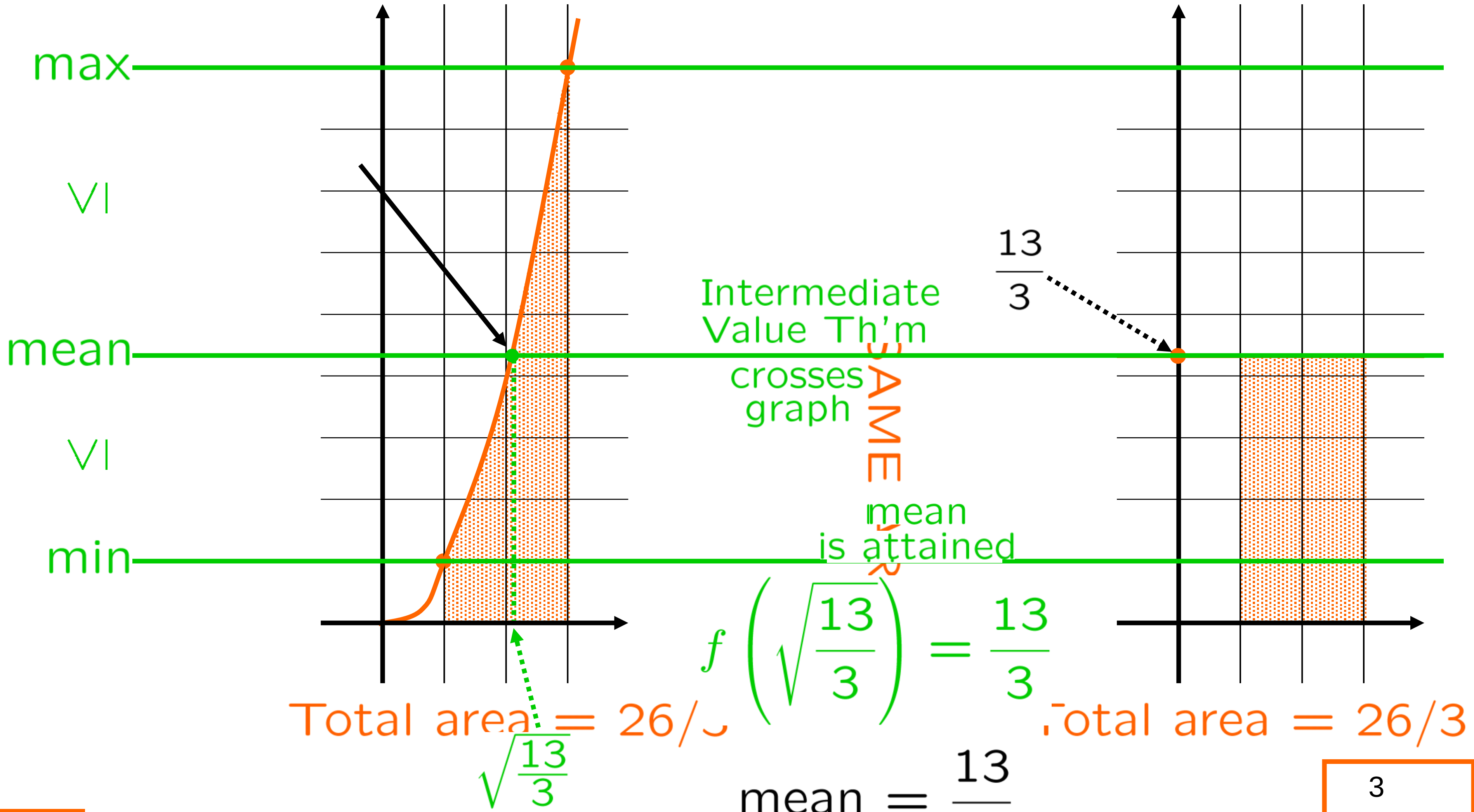
Total area = 24

Total area = 24

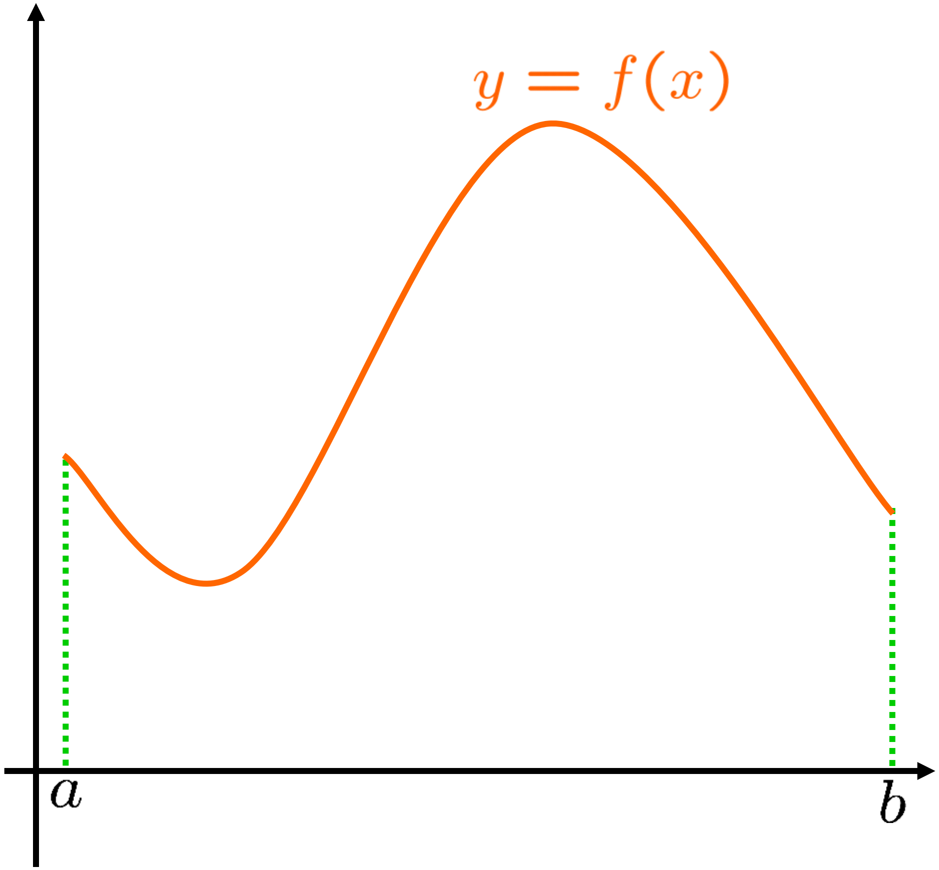
mean = 6

Let $f(x) = x^2$. Calculate the mean (or average) of f on $[1,3]$.

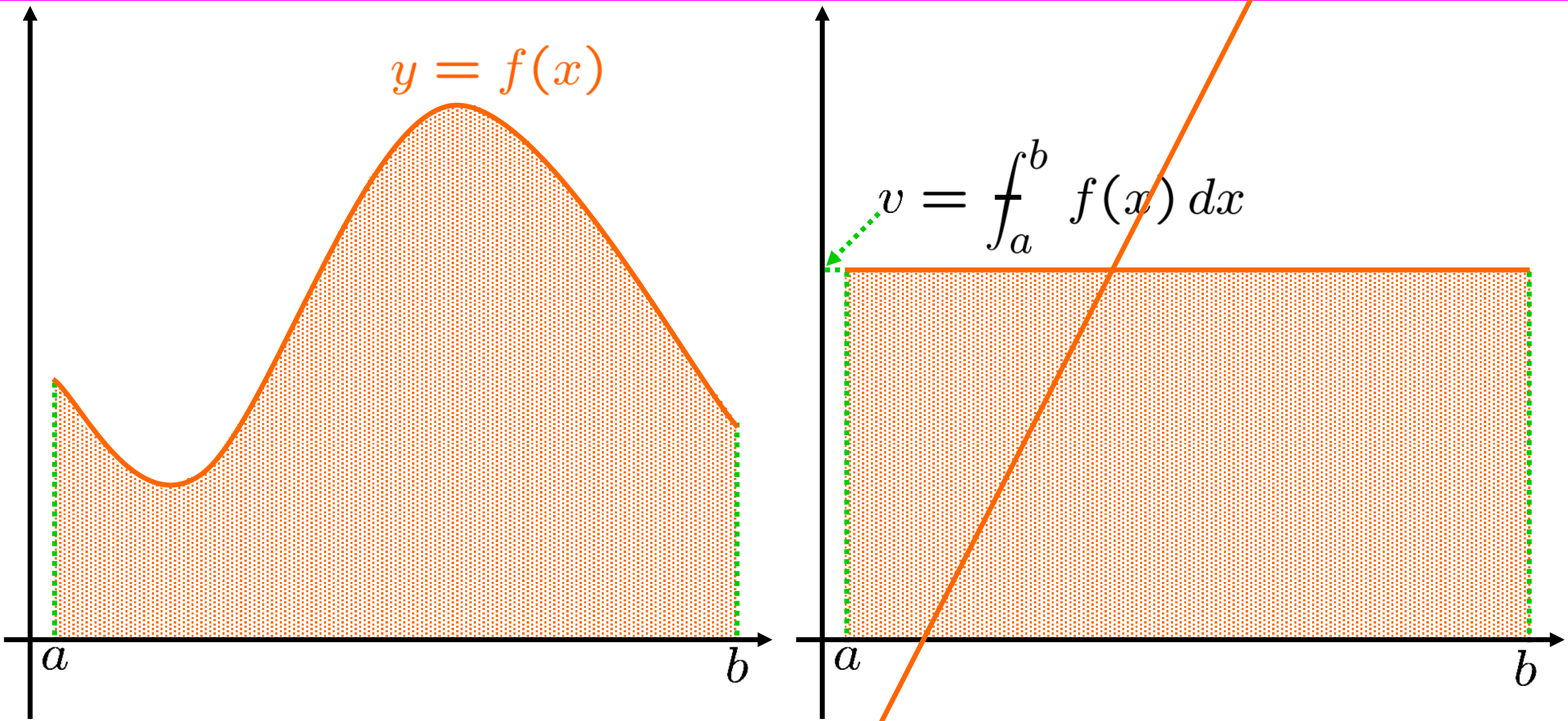
$$\int_1^3 f(x)\,dx = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3} \qquad \left[\frac{1}{3-1}\right]\left[\frac{26}{3}\right] = \frac{13}{3} \qquad \int_1^3 \frac{13}{3}\,dx = \frac{26}{3}$$



$$\text{mean} = \frac{13}{3}$$


$y = f(x)$
a
b

DEFINITION: The **average (or mean)** of f on $[a,b]$ is $\fint_a^b f(x)\,dx := \frac{1}{b-a}\int_a^b f(x)\,dx$.



$$v = \begin{bmatrix}\text{mean}\\ \text{value}\end{bmatrix} \Rightarrow \int_a^b f(x)\,dx = \int_a^b v\,dx = v(b-a)$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = v$$

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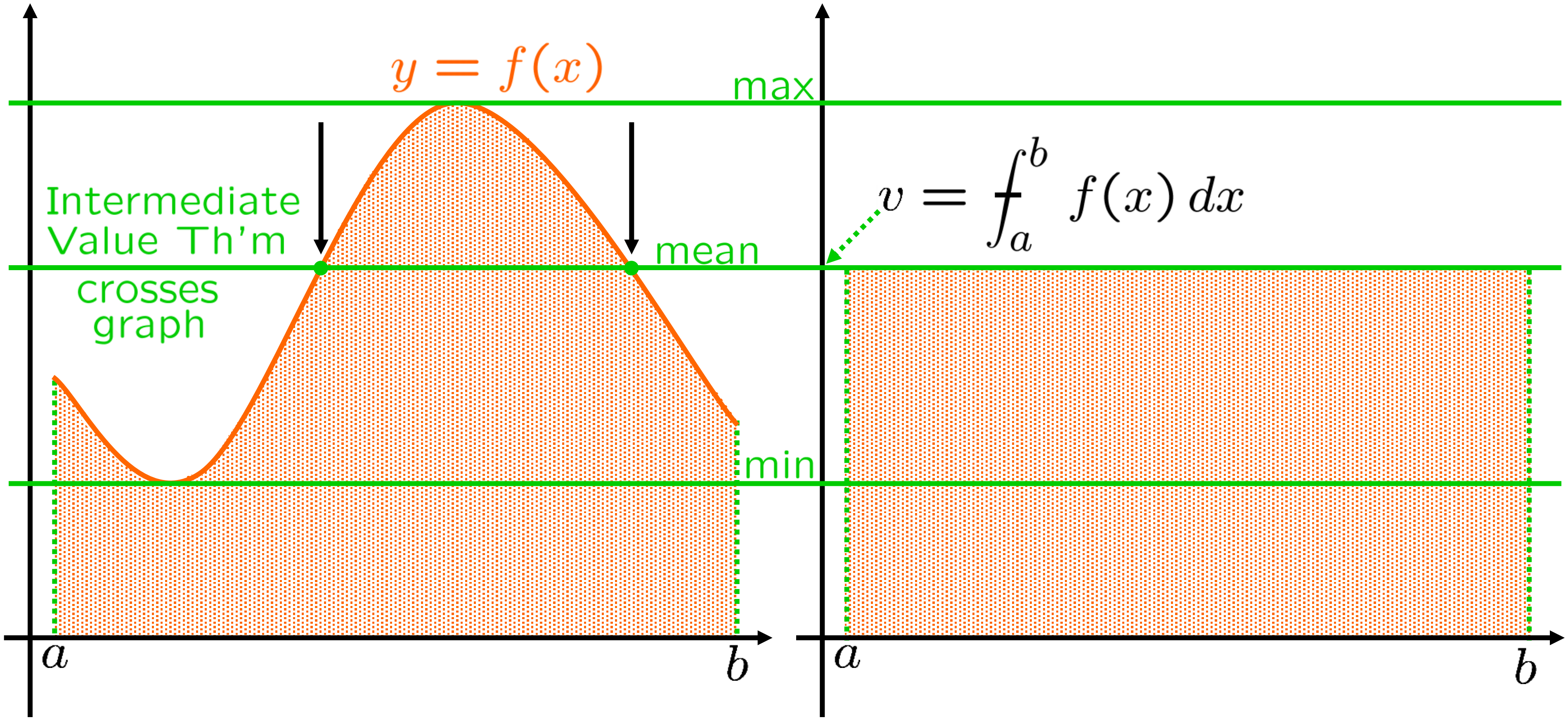

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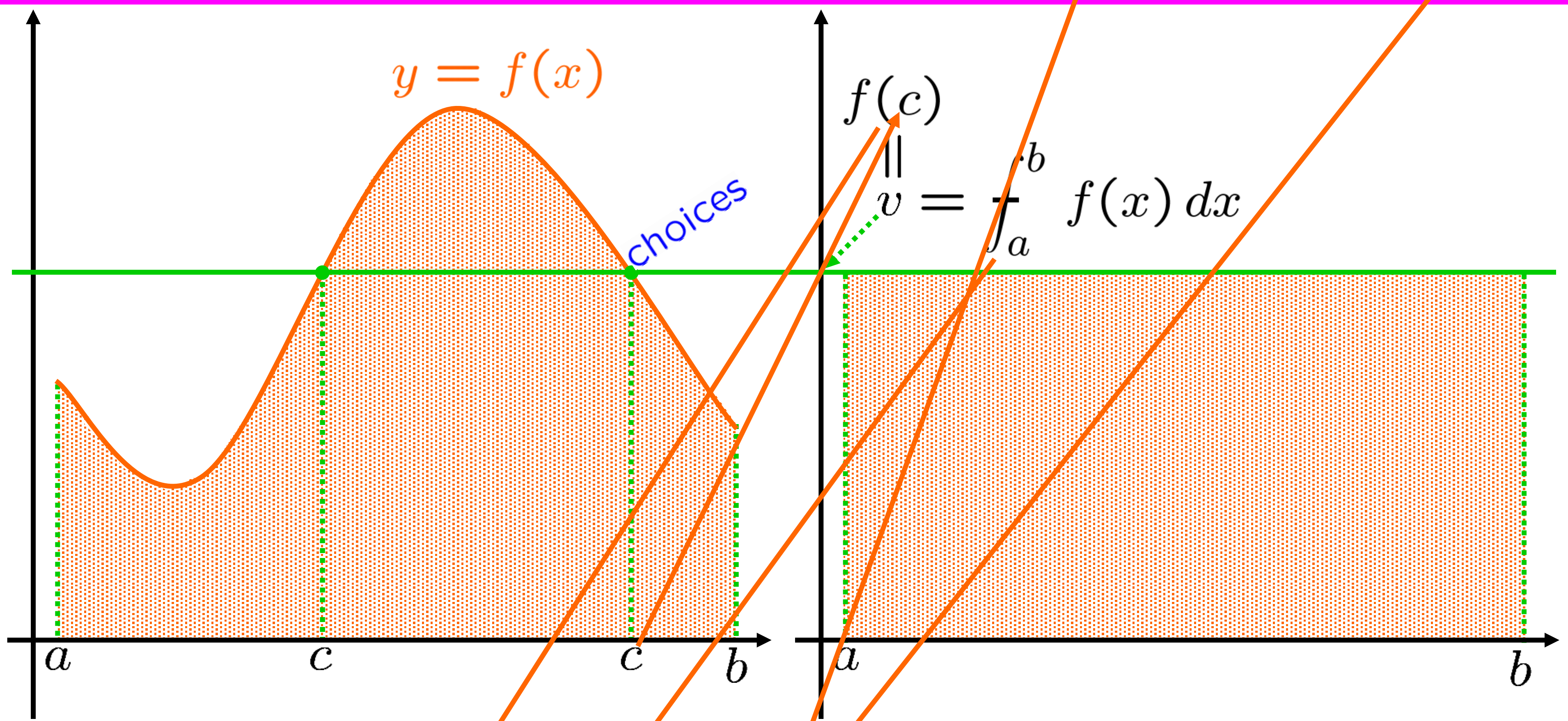

"the mean value is attained"

INTEGRAL MEAN VALUE THEOREM

If f is continuous on $[a, b]$, then there exists $c \in (a, b)$ s.t.

$$f(c) = \fint_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

DEFINITION: The **average** (or **mean**) of f on $[a,b]$ is

$$\fint_a^b f(x)\,dx := \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$\|$$

$$\fint_a^b f(t)\,dt := \frac{1}{b-a}\int_a^b f(t)\,dt$$

$$\|$$

$$\fint_a^b f(s)\,ds := \frac{1}{b-a}\int_a^b f(s)\,ds$$

$$\|$$

$$\fint_a^b f := \frac{1}{b-a}\int_a^b f$$

etc., etc., etc.

$\fint_a^b$ is linear, BUT NOT multiplicative.

$$\fint_a^c \neq \fint_a^b + \fint_b^c$$

$f :\longrightarrow v$ and $x :\longrightarrow t$

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$$\| $$

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$f :\longrightarrow v$ and $x :\longrightarrow t$

INTEGRAL MEAN VALUE THEOREM

If v is continuous on $[a, b]$, then there exists $c \in (a, b)$ s.t.

$$v(c) = \fint_a^b v(t)\,dt = \frac{1}{b-a}\int_a^b v(t)\,dt.$$

INTERPRETATION VIA MOTION ON A LINE

Think of v as the velocity of a particle traveling on a line.

$$\int_a^b v(t)\,dt \text{ is the displacement from time } a \text{ to time } b.$$

$$⨍_a^b v(t)\,dt \text{ is the average velocity from time } a \text{ to time } b.$$

The integral MVT says that average velocity is *ATTAINED* at some time c.

On my trip to Chicago, $a = 0$, $b = 8$ and $\int_a^b v(t)\,dt = 400$,

so $⨍_a^b v(t)\,dt = \dfrac{400}{8-0} = 50$ was my avg velocity.

The $\int$MVT says: I *ATTAINED* that velocity at some time(s).

INTEGRAL MEAN VALUE THEOREM

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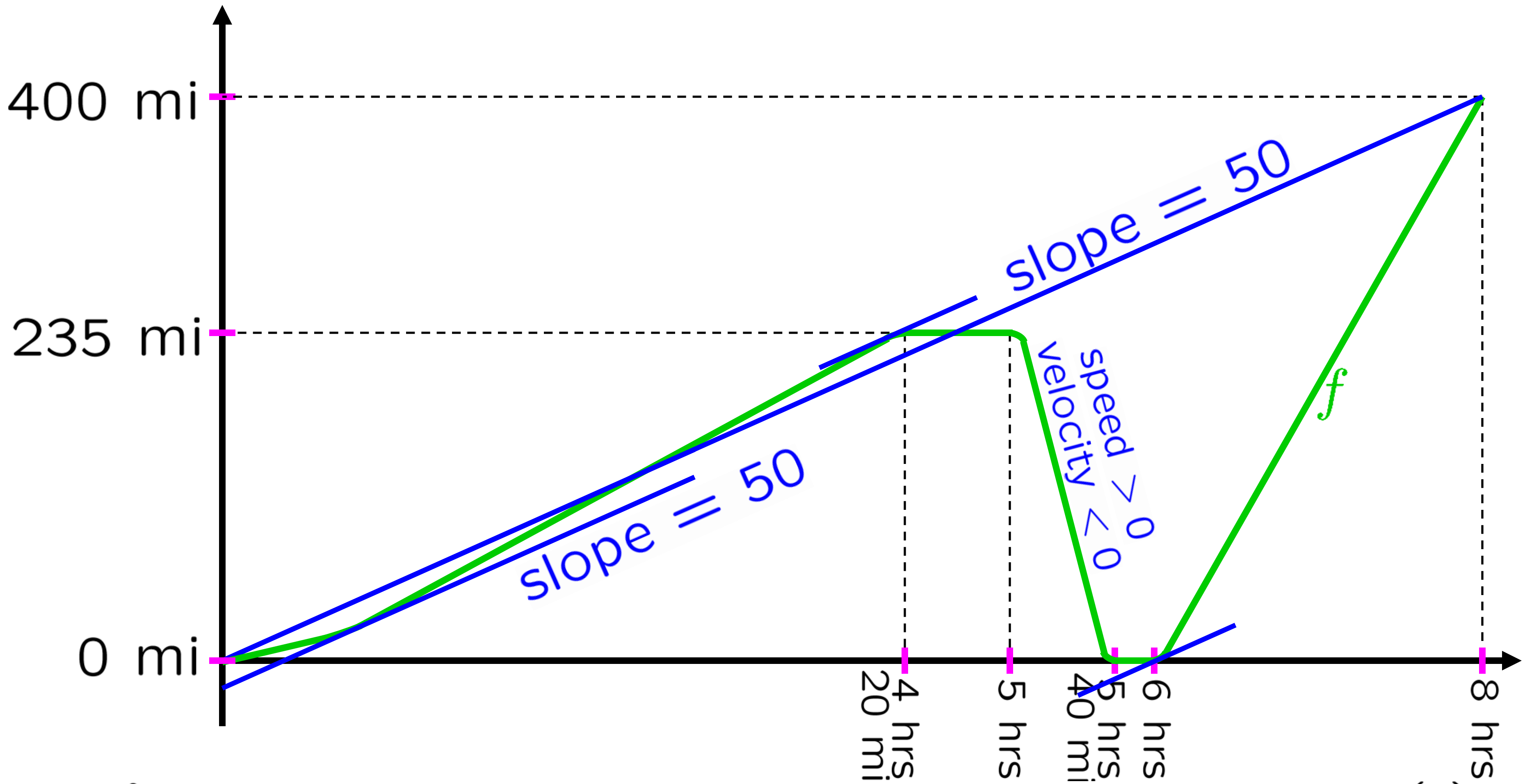

The $\int$MVT says: I *ATTAINED* that velocity at some time(s).

Expect: Every avg. velocity is an instantaneous velocity.

Expect: Every sec. slope is a tangent slope.

Average velocity is 50 mph from 0 hrs to 8 hrs.

instantaneous velocity is 50 mph at some time (45 min)

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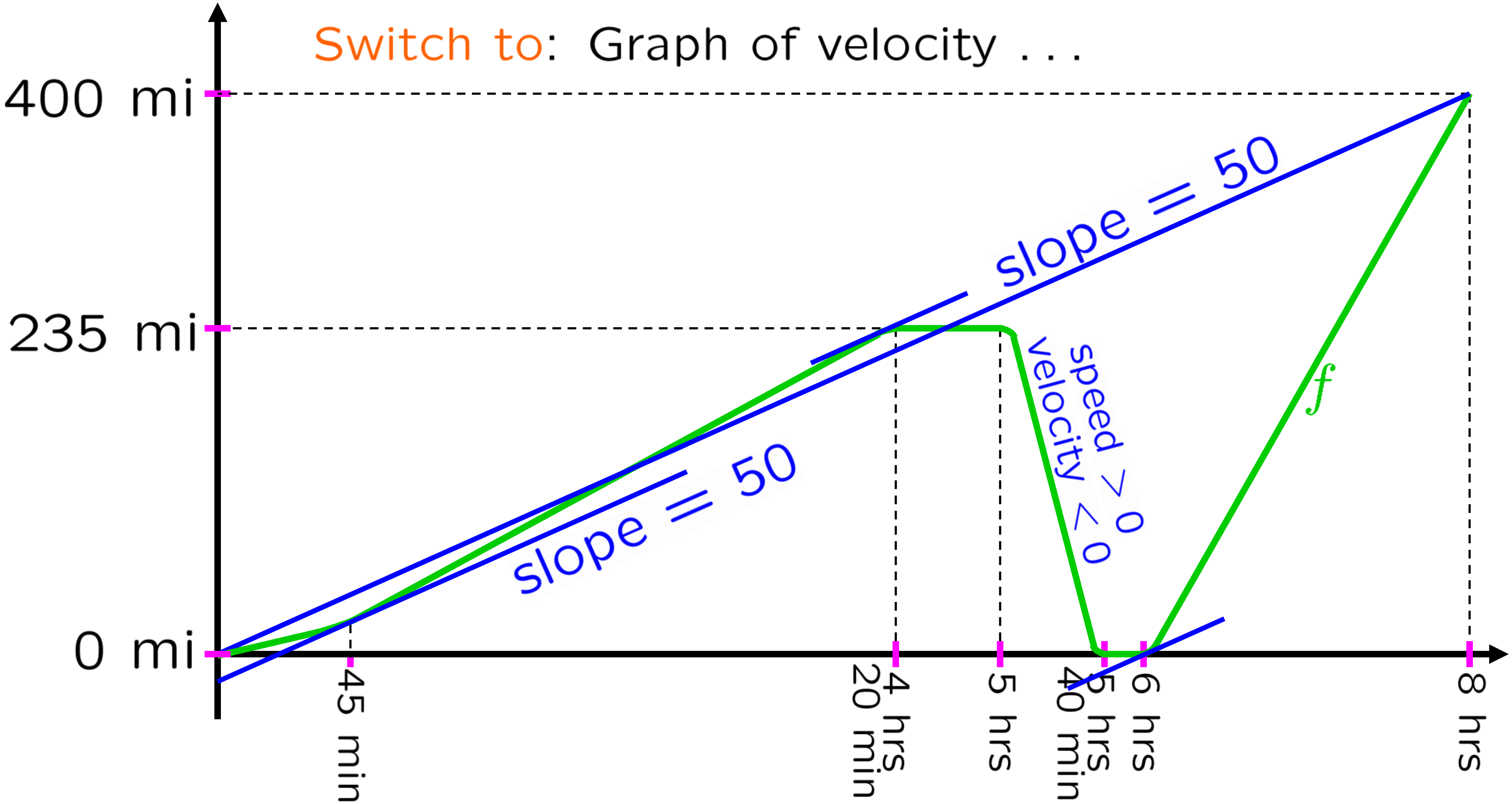


Expect: Every avg. velocity is an instantaneous velocity.

Expect: Every sec. slope is a tangent slope.

Average velocity is 50 mph from 0 hrs to 8 hrs.

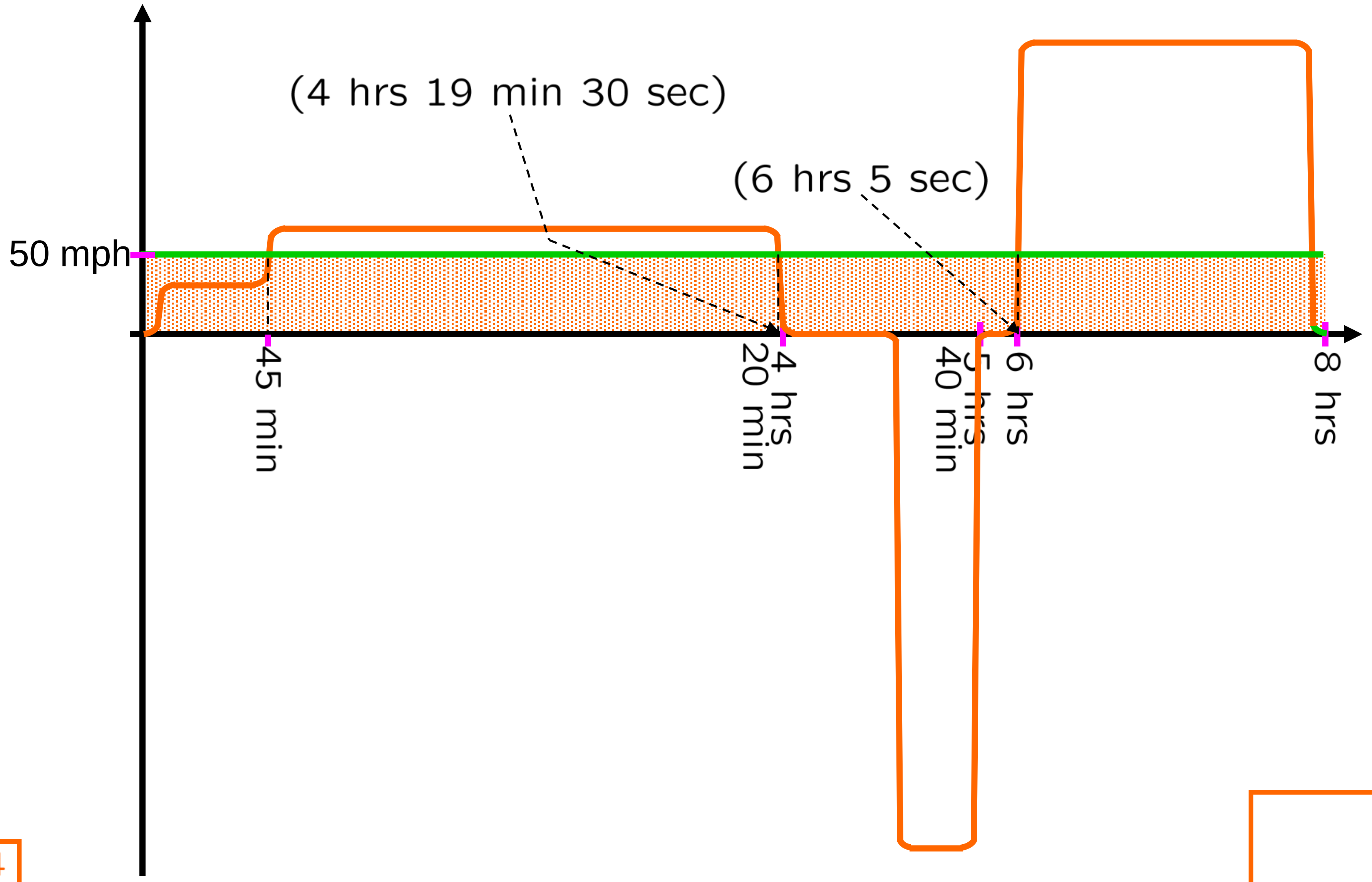
instantaneous velocity is 50 mph at some time (45 min)

(4 hrs 19 min 30 sec)

(6 hrs 5 sec)

The $\int$MVT says: I *ATTAINED* that velocity at some time(s).

Switch to: Graph of velocity . . .



EXAMPLE: Find the average value of the function $2 + x^3$ on the interval $[-1, 5]$.

Solution: $$⨍_{-1}^{5}(2+x^3)\,dx = \frac{1}{5-(-1)}\left[\int_{-1}^{5}(2+x^3)\,dx\right]$$

$$= \frac{1}{6}\left[2x + \frac{x^4}{4}\right]_{x:\to -1}^{x:\to 5}$$

$$= \frac{1}{6}\left[\left(10 + \frac{5^4}{4}\right) - \left(-2 + \frac{(-1)^4}{4}\right)\right]$$

$$= \frac{1}{6}\left[\left(10 + \frac{625}{4}\right) - \left(-2 + \frac{1}{4}\right)\right]$$

$$= \frac{1}{6}\left[12 + \frac{624}{4}\right] = \frac{1}{6}[12 + 156]$$

$$= \frac{1}{6}[168]$$

$$= 28 \blacksquare$$

SKILL
find avg value

EXAMPLE: Find a number $c \in (-1, 5)$ s.t.

$$\fint_{-1}^{5} (2 + x^3)\, dx = 2 + c^3.$$

Solution: $\fint_{-1}^{5} (2 + x^3)\, dx$

Solution: $\fint_{-1}^{5} (2 + x^3)\, dx$ $= 28$

$= 28$

EXAMPLE: Find a number $c \in (-1, 5)$ s.t.

$$\fint_{-1}^{5} (2 + x^3)\, dx = 2 + c^3.$$

Solution: $\fint_{-1}^{5} (2 + x^3)\, dx = 28$

$$28 = 2 + c^3$$

$$26 = c^3$$

$$c = \sqrt[3]{26}$$ ■

SKILL
where avg attained

EXAMPLE: Find a number $c \in (-6, 6)$ s.t.

$$⨍_{-6}^{6} (4 + x^2)\, dx = 4 + c^2.$$

Solution:

$$⨍_{-6}^{6} (4 + x^2)\, dx = \left[⨍_{-6}^{6} 4\, dx\right] + \left[⨍_{-6}^{6} x^2\, dx\right]$$

$$= 4 + \left[\frac{1}{12}\int_{-6}^{6} x^2\, dx\right]$$

$$= 4 + \left[\frac{1}{12}\left[\frac{x^3}{3}\right]_{x:\to -6}^{x:\to 6}\right]$$

$$= 4 + \left[\frac{1}{12}\left(\frac{[x^3]_{x:\to -6}^{x:\to 6}}{3}\right)\right]$$

$$= 4 + \left[\frac{1}{12}\left(\frac{6^3 - (-6)^3}{3}\right)\right] = 16$$

LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$⨍_a^b$ is linear,

BUT NOT

multiplicative.

§9.4

EXAMPLE: Find a number $c \in (-6, 6)$ s.t.

$$\fint_{-6}^{6} (4 + x^2)\, dx = 4 + c^2.$$

Solution: $\fint_{-6}^{6} (4 + x^2)\, dx = 16$ → $16 = 4 + c^2$

$$\fint_{-6}^{6} (4 + x^2)\, dx$$

$$= 16$$

EXAMPLE: Find a number $c \in (-6, 6)$ s.t.

$$⨍_{-6}^{6} (4 + x^2)\, dx = 4 + c^2.$$

Solution: $⨍_{-6}^{6} (4 + x^2)\, dx = 16$

$$16 = 4 + c^2$$

$$12 = c^2$$

$$c = \pm\sqrt{12}$$

$$= \pm 2\sqrt{3}$$ ■

SKILL
where avg attained

EXAMPLE: Find a number $c \in (0,6)$ s.t.

$$\fint_0^6 (4 + x^2)\, dx = 4 + c^2.$$

Solution: $\fint_0^6 (4 + x^2)\, dx = 16$

$$16 = 4 + c^2$$

$$12 = c^2$$

$$c = \pm\sqrt{12}$$

$$= \pm 2\sqrt{3}$$

$$c \in (0,6)$$

$$c = 2\sqrt{3} \quad \blacksquare$$

SKILL
where avg attained

EXAMPLE: Find the average value of the function

$f(x) = \sin(6x)$ on $[-2, 2]$.

odd function — midpt $= 0$

Solution: $⨍_{-2}^{2} \sin(6x)\,dx = \left[\frac{1}{2-(-2)}\right]\left[\int_{-2}^{2} \sin(6x)\,dx\right]$

$$= \left[\frac{1}{4}\right]\left[-\frac{\cos(6x)}{6}\right]_{x:\to -2}^{x:\to 2}$$

LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$= \left[\frac{1}{4}\right]\left[-\frac{[\cos(6x)]_{x:\to -2}^{x:\to 2}}{6}\right]$$

$$= \left[\frac{1}{4}\right]\left[-\frac{[\cos(12)] - [\cos(-12)]}{6}\right]$$

($[\cos(12)] - [\cos(-12)] = 0$)

$= 0$ ■

SKILL
find avg value

EXAMPLE: Find the average value of the function $f(x) = \cos(6x)$ on $[-2, 2]$.

Solution: $$⨍_{-2}^{2} \cos(6x)\,dx = \left[\frac{1}{2-(-2)}\right]\left[\int_{-2}^{2} \cos(6x)\,dx\right]$$

$$= \left[\frac{1}{4}\right]\left[\frac{\sin(6x)}{6}\right]_{x:\to -2}^{x:\to 2}$$

LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$= \left[\frac{1}{4}\right]\left[\frac{[\sin(6x)]_{x:\to -2}^{x:\to 2}}{6}\right]$$

sin is odd

$$= \left[\frac{1}{4}\right]\left[\frac{[\sin(12)] + [\sin(+12)]}{6}\right]$$

$$= \left[\frac{1}{4}\right]\left[\frac{2[\sin(12)]}{6}\right]$$

SKILL

find avg value

$$= \frac{\sin(12)}{12} \doteq -0.0447 \blacksquare$$

EXAMPLE: Find the average value of the function

$f(x) = \cos(6x)$ on $[0, \pi]$.

Exercise: Graph $y = \cos(6x)$ on $[0, \pi]$.

Hint: First, graph $y = \cos(x)$ on $[0, 6\pi]$.

Solution: $$⨍_0^\pi \cos(6x)\,dx = \left[\frac{1}{\pi - 0}\right]\left[\int_0^\pi \cos(6x)\,dx\right]$$

$$= \left[\frac{1}{\pi}\right]\left[\frac{\sin(6x)}{6}\right]_{x:\to 0}^{x:\to\pi}$$

LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$= \left[\frac{1}{\pi}\right]\left[\frac{[\sin(6x)]_{x:\to 0}^{x:\to\pi}}{6}\right]$$

$$= \left[\frac{1}{\pi}\right]\left[\frac{[\sin(6\pi)] - [\sin(0)]}{6}\right]$$

$$= \left[\frac{1}{\pi}\right]\left[\frac{[0] - [0]}{6}\right] = 0 \quad \blacksquare$$

SKILL
find avg value

EXAMPLE: Find the average value of the function

$f(x) = \cos(6x)$ on $[0, \frac{\pi}{12}]$.

Solution: $$⨍_0^{\pi/12} \cos(6x)\,dx = \left[\frac{1}{(\pi/12) - 0}\right]\left[\int_0^{\pi/12} \cos(6x)\,dx\right]$$

$$= \left[\frac{12}{\pi}\right]\left[\frac{\sin(6x)}{6}\right]_{x:\to 0}^{x:\to \pi/12}$$

LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$= \left[\frac{12}{\pi}\right]\left[\frac{[\sin(6x)]_{x:\to 0}^{x:\to \pi/12}}{6}\right]$$

$$= \left[\frac{12}{\pi}\right]\left[\frac{[\sin(\pi/2)] - [\sin(0)]}{6}\right]$$

$$= \left[\frac{12}{\pi}\right]\left[\frac{[1] - [0]}{6}\right] = \frac{2}{\pi} \blacksquare$$

SKILL
find avg value

EXAMPLE: Find the average value of the function

$$f(\theta) = \sec^2(\theta/4) \text{ on } [0,\pi].$$

Solution: $$⨍_0^\pi \sec^2(\theta/4)\,d\theta = \left[\frac{1}{\pi-0}\right]\left[\int_0^\pi \sec^2(\theta/4)\,d\theta\right]$$

$$= \left[\frac{1}{\pi}\right]\left[\frac{\tan(\theta/4)}{1/4}\right]_{\theta:\to 0}^{\theta:\to\pi}$$

LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$= \left[\frac{1}{\pi}\right]\left[\frac{[\tan(\theta/4)]_{\theta:\to 0}^{\theta:\to\pi}}{1/4}\right]$$

$$= \left[\frac{1}{\pi}\right]\left[\frac{[\tan(\pi/4)] - [\tan(0)]}{1/4}\right]$$

$$= \left[\frac{1}{\pi}\right]\left[\frac{[1]-[0]}{1/4}\right] = \frac{4}{\pi}$$ ■

SKILL
find avg value

EXAMPLE: Find the average value of the function

$$h(w) = (5-2w)^{-1} \text{ on } [-2,2].$$

$$\| \quad \frac{1}{5-2w}$$

Solution: $$⨍_{-2}^{2}\left[\frac{1}{5-2w}\right]dw = \left[\frac{1}{2-(-2)}\right]\left[\int_{-2}^{2}\left[\frac{1}{5-2w}\right]dw\right]$$

$$= \left[\frac{1}{4}\right]\left[\frac{\ln(|5-2w|)}{-2}\right]_{w:\to -2}^{w:\to 2}$$

LINEARITY OF $[\bullet]_{x:\to a}^{x:\to b}$

$$= \left[\frac{1}{4}\right]\left[\frac{[\ln(|5-2w|)]_{w:\to -2}^{w:\to 2}}{-2}\right]$$

$$= \left[\frac{1}{4}\right]\left[\frac{[\ln(|1|)] + [\ln(|9|)]}{+2}\right]$$

$$= \left[\frac{1}{4}\right]\left[\frac{\ln(9)}{2}\right] = \frac{\ln(9)}{8} \doteq 0.2747 \blacksquare$$

SKILL
find avg value

EXAMPLE: Let $f(x) = \sqrt[3]{x}$.

(a) Find $\displaystyle ⨍_0^8 f(x)\,dx$.

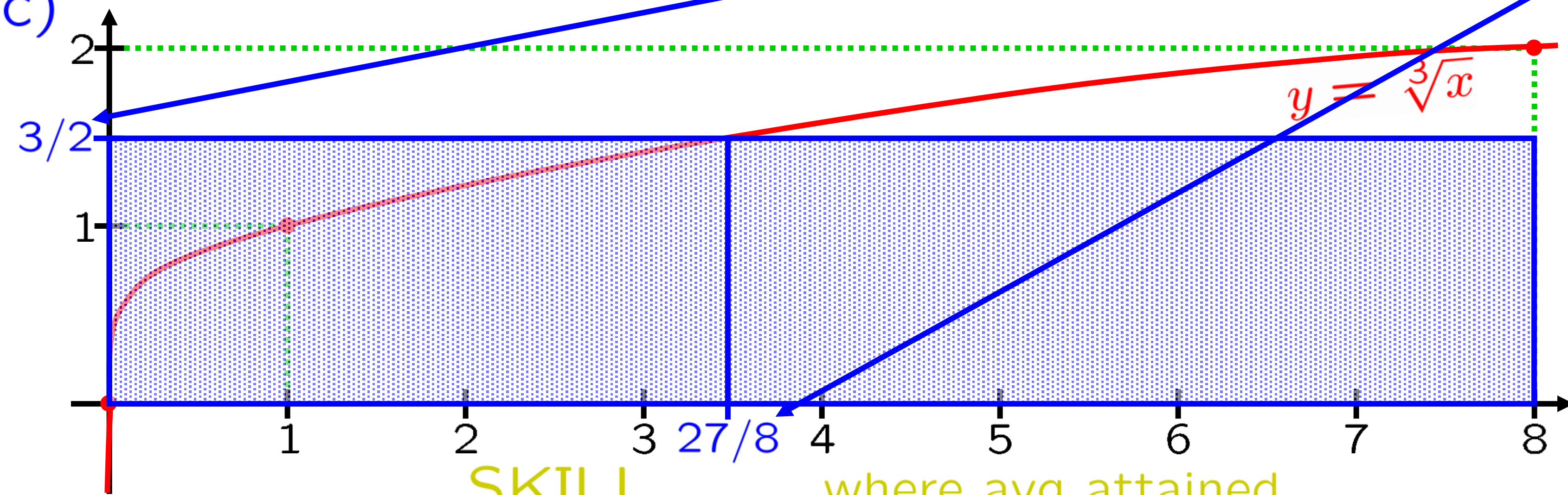
(b) Find $c \in (0,8)$ such that $\displaystyle ⨍_0^8 f(x)\,dx = f(c)$.

(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

Sol'n: (a) $\displaystyle \frac{1}{8}\int_0^8 x^{1/3}\,dx = \frac{1}{8}\left[\frac{x^{4/3}}{4/3}\right]_{x:\to 0}^{x:\to 8} = \frac{1}{8}\left[\frac{8^{4/3}}{4/3}\right] = \frac{1}{8}\left[\frac{2^4}{4/3}\right] = \frac{3}{2}$

(b) $3/2 = c^{1/3} \quad \Rightarrow \quad c = 27/8$

(c)



SKILL
find avg value

where avg attained
show avg rectangle

EXAMPLE: Let $f(x) = 2x/(1+x^2)^2$.

(a) Find $\displaystyle \int_0^2 f(x)\,dx$.

(b) Find $c \in (0,2)$ such that $\displaystyle \int_0^2 f(x)\,dx = f(c)$.

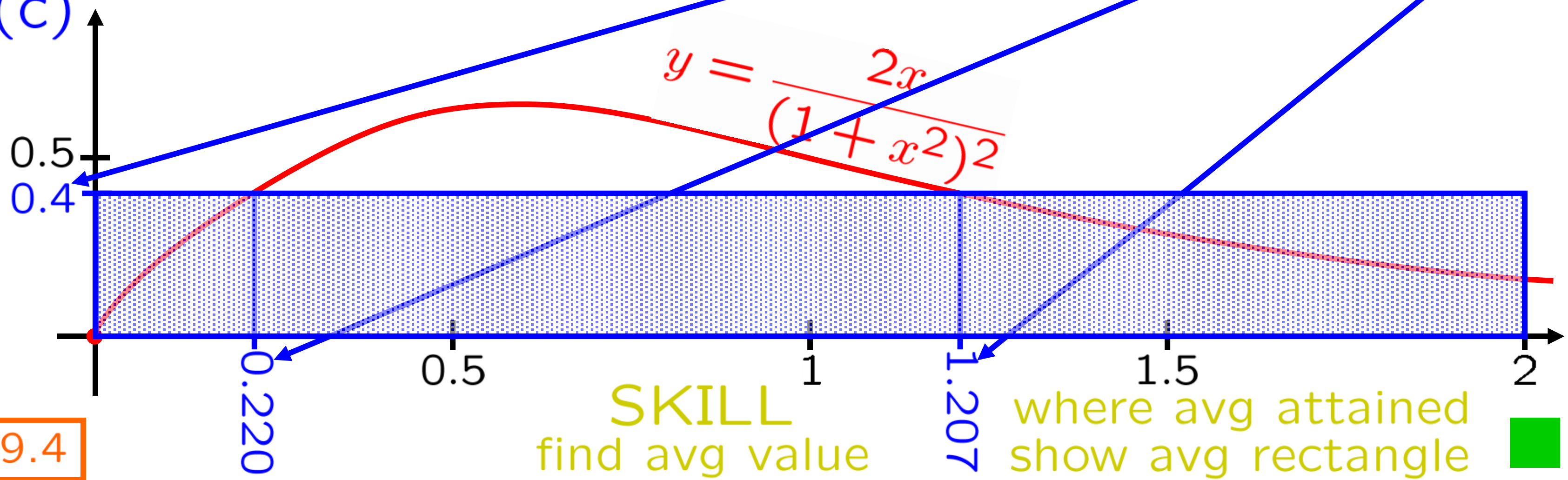
(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

Sol'n: (a) $$\frac{1}{2}\int_0^2 \frac{2x}{(1+x^2)^2}\,dx = \frac{1}{2}\left[\frac{(1+x^2)^{-1}}{-1}\right]_{x:\to 0}^{x:\to 2} = \frac{2}{5} = 0.4$$

(b) $$\frac{2}{5} = \frac{2c}{(1+c^2)^2} \Rightarrow c \in \{0.220, 1.207\}$$

online root finder

(c)



SKILL
find avg value

where avg attained
show avg rectangle

SKILL

average of a function

Whitman problems

§9.4, p. 195, #1-6