Paul Cusson's question

The main results in this note are:

Theorem 30, due to T. Tao,

and Theorem 42, and Theorem 57.

DEFINITION 1. *Let* $a, b \in \mathbb{R}$.

Then: $\boxed{(a;b)} := \{x \in \mathbb{R} \,|\, a < x < b\}$, $\boxed{[a;b)} := \{x \in \mathbb{R} \,|\, a \leqslant x < b\}$,

$\boxed{(a;b]} := \{x \in \mathbb{R} \,|\, a < x \leqslant b\}$, $\boxed{[a;b]} := \{x \in \mathbb{R} \,|\, a \leqslant x \leqslant b\}$.

DEFINITION 2. *Let* f *be a function.*

Then $\boxed{\mathbb{D}_f}$ *denotes the domain of* f.

Also, $\boxed{\mathbb{I}_f} := \{f(x) \,|\, x \in \mathbb{D}_f\}$ *denotes the image of* f.

DEFINITION 3. *Let* A *and* B *be sets.*

By $\boxed{f : A \to B}$, *we mean:* f *is a function and* $\mathbb{D}_f = A$ *and* $\mathbb{I}_f \subseteq B$.

By $\boxed{f : A \dashrightarrow B}$, *we mean:* f *is a function and* $\mathbb{D}_f \subseteq A$ *and* $\mathbb{I}_f \subseteq B$.

DEFINITION 4. $\boxed{\mathbb{N}} := \{1, 2, 3, \ldots\}$ *and* $\boxed{\mathbb{N}_0} = \{0, 1, 2, 3, \ldots\}$.

Convention: Any subset of $\mathbb{R}$ is given the relative topology inherited from the standard topology on $\mathbb{R}$.

NOTE: Any open subset of $\mathbb{R}$ is locally compact and Hausdorff.

NOTE: Any closed subset of any open subset of $\mathbb{R}$ is locally compact and Hausdorff.

THEOREM 5. *Let* W *be a non*$\varnothing$ *bounded open subset of* $\mathbb{R}$.

Let U *be a connected component of* W.

Then: $\exists s, t \in \mathbb{R} \backslash W$ *s.t.* $s < t$ *and s.t.* $U = (s;t)$.

Proof. Since U is a connected component of W, we get: $\varnothing \neq U \subseteq W$.

Since W is bounded and since $U \subseteq W$, we get: U is bounded.

The topological space $\mathbb{R}$ is locally connected, so,

since W is open in $\mathbb{R}$ and

since U is a connected component of W,

we get: U is a connected open subset of $\mathbb{R}$.

Since U is a non$\varnothing$ bounded connected open subset of $\mathbb{R}$,

choose $s, t \in \mathbb{R}$ s.t. $s < t$ and s.t. $U = (s;t)$.

Want: $s, t \notin W$. **Want:** $\{s, t\} \bigcap W = \varnothing$.

Assume: $\{s, t\} \bigcap W \neq \varnothing$. **Want:** Contradiction.

Choose $r \in \{s, t\} \bigcap W$. Then: $r \in \{s, t\}$ and $r \in W$.

Since W is open in $\mathbb{R}$ and since $r \in W$,
choose $\delta > 0$ s.t. $(r - \delta; r + \delta) \subseteq W$.
Since $r \in \{s, t\}$ and since $\delta > 0$,
we get: $(s; t) \bigcap (r - \delta; r + \delta) \neq \varnothing$.
Let $I := (r - \delta; r + \delta)$. Then: I is connected and $r \in I \subseteq W$.
Since $r \in I$, we get: $I \neq \varnothing$.
Since $I \subseteq W$ and since I is non$\varnothing$ and connected,
let V be the connected component of W s.t. $I \subseteq V$.
We have: $U \bigcap V \supseteq U \bigcap I = (s; t) \bigcap (r - \delta; r + \delta) \neq \varnothing$,
so, since U and V are both connected components of W,
we conclude: $U = V$. Then: $r \in I \subseteq V = U$, so $r \in U$.
So, since $r \in \{s, t\}$, we get: $r \in \{s, t\} \bigcap U$. Then $\{s, t\} \bigcap U \neq \varnothing$.
However, $\{s, t\} \bigcap U = \{s, t\} \bigcap (s; t) = \varnothing$. Contradiction. $\square$

THEOREM 6. *Let $c, d, p, r, w \in \mathbb{R}$. Assume: $c < p < w < r < d$. Let W be an open subset of $(c; d)$. Assume: $w \in W$ and $p, r \notin W$. Let U be the connected component of W s.t. $w \in U$. Then there exist $s, t \in [p; r] \backslash W$ s.t. $s < t$ and s.t. $U = (s; t)$.*

Proof. We have $w \in U \subseteq W$. Since $w \in W$, we get: $W \neq \varnothing$.
Since W open in $(c; d)$, and since $(c; d)$ is bounded and open in $\mathbb{R}$,
we get: W is a bounded open subset of $\mathbb{R}$.
So, since U is a connected component of W, by Theorem 5,
choose $s, t \in \mathbb{R} \backslash W$ s.t. $s < t$ and s.t. $U = (s; t)$.
Want: $s, t \in [p; r]$. **Want:** $p \leqslant s < t \leqslant r$.
Since $U = (s; t)$ and $w \in U$, we get: $(s; w) \subseteq U$.
By hypothesis, $p \notin W$, so, since $(s; w) \subseteq U \subseteq W$, we get: $p \notin (s; w)$.
By hypothesis, $p < w$. Since $p < w$ and $p \notin (s; w)$, we get: $p \leqslant s$.
By choice of s and t, we have: $s < t$. **It remains to show:** $t \leqslant r$.
Want: $r \geqslant t$. Since $U = (s; t)$ and $w \in U$, we get: $(w; t) \subseteq U$.
By hypothesis, $r \notin W$, so, since $(w; t) \subseteq U \subseteq W$, we get: $r \notin (w; t)$.
By hypothesis, $w < r$. Since $r > w$ and $r \notin (w; t)$, we get: $r \geqslant t$. $\square$

THEOREM 7. *Let $a, b \in \mathbb{R}$. Assume $a < b$. Let $X \subseteq (a; b)$. Let $X' \subseteq X$. Assume X' has non$\varnothing$ interior in X. Then: $\exists c, d \in [a; b]$ s.t. $c < d$ and s.t. $\varnothing \neq (c; d) \bigcap X \subseteq X'$.*

Proof. Let W denote the interior in X of X'. By hypothesis, $W \neq \varnothing$.
Also, W is open in X and $W \subseteq X'$. Since $W \neq \varnothing$, choose $w \in W$.
Since W is open in X, choose an open subset V of $\mathbb{R}$ s.t. $W = V \bigcap X$.

By hypothesis, $X \subseteq (a;b)$, so: $X = (a;b) \bigcap X$.
Since V and $(a;b)$ are open in $\mathbb{R}$, we get: $V \bigcap (a;b)$ is open in $\mathbb{R}$.
Let $U := V \bigcap (a;b)$. Then U is open in $\mathbb{R}$.
Also, $W = V \bigcap X = V \bigcap (a;b) \bigcap X = U \bigcap X$, so $W = U \bigcap X$.
Since $w \in W = U \bigcap X$, we get: $w \in U$ and $w \in X$.
Since $w \in U$ and since U is open in $\mathbb{R}$,
choose $c, d \in \mathbb{R}$ s.t. $c < d$ and s.t. $w \in (c;d) \subseteq U$.
Since $(c;d) \subseteq U = V \bigcap (a;b) \subseteq (a;b)$, we get: $(c;d) \subseteq (a;b)$.
It follows that $[c;d] \subseteq [a;b]$. Then $c, d \in [a;b]$.
It remains to show: $\varnothing \neq (c;d) \bigcap X \subseteq X'$.
Since $w \in (c;d)$ and since $w \in X$, we get: $w \in (c;d) \bigcap X$.
Then $\varnothing \neq (c;d) \bigcap X$. **Want:** $(c;d) \bigcap X \subseteq X'$.
Since $(c;d) \subseteq U$, we get: $(c;d) \bigcap X \subseteq U \bigcap X$.
Recall: $W \subseteq X'$ and $W = U \bigcap X$.
Then: $(c;d) \bigcap X \subseteq U \bigcap X = W \subseteq X'$. □

DEFINITION 8. $\forall S \subseteq \mathbb{R}$, *let* $\boxed{S^\circ}$ *denote the interior in* $\mathbb{R}$ *of* S.

DEFINITION 9. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.

Then: $$\boxed{\mathbb{D}'_f} := \left\{ x \in (\mathbb{D}_f)^\circ \;\middle|\; \lim_{h \to 0} \frac{(f(x+h)) - (f(x))}{h} \textit{ exists} \right\}.$$

Also, the $\boxed{\textbf{derivative } \textit{of } f}$ *is the function* $\boxed{f'} : \mathbb{D}'_f \to \mathbb{R}$

defined by: $$\forall x \in \mathbb{D}'_f, \quad f'(x) = \lim_{h \to 0} \frac{(f(x+h)) - (f(x))}{h}.$$

DEFINITION 10. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $j \in \mathbb{N}_0$.
Then: $\boxed{f^{(j)}}$ *denotes the* j*th derivative of* f.
Also, $\boxed{\mathbb{D}_f^{(j)}} := \mathbb{D}_{f^{(j)}}$ *denotes the domain of* $f^{(j)}$.

Note: $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $f^{(0)} = f$ and $\mathbb{D}_f^{(0)} = \mathbb{D}_f$.
Also, $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $f^{(1)} = f'$ and $\mathbb{D}_f^{(1)} = \mathbb{D}_{f'} = \mathbb{D}'_f$.

Also, $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\mathbb{D}_f^{(0)} \supseteq \mathbb{D}_f^{(1)} \supseteq \mathbb{D}_f^{(2)} \supseteq \mathbb{D}_f^{(3)} \supseteq \cdots$.
In fact, each set is contained in the *interior* in $\mathbb{R}$ of the preceding one.

DEFINITION 11. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Then: $\boxed{\mathbb{D}_f^{(\infty)}} := \mathbb{D}_f^{(0)} \bigcap \mathbb{D}_f^{(1)} \bigcap \mathbb{D}_f^{(2)} \bigcap \mathbb{D}_f^{(3)} \bigcap \cdots$.

Note that, $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\quad \mathbb{D}_f^{(0)} \bigcap \mathbb{D}_f^{(2)} \bigcap \mathbb{D}_f^{(4)} \bigcap \mathbb{D}_f^{(6)} \bigcap \cdots = \mathbb{D}_f^{(\infty)}$.
Also, $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall j \in \mathbb{N}_0$, $\quad \mathbb{D}_{f^{(j)}}^{(\infty)} = \mathbb{D}_f^{(\infty)}$.

Convention: $0^0 = 1$. Then: $\forall x \in \mathbb{R}$, $x^0 = 1$.

DEFINITION 12. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $k \in \mathbb{N}_0$, $c \in \mathbb{D}_f^{(k)}$.
Then: $\boxed{P_k^{f,c}} : \mathbb{R} \to \mathbb{R}$ *is defined by:*

$$\forall x \in \mathbb{R}, \quad P_k^{f,c}(x) \quad = \quad \sum_{i=0}^{k} \left[(f^{(i)}(c)) \cdot \frac{(x-c)^i}{i!} \right].$$

DEFINITION 13. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $c \in \mathbb{R}$.
By $\boxed{f \textit{ is } \textbf{real-analytic} \textit{ at } c}$, *we mean:*
$\exists \delta > 0$ *s.t.* $P_k^{f,c} \to f$ *pointwise on* $(c - \delta; c + \delta)$, *as* $k \to \infty$.

It is well-known that: $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall c \in \mathbb{R}$,
$(\, f$ is real-analytic at $c\,) \Rightarrow (\, c \in \mathbb{D}_f^{(\infty)}\,)$.

DEFINITION 14. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$.
By $\boxed{f \textit{ is } \textbf{real-analytic} \textit{ on } S}$, *we mean:*
$\forall x \in S$, $\quad f$ *is real-analytic at* x.

THEOREM 15. *Let* $\sigma, \tau : \mathbb{R} \dashrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$, $q \in I$.
Assume: I *is an interval.*
Assume: σ *and* τ *are both real-analytic on* I.
Assume: $\forall j \in \mathbb{N}_0$, $\sigma^{(j)}(q) = \tau^{(j)}(q)$.
Then: $\sigma = \tau$ *on* I.

Theorem 15 is well-known. Its proof is omitted.

THEOREM 16. *Let* $\beta_0, \beta_1, \beta_2, \ldots \in \mathbb{R}$. *Let* $c \in \mathbb{R}$.
Assume $\{\beta_0, \beta_1, \beta_2, \ldots\}$ *is bounded.*
Define $\phi : \mathbb{R} \to \mathbb{R}$ *by:* $\forall x \in \mathbb{R}$, $\displaystyle \phi(x) = \sum_{i=0}^{\infty} \left[\beta_i \cdot \frac{(x-c)^i}{i!} \right]$.

Then: ϕ *is real-analytic on* $\mathbb{R}$.

Also, $\forall j \in \mathbb{N}_0$, $\forall x \in \mathbb{R}$, $\displaystyle \phi^{(j)}(x) = \sum_{i=0}^{\infty} \left[\beta_{i+j} \cdot \frac{(x-c)^i}{i!} \right]$.

Theorem 16 is well-known. Its proof is omitted.

DEFINITION 17. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $x \in \mathbb{R}$, $M \geqslant 0$.
By f *has* $\boxed{M\textbf{-BD} \textit{ at } x}$, *we mean:*

$x \in \mathbb{D}_f^{(\infty)}$ *and* $\forall j \in \mathbb{N}_0, \quad |f^{(j)}(x)| \leqslant M.$

By f has $\boxed{M\text{-}\mathbf{BED}\ \textit{at}\ x}$*, we mean:*

$x \in \mathbb{D}_f^{(\infty)}$ *and* $\forall j \in \mathbb{N}_0, \quad |f^{(2j)}(x)| \leqslant M.$

BD stands for "bounded derivatives".
BED stands for "bounded even derivatives".

DEFINITION 18. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}, \quad x \in \mathbb{R}.$
By f has $\boxed{\mathbf{BD}\ \textit{at}\ x}$*, we mean:*
$\exists M \geqslant 0$ *s.t.* *f has M-BD at x.*
By f has $\boxed{\mathbf{BED}\ \textit{at}\ x}$*, we mean:*
$\exists M \geqslant 0$ *s.t.* *f has M-BED at x.*

Note: $\forall f : \mathbb{R} \dashrightarrow \mathbb{R}$, $\forall x \in \mathbb{R}$,
(f has BD at x) $\Rightarrow$ (f has BED at x) $\Rightarrow$ ($x \in \mathbb{D}_f^{(\infty)}$).

DEFINITION 19. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}, \quad S \subseteq \mathbb{R}, \quad M \geqslant 0.$
By f has $\boxed{M\text{-}\mathbf{BD}\ \textit{on}\ S}$*, we mean:*
$\forall x \in S,$ *f has M-BD at x.*
By f has $\boxed{M\text{-}\mathbf{BED}\ \textit{on}\ S}$*, we mean:*
$\forall x \in S,$ *f has M-BED at x.*

DEFINITION 20. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}, \quad S \subseteq \mathbb{R}.$
By f has $\boxed{\mathbf{PBD}\ \textit{on}\ S}$*, we mean:*
$\forall x \in S,$ *f has BD at x.*
By f has $\boxed{\mathbf{PBED}\ \textit{on}\ S}$*, we mean:*
$\forall x \in S,$ *f has BED at x.*
By f has $\boxed{\mathbf{UBD}\ \textit{on}\ S}$*, we mean:*
$\exists M \geqslant 0$ *s.t.* *f has M-BD on S.*
By f has $\boxed{\mathbf{UBED}\ \textit{on}\ S}$*, we mean:*
$\exists M \geqslant 0$ *s.t.* *f has M-BED on S.*

PBD stands for "pointwise bounded derivatives".
PBED stands for "pointwise bounded even derivatives".
UBD stands for "uniformly bounded derivatives".
UBED stands for "uniformly bounded even derivatives".

DEFINITION 21. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}.$
Then $\boxed{\mathrm{BD}_f} := \{x \in \mathbb{D}_f^{(\infty)} \mid f \textit{ has BD at } x\}.$

DEFINITION 22. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $c \in \mathrm{BD}_f$.
Then: $\boxed{P_\infty^{f,c}} : \mathbb{R} \to \mathbb{R}$ *is defined by:*

$$\forall x \in \mathbb{R}, \qquad P_\infty^{f,c}(x) \;=\; \sum_{i=0}^{\infty} \left[(f^{(i)}(c)) \cdot \frac{(x-c)^i}{i!} \right].$$

THEOREM 23. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $c \in \mathrm{BD}_f$, $g = P_\infty^{f,c}$.
Then: g *is real-analytic on* $\mathbb{R}$. *Also:* $\forall j \in \mathbb{N}_0$, $f^{(j)}(c) = g^{(j)}(c)$.

Proof. For all $i \in \mathbb{N}_0$, let $\beta_i := f^{(i)}(c)$.
Since $c \in \mathrm{BD}_f$, we get: $\{\beta_0, \beta_1, \beta_2, \ldots\}$ is bounded.
Since $g = P_\infty^{f,c}$, we get: $\forall x \in \mathbb{R}$, $g(x) = \displaystyle\sum_{i=0}^{\infty} \left[\beta_i \cdot \frac{(x-c)^i}{i!} \right]$.
Then, by Theorem 16, we get: g is real-analytic on $\mathbb{R}$.
It remains to show: $\forall j \in \mathbb{N}_0$, $f^{(j)}(c) = g^{(j)}(c)$.
Given $j \in \mathbb{N}_0$, **want:** $f^{(j)}(c) = g^{(j)}(c)$. **Want:** $g^{(j)}(c) = \beta_j$.

By Theorem 16, we get: $g^{(j)}(c) = \displaystyle\sum_{i=0}^{\infty} \left(\beta_{i+j} \cdot \frac{(c-c)^i}{i!} \right)$.

Then $g^{(j)}(c) = \displaystyle\sum_{i=0}^{\infty} \left(\beta_{i+j} \cdot \frac{0^i}{i!} \right) = \left[\beta_{0+j} \cdot \frac{0^0}{0!} \right] + \left[\sum_{i=1}^{\infty} \left(\beta_{i+j} \cdot \frac{0^i}{i!} \right) \right]$.

Then $g^{(j)}(c) = [\beta_j \cdot 1] + \left[\displaystyle\sum_{i=1}^{\infty} (\beta_{i+j} \cdot 0) \right] = \beta_j + 0 = \beta_j$. □

THEOREM 24. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $B \subseteq \mathbb{R}$, $c, x \in B$, $M \geqslant 0$.
Assume: B *is an interval.* *Assume:* f *has* M*-BD on* B.
Let $j \in \mathbb{N}_0$. *Then:* $|(f(x)) - (P_j^{f,c}(x))| \leqslant M \cdot \dfrac{|x-c|^{j+1}}{(j+1)!}$.

Proof. Since f has M-BD on B, we get: $B \subseteq \mathbb{D}_f^{(\infty)}$.
By Taylor's Theorem, choose ξ strictly between c and x s.t.

$$f(x) \;=\; (P_j^{f,c}(x)) \;+\; \left((f^{(j+1)}(\xi)) \cdot \frac{(x-c)^{j+1}}{(j+1)!} \right).$$

Then: $\quad (f(x)) - (P_j^{f,c}(x)) \;=\; (f^{(j+1)}(\xi)) \cdot \dfrac{(x-c)^{j+1}}{(j+1)!}$.

Then: $\quad |(f(x)) - (P_j^{f,c}(x))| \;=\; |f^{(j+1)}(\xi)| \cdot \dfrac{|x-c|^{j+1}}{(j+1)!}$.
Since B is an interval and $c, x \in B$, we get: $\xi \in B$.
So, since f has M-BD on B, we get: $|f^{(j+1)}(\xi)| \leqslant M$.

Then: $\quad |(f(x)) - (P_j^{f,c}(x))| \leqslant M \cdot \dfrac{|x-c|^{j+1}}{(j+1)!}$. □

DEFINITION 25. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $x \in \mathbb{R}$.
By f *has* **UBD** *near* x, *we mean:*
$\exists \delta > 0$ *s.t.* f *has UBD on* $(x - \delta; x + \delta)$.

THEOREM 26. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $U \subseteq \mathbb{R}$.
Assume: $\forall x \in U$, f *has UBD near* x.
Then: f *is real-analytic on* U.

Proof. Given $c \in U$, **want:** f is real-analytic at c.
Want: $\exists \delta > 0$ s.t. $P_j^{f,c} \to f$ pointwise on $(c - \delta; c + \delta)$, as $j \to \infty$.
Since $c \in U$, by hypothesis, f has UBD near c, so
choose $\delta > 0$ s.t. f has UBD on $(c - \delta; c + \delta)$.
Want: $P_j^{f,c} \to f$ pointwise on $(c - \delta; c + \delta)$, as $j \to \infty$.
Let $B := (c - \delta; c + \delta)$.
Then: B is an interval and $c \in B$ and f has UBD on B.
Want: $P_j^{f,c} \to f$ pointwise on B, as $j \to \infty$.
Given $x \in B$, **want:** $P_j^{f,c}(x) \to f(x)$, as $j \to \infty$.
Want: $|(f(x)) - (P_j^{f,c}(x))| \to 0$, as $j \to \infty$.
Since f has UBD on B, choose $M \geqslant 0$ s.t. f has M-BD on B.
Then, by Theorem 24, $\forall j \in \mathbb{N}_0$, $|(f(x)) - (P_j^{f,c}(x))| \leqslant M \cdot \dfrac{|x-c|^{j+1}}{(j+1)!}$.
So, since $M \cdot \dfrac{|x-c|^{j+1}}{(j+1)!} \to 0$, as $j \to \infty$,
we conclude: $|(f(x)) - (P_j^{f,c}(x))| \to 0$, as $j \to \infty$. □

THEOREM 27. *Let* $f, g : \mathbb{R} \dashrightarrow \mathbb{R}$, $r, s, t \in \mathbb{R}$.
Assume: $s < t$ *and* $r \in [s; t]$.
Assume: $r \in \mathbb{D}_f^{(\infty)} \bigcap \mathbb{D}_g^{(\infty)}$ *and* $(s; t) \subseteq \mathbb{D}_f^{(\infty)} \bigcap \mathbb{D}_g^{(\infty)}$.
Assume: $f = g$ *on* $(s; t)$.
Then: $\forall j \in \mathbb{N}_0$, $f^{(j)}(r) = g^{(j)}(r)$.

Proof. Given $j \in \mathbb{N}_0$, **want:** $f^{(j)}(r) = g^{(j)}(r)$.
Since $f = g$ on $(s; t)$, we get: $f^{(j)} = g^{(j)}$ on $(s; t)$.
Let $\phi := f^{(j)}$ and $\psi := g^{(j)}$.
Then: $\phi = \psi$ on $(s; t)$. **Want:** $\phi(r) = \psi(r)$.
We have: $\mathbb{D}_\phi^{(\infty)} = \mathbb{D}_f^{(\infty)}$ and $\mathbb{D}_\psi^{(\infty)} = \mathbb{D}_g^{(\infty)}$.
Then: $r \in \mathbb{D}_\phi^{(\infty)} \bigcap \mathbb{D}_\psi^{(\infty)}$ and $(s; t) \subseteq \mathbb{D}_\phi^{(\infty)} \bigcap \mathbb{D}_\psi^{(\infty)}$.

Since $r \in \mathbb{D}_\phi^{(\infty)} \bigcap \mathbb{D}_\psi^{(\infty)} \subseteq \mathbb{D}_\phi^{(1)} \bigcap \mathbb{D}_\psi^{(1)}$,
we get: ϕ and ψ are both differentiable at r.
Then: ϕ and ψ are both continuous at r.
Since $r \in [s;t]$, choose $q_1, q_2, q_3 \cdots \in (s;t)$ s.t. $q_j \to r$, as $j \to \infty$.
By continuity, $\phi(q_j) \to \phi(r)$, as $j \to \infty$ and $\psi(q_j) \to \psi(r)$, as $j \to \infty$.
Since $\phi = \psi$ on $(s;t)$, we get: $\forall j \in \mathbb{N}$, $\phi(q_j) = \psi(q_j)$.
So, letting $j \to \infty$, we get: $\phi(r) = \psi(r)$. □

THEOREM 28. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $s, t \in \mathbb{R}$, $M \geqslant 0$.
Assume: $s < t$. *Assume:* $\forall x \in (s;t)$, f *has UBD near* x.
Let $r \in [s;t]$. *Assume:* f *has* M*-BD at* r.
Let $N := M \cdot e^{t-s}$. *Then:* f *has* N*-BD on* $(s;t)$.

Proof. Let $c := (s+t)/2$. Then $c \in (s;t)$.
So, by hypothesis, we get: f has UBD near c.
Then f has BD at c. Then $c \in \mathrm{BD}_f$. Let $g := P_\infty^{f,c}$.
By Theorem 23, g is real-analytic on $\mathbb{R}$.
Then $\mathbb{D}_g^{(\infty)} = \mathbb{R}$, so: $r \in \mathbb{D}_g^{(\infty)}$ and $(s;t) \subseteq \mathbb{D}_g^{(\infty)}$.
By hypothesis, f has M-BD at r, so we get: $r \in \mathbb{D}_f^{(\infty)}$.
By hypothesis, we have: $\forall x \in (s;t)$, f has UBD near x.
So, by Theorem 26, f is real-analytic on $(s;t)$. Then: $(s;t) \subseteq \mathbb{D}_f^{(\infty)}$.
Then: $r \in \mathbb{D}_f^{(\infty)} \bigcap \mathbb{D}_g^{(\infty)}$ and $(s;t) \subseteq \mathbb{D}_f^{(\infty)} \bigcap \mathbb{D}_g^{(\infty)}$.
By Theorem 23, we get: $\forall j \in \mathbb{N}_0$, $f^{(j)}(c) = g^{(j)}(c)$.
So, since $c \in (s;t)$ and since f and g are both real-analytic on $(s;t)$,
by Theorem 15, we get: $f = g$ on $(s;t)$.
Then, by Theorem 27, we get: $\forall j \in \mathbb{N}_0$, $f^{(j)}(r) = g^{(j)}(r)$.
By hypothesis, f has M-BD at r, so f has BD at r. Then $r \in \mathrm{BD}_f$.
Let $h := P_\infty^{f,r}$. Then, by Theorem 23, h is real-analytic on $\mathbb{R}$.
Also, by Theorem 23, $\forall j \in \mathbb{N}_0$, $f^{(j)}(r) = h^{(j)}(r)$.
Since $\forall j \in \mathbb{N}_0$, $g^{(j)}(r) = f^{(j)}(r) = h^{(j)}(r)$.
and since g and h are both real-analytic on $\mathbb{R}$,
by Theorem 15, we get: $g = h$ on $\mathbb{R}$.
So, since $f = g$ on $(s;t)$, we get: $f = h$ on $(s;t)$.
It therefore suffices to show: h has N-BD on $(s;t)$.
Given $u \in (s;t)$, **want:** h has N-BD at u.
Given $j \in \mathbb{N}_0$, **want:** $|h^{(j)}(u)| \leqslant N$. By hypothesis, $r \in [s;t]$.
Since $r, u \in [s;t]$, we get: $|u - r| \leqslant t - s$. Then $e^{|u-r|} \leqslant e^{t-s}$.
So, since $M \geqslant 0$, we get: $M \cdot e^{|u-r|} \leqslant M \cdot e^{t-s}$.

By hypothesis, f has M-BD at r, so: $\forall i \in \mathbb{N}_0,\ \ |f^{(i)}(r)| \leqslant M$.

Since $h = P_\infty^{f,r}$, we get: $\forall x \in \mathbb{R},\ \ h(x) \ = \ \sum_{i=0}^{\infty} \left[(f^{(i)}(r)) \cdot \frac{(x-r)^i}{i!} \right]$.

Then, by Theorem 16, we have: $\forall x \in \mathbb{R}$,

$$h^{(j)}(x) \ = \ \sum_{i=0}^{\infty} \left[(f^{(i+j)}(r)) \cdot \frac{(x-r)^i}{i!} \right].$$

Then:
$$\begin{aligned} |h^{(j)}(u)| &\leqslant \sum_{i=0}^{\infty} \left[|f^{(i+j)}(r)| \cdot \frac{|u-r|^i}{i!} \right] \\ &\leqslant \sum_{i=0}^{\infty} \left[M \cdot \frac{|u-r|^i}{i!} \right] \ = \ M \cdot \left[\sum_{i=0}^{\infty} \frac{|u-r|^i}{i!} \right] \\ &= \ M \cdot e^{|u-r|} \ \leqslant \ M \cdot e^{t-s} \ = \ N. \end{aligned}$$
□

THEOREM 29. *Let* $I \subseteq \mathbb{R}$, $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Assume: I *is a non*$\varnothing$ *bounded open interval.*
Assume: $\forall x \in I$, f *has UBD near* x. *Then:* f *has UBD on* I.

Proof. Since I is an interval, we get: I is connected.
Since I is a non$\varnothing$ bounded connected open subset of $\mathbb{R}$,
choose $s, t \in \mathbb{R}$ s.t. $s < t$ and s.t. $I = (s;t)$.
Then: $\forall x \in (s;t)$, f has UBD near x.
By Theorem 26, f is real-analytic on $(s;t)$.
Let $r := (s+t)/2$. Then $r \in (s;t)$. Then $r \in I$ and $r \in [s;t]$.
Since $r \in I$, by assumption, f has UBD near r.
Then f has BD at r. Choose $M \geqslant 0$ s.t. f has M-BD at r.
Let $N := M \cdot e^{t-s}$. By Theorem 28, f has N-BD on $(s;t)$.
Then f has UBD on $(s;t)$. Then f has UBD on I. □

Theorem 30 and the proof below are both due to T. Tao. See
https://mathoverflow.net/questions/413165/does-iterating-the-derivative-infinitely-many-times-give-a-smooth-function-whene

THEOREM 30. *(T. Tao) Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume: $a < b$. *Let* $I := (a;b)$.
Assume: f *has PBD on* I. *Then:* f *has UBD on* I.

Proof. Let $V := \{x \in I \,|\, f \text{ has UBD near } x\}$. Then V is open in I.
By Theorem 29, **it suffices to show:** $V = I$.
Let $X := I \backslash V$. Then $V = I \backslash X$. **Want:** $X = \varnothing$.

Assume: $X \neq \varnothing$. **Want:** Contradiction.
Since $I = (a;b)$, we get: I is open in $\mathbb{R}$.
Since V is open in I and since $X = I \backslash V$, we get: X is closed in I.
Since X is closed in I and since I is open in $\mathbb{R}$,
we get: X is locally compact and Hausdorff.
By hypothesis, f has PBD on I, so, since $X = I \backslash V \subseteq I$,
we get: f has PBD on X.
Then: $X \subseteq \mathbb{D}_f^{(\infty)}$. For all $m \in \mathbb{N}$, let $X_m := \{x \in X \mid f \text{ has } m\text{-BD at } x\}$.
By continuity, we get: $\forall m \in \mathbb{N}$, X_m is closed in X.
Since f has PBD on X, we get: $X_1 \bigcup X_2 \bigcup X_3 \bigcup \cdots = X$.
So, since X is non$\varnothing$ and locally compact and Hausdorff,
by the Baire Category Theorem,
choose $M \in \mathbb{N}$ s.t. X_M has non$\varnothing$ interior in X.
So, since $X = I \backslash V \subseteq I = (a;b)$, by Theorem 7, choose $c, d \in [a;b]$
s.t. $c < d$ and s.t. $\varnothing \neq (c;d) \bigcap X \subseteq X_M$.
Since $\varnothing \neq (c;d) \bigcap X$, choose $q \in (c;d) \bigcap X$.
Then $q \in X_M$. Also, $q \in (c;d)$ and $q \in X$.
Since $q \in (c;d)$ and since $(c;d)$ is open in $\mathbb{R}$,
choose $\delta > 0$ s.t. $(q - \delta; q + \delta) \subseteq (c;d)$.
Since $q \in X = I \backslash V$, by definition of V,
we get: f does not have UBD near q.
Then: f does not have UBD on $(q - \delta; q + \delta)$.
So, since $(q - \delta; q + \delta) \subseteq (c;d)$, we get:
f does not have UBD on $(c;d)$.
Let $K := M \cdot e^{d-c}$. Then f does not have K-BD on $(c;d)$.
Choose $p \in (c;d)$ s.t. f does not have K-BD at p.
Since $c < d$, we get: $e^{d-c} \geqslant 1$. Then: $K \geqslant M$.
By definition of X_M, f has M-BD on X_M.
So, since $K \geqslant M$, we get: f has K-BD on X_M.
So, since f does not have K-BD at p, we get: $p \notin X_M$.
Since $I = (a;b)$, we get: I is open in $\mathbb{R}$.
Since X_M is closed in X and since X is closed in I,
we get: X_M is closed in I. Then: $I \backslash X_M$ is open in I.
So, since I is open in $\mathbb{R}$, we get: $I \backslash X_M$ is open in $\mathbb{R}$.
Since $c, d \in [a;b]$, we get: $(c;d) \subseteq (a;b)$.
Since $(c;d) \subseteq (a;b) = I$, we get: $(c;d) \backslash X_M = (c;d) \bigcap (I \backslash X_M)$.
Let $W := (c;d) \backslash X_M$. Then: $W = (c;d) \bigcap (I \backslash X_M)$.
Since $(c;d)$ and $I \backslash X_M$ are both open in $\mathbb{R}$,

we get: $(c;d)\bigcap(I\backslash X_M)$ is open in $\mathbb{R}$. Then W is open in $\mathbb{R}$.
Since $p \in (c;d)$ and $p \notin X_M$, we get: $p \in W$. Then: $W \neq \varnothing$.
Since $W = (c;d)\backslash X_M \subseteq (c;d)$, we get: $W \subseteq (c;d)$.
Then W is bounded. Then W is a non$\varnothing$ bounded open subset of $\mathbb{R}$.
Recall: $(c;d)\bigcap X \subseteq X_M$. Then $[(c;d)\bigcap X]\backslash X_M = \varnothing$.
Then: $W\bigcap X = [(c;d)\backslash X_M]\bigcap X = [(c;d)\bigcap X]\backslash X_M = \varnothing$.
Then: $W\bigcap X = \varnothing$. Also, $W \subseteq (c;d) \subseteq (a;b) = I$, so $W \subseteq I$.
Since $W \subseteq I$ and $W\bigcap X = \varnothing$, we get: $W \subseteq I\backslash X$.
Then $W \subseteq I\backslash X = V$, so, by definition of V,
we get: $\forall x \in W$, f has UBD near x.
Let U be the connected component of W s.t. $p \in U$.
Then: $p \in U \subseteq W$. Then: $\forall x \in U$, f has UBD near x.
By Theorem 5, choose $s, t \in \mathbb{R}\backslash W$ s.t. $s < t$ and s.t. $U = (s;t)$.
Then: $\{s,t\} \subseteq \mathbb{R}\backslash W$. Recall: $W \subseteq (c;d)$.
Then $(s;t) = U \subseteq W \subseteq (c;d)$, so $(s;t) \subseteq (c;d)$, so $[s;t] \subseteq [c;d]$.
Then: $s, t \in [c;d]$. Then: $c \leqslant s < t \leqslant d$.
Then: $t - s \leqslant d - c$. Then: $e^{t-s} \leqslant e^{d-c}$.
Since $M \in \mathbb{N}$, we get: $M > 0$. Then: $M \cdot e^{t-s} \leqslant M \cdot e^{d-c}$.
Let $N := M \cdot e^{t-s}$. Recall: $K = M \cdot e^{d-c}$. Then $N \leqslant K$.
Since $W = (c;d)\backslash X_M$ and since $q \in X_M$, we get: $q \notin W$.
So, since $(s;t) = U \subseteq W$, we get: $q \notin (s;t)$. Recall: $q \in (c;d)$.
Since $q \notin (s;t)$ and since $q \in (c;d)$, we get: $(s;t) \neq (c;d)$.
Since $(s;t) \neq (c;d)$, we get: either $s \neq c$ or $t \neq d$.
Recall: $c \leqslant s < t \leqslant d$.
Then: either $c < s < t \leqslant d$ or $c \leqslant s < t < d$.
Then: either $c < s < d$ or $c < t < d$.
Then: either $s \in (c;d)$ or $t \in (c;d)$.
Then: $\{s,t\}\bigcap(c;d) \neq \varnothing$. Choose $r \in \{s,t\}\bigcap(c;d)$.
Since $r \in \{s,t\} \subseteq \mathbb{R}\backslash W$, we get: $r \in \mathbb{R}\backslash W$. Then: $r \in (c;d)\backslash W$.
By definition of W, we have: $W = (c;d)\backslash X_M$.
Since $r \in (c;d)\backslash W = (c;d)\backslash[(c;d)\backslash X_M] = (c;d)\bigcap X_M \subseteq X_M$,
by definition of X_M, we get: f has M-BD at r.
We have $r \in \{s,t\} \subseteq [s;t]$, so $r \in [s;t]$.
Recall: $\forall x \in U$, f has UBD near x.
Then, by Theorem 28, f has N-BD on $(s;t)$.
So, since $N \leqslant K$, we get: f has K-BD on $(s;t)$.
So, since $p \in U = (s;t)$, we get: f has K-BD at p.
By choice of p, f does not have K-BD at p. Contradiction. $\square$

THEOREM 31. *Let* $g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $M \geqslant 0$.
Assume: $a < b$. *Let* $I := (a; b)$. *Assume:* $I \subseteq \mathbb{D}_g^{(2)}$.
Assume: $|g| \leqslant M$ *on* I *and* $|g''| \leqslant M$ *on* I.
Let $N := M \cdot \left(\dfrac{6}{b-a} + \dfrac{b-a}{6} \right)$. *Then:* $|g'| \leqslant N$ *on* I.

Proof. Given $x \in I$, **want:** $|g'(x)| \leqslant N$.
Let $\delta := \dfrac{b-a}{3}$. Then $\delta > 0$ and $\dfrac{2M}{\delta} + \dfrac{M\delta}{2} = N$.
Choose $h \in \{\delta, -\delta\}$ s.t. $x + h \in I$. Then $|h| = \delta$.
By Taylor's Theorem, choose ξ strictly between x and $x + h$ s.t.

$$g(x+h) \ = \ (g(x)) \ + \ (g'(x)) \cdot h \ + (g''(\xi)) \cdot \frac{h^2}{2}.$$

Then: $g'(x) = \dfrac{(g(x+h)) - (g(x))}{h} - \dfrac{(g''(\xi)) \cdot h}{2}$.

Then: $|g'(x)| \leqslant \dfrac{|g(x+h)| + |g(x)|}{|h|} + \dfrac{|g''(\xi)| \cdot |h|}{2}$.

Since $|g|, |g''| \leqslant M$ on I and since $x, \xi, x+h \in I$, we get:
$|g(x)| \leqslant M$ and $|g''(\xi)| \leqslant M$ and $|g(x+h)| \leqslant M$.
Recall: $|h| = \delta$. Then: $|g'(x)| \leqslant \dfrac{2M}{\delta} + \dfrac{M\delta}{2} = N$. □

THEOREM 32. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$.
Assume: I *is a non*$\varnothing$ *bounded open interval.*
Assume: f *has UBED on* I. *Then:* f *has UBD on* I.

Proof. **Want:** $\exists N \geqslant 0$ s.t. f has N-BD on I.
Since f has UBED on I, choose $M \geqslant 0$ s.t. f has M-BED on I.
Since I is a non$\varnothing$ bounded open interval,
choose $a, b \in \mathbb{R}$ s.t. $a < b$ and s.t. $I = (a; b)$.
Let $N := M \cdot \left(\dfrac{6}{b-a} + \dfrac{b-a}{6} \right)$. Then $M \leqslant N$. Then $N \geqslant 0$.
Want: f has N-BD on I. Given $x \in I$, **want:** f has N-BD at x.
Given $j \in \mathbb{N}_0$, **want:** $|f^{(j)}(x)| \leqslant N$.

Case 1: j is even.
Proof in Case 1:
Since j is even, by choice of M, we have: $|f^{(j)}| \leqslant M$ on I.
So, since $x \in I$, we get: $|f^{(j)}(x)| \leqslant M$. Then $|f^{(j)}(x)| \leqslant M \leqslant N$.

End of proof in Case 1.

Case 2: j is odd.
Proof in Case 2:
Since $j-1$ and $j+1$ are even, by the choice of M, we have:
$|f^{(j-1)}| \leqslant M$ on I and $|f^{(j+1)}| \leqslant M$ on I.
By hypothesis, f has UBED on I, so: $I \subseteq \mathbb{D}_f^{(\infty)}$.
Let $g := f^{(j-1)}$. Then $I \subseteq \mathbb{D}_f^{(\infty)} = \mathbb{D}_g^{(\infty)} \subseteq \mathbb{D}_g^{(2)}$, so $I \subseteq \mathbb{D}_g^{(2)}$.
Also, $g' = f^{(j)}$ and $g'' = f^{(j+1)}$.
Then: $|g| \leqslant M$ on I and $|g''| \leqslant M$ on I.
Then, by Theorem 31, we get: $|g'| \leqslant N$ on I.
So, since $x \in I$, we get: $|g'(x)| \leqslant N$. Then $|f^{(j)}(x)| = |g'(x)| \leqslant N$.
End of proof in Case 2. □

THEOREM 33. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $c, d \in \mathbb{R}$.
Assume $c < d$. *Let* $J := (c; d)$. *Assume* f *has PBED on* J.
Then $\exists$*non*$\varnothing$ *open subintervals* $U_1, U_2, U_3, \ldots$ *of* J
s.t. $\forall i \in \mathbb{N}$, f *has UBD on* U_i *and*
s.t. $U_1 \bigcup U_2 \bigcup U_3 \bigcup \cdots$ *is dense in* J.

Proof. Since J is second-countable,
choose a countable base $\mathcal{W}$ for J s.t., $\forall W \in \mathcal{W}$, $W \neq \varnothing$.
Since $\mathcal{W}$ is countable, **it suffices to prove:**
$\forall W \in \mathcal{W}$, $\exists$non$\varnothing$ open subinterval U of J
s.t. $U \subseteq W$ and s.t. f has UBD on U.
Given $W \in \mathcal{W}$, **want:** $\exists$non$\varnothing$ open subinterval U of J
s.t. $U \subseteq W$ and s.t. f has UBD on U.
Since $W \in \mathcal{W}$, we get: $W \neq \varnothing$ and $W \subseteq J$.
Since $W \in \mathcal{W}$, we get: W is open in J.
So, since J is open in $\mathbb{R}$, we get: W is open in $\mathbb{R}$.
Then: W is locally compact and Hausdorff.
For all $m \in \mathbb{N}$, let $C_m := \{x \in W \mid f \text{ has } m\text{-BED at } x\}$.
Since f has PBED on J and since $W \subseteq J$, we get: f has PBED on W.
Then $W \subseteq \mathbb{D}_f^{(\infty)}$. So, by continuity, $\forall m \in \mathbb{N}$, C_m is closed in W.
Since f has PBED on W, we get: $C_1 \bigcup C_2 \bigcup C_3 \bigcup \cdots = W$.
So, since W is non$\varnothing$ and locally compact and Hausdorff,
by the Baire Category Theorem,
choose $M \in \mathbb{N}$ s.t. C_M has non$\varnothing$ interior in W.
Then, since W is open in $\mathbb{R}$, we get: C_M has non$\varnothing$ interior in $\mathbb{R}$.

So choose $s, t \in \mathbb{R}$ s.t. $s < t$ and s.t. $(s; t) \subseteq C_M$.
Let $U := (s; t)$. Then: U is a non$\varnothing$ open interval and $U \subseteq C_M$.
Since $U \subseteq C_M \subseteq W \subseteq J$ and since U is a non$\varnothing$ open interval,
we get: U is a non$\varnothing$ open subinterval of J.
As $U \subseteq C_M \subseteq W$, **it remains only to show:** f has UBD on U.
Since $U \subseteq C_M$, by definition of C_M, we get: f has M-BED on U.
Then f has UBED on U. Then, by Theorem 32, f has UBD on U. □

DEFINITION 34. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$.
Then $\boxed{\mathrm{IBD}_f} := (\mathrm{BD}_f)^\circ$ *denotes the interior in* $\mathbb{R}$ *of* BD_f.

THEOREM 35. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $c, d \in \mathbb{R}$.
Assume $c < d$. *Let* $J := (c; d)$.
Assume f *has PBED on* J. *Then* $\mathrm{IBD}_f \bigcap J$ *is dense in* J.

Proof. By Theorem 33,
choose non$\varnothing$ open subintervals $U_1, U_2, U_3, \ldots$ of J
s.t. $\forall i \in \mathbb{N}$, f has UBD on U_i and
s.t. $U_1 \bigcup U_2 \bigcup U_3 \bigcup \cdots$ is dense in J.
Then: $\forall i \in \mathbb{N}$, since f has UBD on U_i,
it follows that f has BD on U_i, so $U_i \subseteq \mathrm{BD}_f$.
Let $U := U_1 \bigcup U_2 \bigcup U_3 \bigcup \cdots$. Then $U \subseteq \mathrm{BD}_f$, so $U^\circ \subseteq (BD_f)^\circ$.
Since $\forall i \in \mathbb{N}$, $U_i \subseteq J$, we get: $U \subseteq J$.
Since $\forall i \in \mathbb{N}$, U_i is open in J, we get: U is open in J.
So, since J is open in $\mathbb{R}$, we get: U is open in $\mathbb{R}$. Then $U^\circ = U$.
Since $U_1 \bigcup U_2 \bigcup U_3 \bigcup \cdots$ is dense in J, we get: U is dense in J.
Since $U = U^\circ \subseteq (\mathrm{BD}_f)^\circ = \mathrm{IBD}_f$ and since $U \subseteq J$,
we get: $U \subseteq \mathrm{IBD}_f \bigcap J$.
So, since U is dense in J, we get: $\mathrm{IBD}_f \bigcap J$ is dense in J. □

THEOREM 36. *Let* $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$, $s, t \in \mathbb{R}$, $L \geqslant 0$. *Assume:* $s < t$.
Assume: $(s; t) \subseteq \mathbb{D}_\phi^{(2)}$ *and* ϕ *is continuous both at* s *and at* t.
Assume: $\phi'' > 0$ *on* $(s; t)$. *Assume:* $\phi \leqslant L$ *on* $\{s, t\}$.
Then: $\phi < L$ *on* $(s; t)$.

Theorem 36 is a special case of the Maximum Principle.
This particular special case follows from the Mean Value Theorem.
We omit the proof.

THEOREM 37. *Let* $g : \mathbb{R} \dashrightarrow \mathbb{R}$, $s, t \in \mathbb{R}$, $L \geqslant 0$.
Assume: $s < t$ *and* $t - s \leqslant 1$.

Assume: $(s;t) \subseteq \mathbb{D}_g^{(2)}$ *and* g *is continuous both at* s *and at* t.
Assume: $|g| \leqslant L$ *on* $\{s,t\}$. *Let* $w \in (s;t)$. *Assume* $|g(w)| \geqslant 2L$.
Then: $\exists x \in (s;t)$ *s.t.* $|g''(x)| \geqslant 8L$.

Proof. Choose $h \in \{g, -g\}$ s.t. $|g(w)| = h(w)$. Then $h(w) \geqslant 2L$.
Also, $|h| = |g|$ and $|h'| = |g'|$ and $|h''| = |g''|$.
Also, $(s;t) \subseteq \mathbb{D}_h^{(2)}$ and h is continuous both at s and at t.
Want: $\exists x \in (s;t)$ s.t. $|h''(x)| \geqslant 8L$.
Assume: $|h''| < 8L$ on $(s;t)$. **Want:** Contradiction.
We have: $-8L < h'' < 8L$ on $(s;t)$.
Since $h'' > -8L$ on $(s;t)$, we get: $8L + h'' > 0$ on $(s;t)$.
Define $Q : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}$, $Q(x) = 4L \cdot (x - s) \cdot (x - t)$.
Then: $Q'' = 8L$ on $\mathbb{R}$. Then: $(Q + h)'' > 0$ on (s,t).
Let $\phi := Q + h$. Then $\phi'' > 0$ on $(s;t)$.
Since $Q = 0$ on $\{s,t\}$ and since $h \leqslant |h| = |g| \leqslant L$ on $\{s,t\}$,
we get: $Q + h \leqslant L$ on $\{s,t\}$. Then: $\phi \leqslant L$ on $\{s,t\}$.
Also, $(s;t) \subseteq \mathbb{D}_\phi^{(2)}$ and ϕ is continuous both at s and at t.
Then, by Theorem 36 (Maximum Principle), we get: $\phi < L$ on $(s;t)$.
By hypothesis, we have: $w \in (s;t)$. Then $\phi(w) < L$.
Since $(Q(w)) + (h(w)) = \phi(w) < L$, we get: $h(w) < L - (Q(w))$.
Let $c := (s + t)/2$. The minimum value of Q is $Q(c)$.
Then $Q(w) \geqslant Q(c)$. We calculate: $Q(c) = -L \cdot (t - s)^2$.
Since $0 < t - s \leqslant 1$, we get: $(t - s)^2 \leqslant 1$.
So, since $L \geqslant 0$, we get: $-L \cdot (t - s)^2 \geqslant -L$.
Then $Q(w) \geqslant Q(c) = -L \cdot (t - s)^2 \geqslant -L$, so $-(Q(w)) \leqslant L$.
Then $h(w) < L - (Q(w)) \leqslant L + L = 2L$, so $h(w) < 2L$.
Recall, from the start of the proof: $h(w) \geqslant 2L$. Contradiction. ☐

THEOREM 38. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $s, t \in \mathbb{R}$, $M > 0$.
Assume $s < t$. *Assume* $t - s \leqslant 1$.
Assume f *has* M*-BED on* $\{s,t\}$. *Assume* f *has UBED on* $(s;t)$.
Then f *has* $2M$*-BED on* $(s;t)$.

Proof. Given $p \in (s;t)$, **want:** f has $2M$-BED at p.
Given $j \in \mathbb{N}_0$, **want:** $|f^{(2j)}(p)| \leqslant 2M$.
Assume: $|f^{(2j)}(p)| > 2M$. **Want:** Contradiction.
Since $|f^{(2j)}(p)| > 2M$, we get: $|f^{(2j)}(p)| \geqslant 2M$.
For all $i \in \mathbb{N}_0$, let $L_i := 4^i \cdot M$. Then: $\forall i \in \mathbb{N}_0$, $L_i \geqslant 0$.
Also, $L_0 = M$ and $\forall i \in \mathbb{N}_0$, $L_{i+1} = 4L_i$.
For all $i \in \mathbb{N}_0$, let $B_i := \{q \in (s;t)$ s.t. $|f^{(2j+2i)}(q)| \geqslant 2L_i\}$.

Claim: $\forall i \in \mathbb{N}_0, \quad B_i \neq \emptyset$.
Proof of Claim: We have $|f^{(2j+2\cdot 0)}(p)| = |f^{(2j)}(p)| \geqslant 2M = 2L_0$.
Also, $p \in (s;t)$. Then $p \in B_0$. Then $B_0 \neq \emptyset$.
We proceed by mathematical induction:
Given $i \in \mathbb{N}_0$, assume $B_i \neq \emptyset$, **want:** $B_{i+1} \neq \emptyset$.
Choose $w \in B_i$. Then $w \in (s;t)$ and $|f^{(2j+2i)}(w)| \geqslant 2L_i$.
By hypothesis, f has M-BED on $\{s,t\}$, so $s,t \in \mathbb{D}_f^{(\infty)}$.
By hypothesis, f has M-BED on $\{s,t\}$, so $|f^{(2j+2i)}| \leqslant M$ on $\{s,t\}$.
By hypothesis, f has UBED on $(s;t)$, so $(s;t) \subseteq \mathbb{D}_f^{(\infty)}$.
Let $g := f^{(2j+2i)}$. Then $(s;t) \subseteq \mathbb{D}_f^{(\infty)} = \mathbb{D}_g^{(\infty)} \subseteq \mathbb{D}_g^{(2)}$, so $(s;t) \subseteq \mathbb{D}_g^{(2)}$.
Since $s,t \in \mathbb{D}_f^{(\infty)} = D_g^{(\infty)} \subseteq \mathbb{D}_g^{(2)} \subseteq \mathbb{D}_g^{(1)}$,
we get: g is differentiable both at s and at t.
Then g is continuous both at s and at t.
Also, $|g(w)| = |f^{(2j+2i)}(w)| \geqslant 2L_i$, so $|g(w)| \geqslant 2L_i$.
Also, $|g| = |f^{(2j+2i)}| \leqslant M$ on $\{s,t\}$, so $|g| \leqslant M$ on $\{s,t\}$.
We have: $M \leqslant 4^i \cdot M = L_i$. Then $|g| \leqslant L_i$ on $\{s,t\}$.
By Theorem 37, choose $x \in (s;t)$ s.t. $|g''(x)| \geqslant 8L_i$.
Since $g'' = (f^{(2j+2i)})'' = f^{(2j+2i+2)} = f^{(2j+2\cdot(i+1))}$,
we get: $|f^{(2j+2\cdot(i+1))}(x)| = |g''(x)|$.
Then $|f^{(2j+2\cdot(i+1))}(x)| = |g''(x)| \geqslant 8L_i = 2\cdot 4L_i = 2L_{i+1}$,
so $|f^{(2j+2\cdot(i+1))}(x)| \geqslant 2L_{i+1}$.
Also, $x \in (s;t)$. Then $x \in B_{i+1}$. Then $B_{i+1} \neq \emptyset$.
End of proof of Claim.

By hypothesis, f has UBED on $(s;t)$, so
choose $K \geqslant 0$ s.t. f has K-BED on $(s;t)$.
By hypothesis, $M > 0$, so choose $n \in \mathbb{N}_0$ s.t. $2\cdot 4^n \cdot M > K$.
By the Claim, $B_n \neq \emptyset$, so choose $z \in B_n$.
Then, by definition of B_n, we get: $z \in (s;t)$ and $|f^{(2j+2n)}(z)| \geqslant 2L_n$.
Then $|f^{(2j+2n)}(z)| \geqslant 2L_n = 2\cdot 4^n \cdot M > K$, so $|f^{(2j+2n)}(z)| > K$.
On the other hand, since f has K-BED on $(s;t)$ and since $z \in (s;t)$,
we get: $|f^{(2j+2n)}(z)| \leqslant K$. Contradiction. $\square$

THEOREM 39. *Let* $c,d \in \mathbb{R}$. *Assume:* $c < d$. *Let* $J := (c;d)$.
Let $T \subseteq J$. *Assume:* T *is finite.* *Let* $q \in T$.
Then: $\exists \delta > 0$ *s.t.* $(q-\delta;q) \subseteq J \backslash T$.

The preceding result is basic. Its proof is left as an exercise.

THEOREM 40. *Let* $f:\mathbb{R}\dashrightarrow\mathbb{R}$, $c,d\in\mathbb{R}$.
Assume: $c<d$. *Let* $J:=(c;d)$. *Assume:* $J\subseteq\mathbb{D}_f^{(\infty)}$.
Let $T:=J\backslash\mathrm{BD}_f$. *Assume:* $T\neq\varnothing$. *Then:* T *is infinite.*

Proof. Assume: T is finite. **Want:** Contradiction.
Since $T\neq\varnothing$, choose $q\in T$. Then $q\in J$ and $q\notin\mathrm{BD}_f$.
By Theorem 39, choose $\delta>0$ s.t. $(q-\delta;q)\subseteq J\backslash T$.
Since $(q-\delta;q)\subseteq J\backslash T\subseteq J$ and since $q\in J$, we get: $(q-\delta;q]\subseteq J$.
We have: $(q-\delta;q)\subseteq J\backslash T=J\backslash(J\backslash\mathrm{BD}_f)=J\bigcap\mathrm{BD}_f\subseteq\mathrm{BD}_f$,
so $(q-\delta;q)\subseteq\mathrm{BD}_f$, so f has PBD on $(q-\delta;q)$.
So, by Tao's Theorem (Theorem 30), we get: f has UBD on $(q-\delta;q)$.
Choose $M\geqslant 0$ s.t. f has M-BD on $(q-\delta;q)$.
So, since $(q-\delta;q]\subseteq J\subseteq\mathbb{D}_f^{(\infty)}$, by continuity, f has M-BD at q.
Then f has BD at q, so $q\in\mathrm{BD}_f$. Recall: $q\notin\mathrm{BD}_f$. Contradiction. ☐

THEOREM 41. *Let* $T\subseteq\mathbb{R}$, $\varepsilon>0$.
Assume: T *is bounded and infinite.*
Then: $\exists p,q,r\in T$ *s.t.* $p<q<r$ *and s.t.* $r-p\leqslant\varepsilon$.

Proof. Since T is bounded and infinite, choose a limit point x of T.
Let $C:=[x-(\varepsilon/2);x+(\varepsilon/2)]$. Then $C\bigcap T$ is infinite.
Choose $p,q,r\in C\bigcap T$ s.t. $p<q<r$. **Want:** $r-p\leqslant\varepsilon$.
Since $p,r\in C\bigcap T\subseteq C=[x-(\varepsilon/2);x+(\varepsilon/2)]$, we get: $r-p\leqslant\varepsilon$. ☐

THEOREM 42. *Let* $f:\mathbb{R}\dashrightarrow\mathbb{R}$, $a,b\in\mathbb{R}$.
Assume: $a<b$. *Let* $I:=(a;b)$.
Assume: f *has PBED on* I. *Then:* f *has PBD on* I.

Proof. **Want:** $I\subseteq\mathrm{BD}_f$. Let $V:=\mathrm{IBD}_f\bigcap I$.
Since IBD_f is open in $\mathbb{R}$, we get: V is open in I.
Since $V\subseteq\mathrm{IBD}_f\subseteq\mathrm{BD}_f$, **it suffices to show:** $I\subseteq V$.
Let $X:=I\backslash V$. **Want:** $X=\varnothing$.
Assume $X\neq\varnothing$. **Want:** Contradiction.
Since V is open in I and since $X=I\backslash V$, we get: X is closed in I.
Since $I=(a;b)$, we get: I is open in $\mathbb{R}$.
Since X is closed in I and since I is open in $\mathbb{R}$,
we get: X is locally compact and Hausdorff.
By hypothesis, f has PBED on I, so, since $X=I\backslash V\subseteq I$,
it follows that: f has PBED on X. Then: $X\subseteq\mathbb{D}_f^{(\infty)}$.
For all $m\in\mathbb{N}$, let $X_m:=\{x\in X\,|\,f$ has m-BED at $x\}$.
Then, by continuity, we get: $\forall m\in\mathbb{N}$, X_m is closed in X.

Since f has PBED on X, we get: $X_1 \bigcup X_2 \bigcup X_3 \bigcup \cdots = X$.
So, since X is non$\varnothing$ and locally compact and Hausdorff,
by the Baire Category Theorem,
choose $M \in \mathbb{N}$ s.t. X_M has non$\varnothing$ interior in X.
So, since $X = I \backslash V \subseteq I = (a;b)$, by Theorem 7, choose $c, d \in [a;b]$
s.t. $c < d$ and s.t. $\varnothing \neq (c;d) \bigcap X \subseteq X_M$.
Then: $a \leqslant c < d \leqslant b$. Then: $(c;d) \subseteq (a;b)$.
Let $J := (c;d)$. Then: J is open in $\mathbb{R}$, so $J^\circ = J$.
Also, $J = (c;d) \subseteq (a;b) = I$, so: $J \subseteq I$. Then $J \backslash V = J \bigcap (I \backslash V)$.
Since $J \backslash V = J \bigcap (I \backslash V) = J \bigcap X = (c;d) \bigcap X$,
we get: $J \backslash V = (c;d) \bigcap X$.
So, since $\varnothing \neq (c;d) \bigcap X \subseteq X_M$, we get: $\varnothing \neq J \backslash V \subseteq X_M$.
Since $J \backslash V \neq \varnothing$, we get: $J \not\subseteq V$.
Since $J \not\subseteq V = \mathrm{IBD}_f \bigcap I$ and since $J \subseteq I$, we get: $J \not\subseteq \mathrm{IBD}_f$.
Since $J^\circ = J \not\subseteq \mathrm{IBD}_f = (\mathrm{BD}_f)^\circ$, we get $J^\circ \not\subseteq (\mathrm{BD}_f)^\circ$, and so $J \not\subseteq \mathrm{BD}_f$.
Then: $J \backslash \mathrm{BD}_f \neq \varnothing$. Let $T := J \backslash \mathrm{BD}_f$. Then $T \neq \varnothing$.
By hypothesis, f has PBED on I, so, since $J \subseteq I$,
it follows that: f has PBED on J. Then $J \subseteq \mathbb{D}_f^{(\infty)}$.
Then, by Theorem 40, we get: T is infinite.
Also, $T = J \backslash \mathrm{BD}_f \subseteq J = (c;d)$, so $T \subseteq (c;d)$. Then T is bounded.
By Theorem 41, choose $p, q, r \in T$ s.t. $p < q < r$ and s.t. $r - p \leqslant 1$.
Then: $p, q, r \in T \subseteq (c;d)$. Then: $a \leqslant c < p < q < r < d \leqslant b$.
Then: $[p;r] \subseteq (c;d)$. By Theorem 35, $\mathrm{IBD}_f \bigcap J$ is dense in J.
Let $W := \mathrm{IBD}_f \bigcap J$. Then: W is dense in J.
Since $J \subseteq I$, we get: $J = I \bigcap J$. Then $W = J \bigcap \mathrm{IBD}_f \bigcap I$.
By definition of V, we have: $V = \mathrm{IBD}_f \bigcap I$. Then: $W = J \bigcap V$.
So, since $J \backslash V = J \backslash (J \bigcap V)$, we get: $J \backslash V = J \backslash W$.
Recall: $\varnothing \neq J \backslash V \subseteq X_M$.
Since $J \backslash W = J \backslash V \subseteq X_M$, we get: $J \backslash W \subseteq X_M$.
We have $(p;r) \subseteq [p;r] \subseteq (c;d) = J$, so $(p;r) \subseteq J$.
Then: $(p;r)$ is an open subset of J.
So, since W is dense in J, we get: $W \bigcap (p;r)$ is dense in $(p;r)$.
We have $p, q, r \in T = J \backslash \mathrm{BD}_f$. Then $p, q, r \notin \mathrm{BD}_f$.
Since $p < q < r$, we get: $q \in (p;r)$.
Since $q \notin \mathrm{BD}_f$, we get: f does not have BD at q.
So, since $q \in (p;r)$, we get: f does not have PBD on $(p;r)$.
Then f does not have UBD on $(p;r)$.
Then, by Theorem 32, f does not have UBED on $(p;r)$.

Then: f does not have $2M$-BED on $(p;r)$.
So, since $(p;r) \subseteq J \subseteq \mathbb{D}_f^{(\infty)}$ and
since $W \bigcap (p;r)$ is dense in $(p;r)$, by continuity,
we get: f does not have $2M$-BED on $W \bigcap (p;r)$.
Choose $w \in W \bigcap (p;r)$ s.t. f does not have $2M$-BED at w.
Then: $a \leqslant c < p < w < r < d \leqslant b$. Also, $w \in W$.
By definition of W, we have: $W = \text{IBD}_f \bigcap J$.
So, since IBD_f is open in $\mathbb{R}$, we get: W is an open subset of J.
So, since $J = (c;d)$, we get: W is an open subset of $(c;d)$.
Since $p, r \notin \text{BD}_f \supseteq \text{IBD}_f \supseteq \text{IBD}_f \bigcap J = W$, we get: $p, r \notin W$.
Let U be the connected component of W s.t. $w \in U$. Then: $w \in U \subseteq W$.
By Theorem 6, choose $s, t \in [p;r] \backslash W$ s.t. $s < t$ and s.t. $U = (s;t)$.
Then $p \leqslant s < t \leqslant r$. Since $w \in U = (s;t)$, we get: $s < w < t$.
Then: $a \leqslant c < p \leqslant s < w < t \leqslant r < d \leqslant b$.
Since $p \leqslant s < t \leqslant r$, we get: $t - s \leqslant r - p$.
So, since $r - p \leqslant 1$, we get: $t - s \leqslant 1$.
Since $(s;t) = U \subseteq W = \text{IBD}_f \bigcap J \subseteq \text{IBD}_f \subseteq \text{BD}_f$,
we get: f has PBD on $(s;t)$.
Then, by Tao's Theorem (Theorem 30), we get: f has UBD on $(s;t)$.
Then: f has UBED on $(s;t)$. Since $M \in \mathbb{N}$, we get: $M > 0$.
Recall: $J \backslash W \subseteq X_M$ and $J = (c;d)$ and $[p;r] \subseteq (c;d)$.
Since $s, t \in [p;r] \backslash W \subseteq (c;d) \backslash W = J \backslash W \subseteq X_M$,
by definition of X_M, we get: f has M-BED on $\{s, t\}$.
Then, by Theorem 38, we get: f has $2M$-BED on $(s;t)$.
So, since $w \in U = (s;t)$, we get: f has $2M$-BED at w.
By choice of w, f does not have $2M$-BED at w. Contradiction. $\square$

DEFINITION 43. *Let* $\mu : \mathbb{R} \dashrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$.
By μ *is* **affine** *on* I, *we mean:* $I \subseteq \mathbb{D}_\mu$ *and*
$\exists m, c \in \mathbb{R}$ *s.t.,* $\forall x \in I$, $\mu(x) = mx + c$.

THEOREM 44. *Let* $\mu : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume $a < b$. *Let* $I := (a;b)$. *Assume:* $I \subseteq \mathbb{D}_\mu$.
Then: $\big(\mu \text{ is affine on } I \big)$
$\Leftrightarrow \big(\mu'' = 0 \text{ on } I \big)$
$\Leftrightarrow \big(\forall p, q \in I,\ \forall t \in [0;1],$
$\mu(\,(1-t) \cdot p + t \cdot q\,) = (1-t) \cdot (\mu(p)) + t \cdot (\mu(q))\,\big)$.

The preceding result is basic. Its proof is left as an exercise.

THEOREM 45. *Let $a, b \in \mathbb{R}$. Assume $a < b$. Let $I := (a; b)$. Let $\lambda_0, \lambda_1, \lambda_2 \ldots : I \to \mathbb{R}$. Assume: $\forall j \in \mathbb{N}$, λ_j is affine on I. Let $\mu : I \to \mathbb{R}$. Assume: $\lambda_j \to \mu$ pointwise, as $j \to \infty$. Then: μ is affine on I.*

Proof. Given $p, q \in I$, $t \in [0; 1]$, **want:**

$$\mu(\,(1-t)p + tq\,) \;=\; (1-t)\cdot(\mu(p)) \;+\; t\cdot(\mu(q)).$$

Since, $\forall j \in \mathbb{N}_0$, λ_j is affine on I, we get:

$$\forall j \in \mathbb{N}_0, \quad \lambda_j(\,(1-t)p + tq\,) \;=\; (1-t)\cdot(\lambda_j(p)) \;+\; t\cdot(\lambda_j(q)).$$

So, letting $j \to \infty$, by pointwise convergence, we get:

$$\mu(\,(1-t)p + tq\,) \;=\; (1-t)\cdot(\mu(p)) \;+\; t\cdot(\mu(q)).$$ □

THEOREM 46. *Let $\mu : \mathbb{R} \dashrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$. Assume: μ is affine on I. Then: μ is Lipschitz on I.*

Proof. Choose $m, c \in \mathbb{R}$ s.t., $\forall x \in I$, $\mu(x) = mx + c$.
Want: μ is $|m|$-Lipschitz on I.
Given $p, q \in I$, **want:** $|(\mu(q)) - (\mu(p))| \leqslant |m| \cdot |q - p|$.
We have: $(\mu(q)) - (\mu(p)) = (mq + c) - (mp + c) = m \cdot (q - p)$.
Then: $|(\mu(q)) - (\mu(p))| = |m \cdot (q - p)| = |m| \cdot |q - p|$.
Then: $|(\mu(q)) - (\mu(p))| \leqslant |m| \cdot |q - p|$. □

THEOREM 47. *Let $\phi : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $M \geqslant 0$. Assume: $a < b$. Let $I := (a; b)$. Assume: ϕ is M-Lipschitz on I. Let $c \in I$. Let $M' := |\phi(c)| + M \cdot (b - a)$. Then: $|\phi| \leqslant M'$ on I.*

Proof. Given $x \in I$, **want:** $|\phi(x)| \leqslant M'$.
Since $c, x \in I = (a; b)$, we get: $|x - c| < b - a$.
So, since $M \geqslant 0$, we get: $M \cdot |x - c| \leqslant M \cdot (b - a)$.
Since ϕ is M-Lipschitz on I, we get: $|(\phi(x)) - (\phi(c))| \leqslant M \cdot |x - c|$.
Then: $|\phi(x)| = |[\phi(c)] + [(\phi(x)) - (\phi(c))]| \leqslant |\phi(c)| + |(\phi(x)) - (\phi(c))|$
$\leqslant |\phi(c)| + M \cdot |x - c| \leqslant |\phi(c)| + M \cdot (b - a) = M'$. □

THEOREM 48. *Let $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $M \geqslant 0$. Assume: $a < b$. Let $I := (a; b)$. Assume: ϕ is Lipschitz on I. Then: ϕ is bounded and continuous on I.*

Proof. Since ϕ is Lipschitz on I, we get: ϕ is continuous on I.
It remains to show: ϕ is bounded on I.
Since ϕ is Lipschitz on I, choose $M \geqslant 0$ s.t. ϕ is M-Lipschitz on I.
Let $c := (a + b)/2$. Then $c \in I$. Let $M' := |\phi(c)| + M \cdot (b - a)$.
By Theorem 47, we get: $|\phi| \leqslant M'$ on I. Then ϕ is bounded on I. □

DEFINITION 49. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume: $a < b$. *Let* $I := (a; b)$. *Let* $c := (a + b)/2$.
Assume: f *is bounded and measurable on* I.
Then $\boxed{f_I^\#} : I \to \mathbb{R}$ *is defined by:* $\forall x \in I$, $f_I^\#(x) = \int_c^x f$.

THEOREM 50. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume: $a < b$. *Let* $I := (a; b)$.
Assume: f *is bounded and continuous on* I.
Then: $(f^\#)' = f$ *on* I.

Theorem 50 is a case of the Fundamental Theorem of Calculus.

THEOREM 51. *Let* $a, b \in \mathbb{R}$. *Assume:* $a < b$. *Let* $I := (a; b)$.
Let $f_0, f_1, f_2, \ldots : I \to \mathbb{R}$ *be measurable.* *Let* $g : I \to \mathbb{R}$.
Let $M \geqslant 0$. *Assume:* $\forall j \in \mathbb{N}_0$, $|f_j| \leqslant M$ *on* I.
Assume: $f_j \to g$ *pointwise on* I, *as* $j \to \infty$.
Then: g *is bounded and measurable on* I *and*
$(f_j)_I^\# \to g_I^\#$ *pointwise on* I, *as* $j \to \infty$.

Proof. Since $\forall j \in \mathbb{N}_0$, $|f_j| \leqslant M$ on I
and since $f_j \to g$ pointwise on I, as $j \to \infty$,
we get $|g| \leqslant M$ on I, so g is bounded on I.
Since a pointwise limit of measurable functions is measurable,
we get: g is measurable on I.
It remains to show: $(f_j)_I^\# \to g_I^\#$ pointwise on I, as $j \to \infty$.
Given $x \in I$, **want:** $(f_j)_I^\#(x) \to g_I^\#(x)$, as $j \to \infty$.

Let $c := (a + b)/2$. Then: $g_I^\#(x) = \int_c^x g$.

Also, we have: $\forall j \in \mathbb{N}_0$, $(f_j)_I^\#(x) = \int_c^x f_j$

Since $\forall j \in \mathbb{N}_0$, $|f_j| \leqslant M$ on I and
since $f_j \to g$ pointwise on I, as $j \to \infty$,
by the Dominated Convergence Theorem, we get:

$$\int_c^x f_j \to \int_c^x g, \quad \text{as } j \to \infty.$$

Then: $(f_j)_I^\#(x) \to g_I^\#(x)$, as $j \to \infty$. □

THEOREM 52. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $M \geqslant 0$.
Assume: $a < b$. *Let* $I := (a; b)$.

Assume: f *is measurable on* I. *Assume:* $|f| \leqslant M$ *on* I.
Then: $f_I^\#$ *is* M*-Lipschitz on* I.

Proof. Given $s, t \in I$, assume $s < t$,

want: $|(f_I^\#(t)) - (f_I^\#(s))| \leqslant M \cdot (t - s)$.

Since $s, t \in I$ and since I is an interval, we get: $[s; t] \subseteq I$.
Then: $|f| \leqslant M$ on $[s; t]$. Let $c := (a + b)/2$.

Then: $$(f_I^\#(t)) - (f_I^\#(s)) = \left(\int_c^t f \right) - \left(\int_c^s f \right) = \int_s^t f.$$

Then: $$|(f_I^\#(t)) - (f_I^\#(s))| \leqslant \int_s^t |f|.$$

So, since $|f| \leqslant M$ on $[s; t]$, we get: $$|(f_I^\#(t)) - (f_I^\#(s))| \leqslant \int_s^t M.$$

Then: $|(f_I^\#(t)) - (f_I^\#(s))| \leqslant M \cdot (t - s)$. □

THEOREM 53. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume $a < b$. *Let* $I := (a; b)$.
Assume: f *is bounded and measurable on* I.
Then: $f_I^\#$ *is bounded and continuous on* I.

Proof. Since f is bounded on I, choose $M \geqslant 0$ s.t. $|f| \leqslant M$ on I.
By Theorem 52, $f_I^\#$ is M-Lipschitz on I, so $f_I^\#$ is Lipschitz on I.
Then, by Theorem 48, $f_I^\#$ is bounded and continuous on I. □

DEFINITION 54. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume $a < b$. *Let* $I := (a; b)$.
Assume: f *is bounded and measurable on* I.
Then: $\boxed{f_I^{\#\#}} := (f_I^\#)_I^\#$.

Implicit in Definition 54 is that, by Theorem 53,
$f_I^\#$ is bounded and continuous on I,
and so $f_I^\#$ is bounded and measurable on I.

THEOREM 55. *Let* $g : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume: $a < b$. *Let* $I := (a; b)$.
Assume: g *is bounded and continuous on* I.
Then: $(g_I^{\#\#})'' = g$ *on* I.

Proof. By Theorem 50, we get: $(g_I^\#)' = g$ on I.
Let $h := g_I^\#$. Then $h' = g$.
Since g is continuous on I, we get: g is measurable on I.

Then, by Theorem 53, we get: $g_I^{\#}$ is bounded and continuous on I.
So, since $h = g_I^{\#}$, we get: h is bounded and continuous on I.
So, by Theorem 50, we get: $(h_I^{\#})' = h$ on I.
So, since $h' = g$ on I, we get: $(h_I^{\#})'' = g$ on I.
Then: $(g_I^{\#\#})'' = ((g_I^{\#})_I^{\#})'' = (h_I^{\#})'' = g$ on I. ☐

THEOREM 56. *Let* $f : \mathbb{R} \dashrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume: $a < b$. *Let* $I := (a; b)$. *Assume:* $I \subseteq \mathbb{D}_f^{(2)}$.
Assume: f'' *is bounded and continuous on* I.
Then: $(f'')_I^{\#\#} - f$ *is affine on* I.

Proof. Let $\phi := (f'')_I^{\#\#}$. **Want:** $\phi - f$ is affine on I.
Want: $(\phi - f)'' = 0$ on I. **Want:** $\phi'' = f''$ on I.
Let $g := f''$. By hypothesis, g is bounded and continuous on I.
Then, by Theorem 55, we get: $(g_I^{\#\#})'' = g$ on I.
Then: $\phi'' = ((f'')_I^{\#\#})'' = (g_I^{\#\#})'' = g = f''$ on I. ☐

THEOREM 57. *Let* $a, b \in \mathbb{R}$. *Assume* $a < b$. *Let* $I := (a; b)$.
Let $S := C^\infty(I, \mathbb{R})$. *Define* $L : S \to S$ *by:* $\forall h \in S$, $Lh = h''$.
Let $f \in S$. *Let* $g : I \to \mathbb{R}$. *Assume* $f, Lf, L^2 f, \ldots \to g$ *pointwise on* I.
Then: $g \in S$ *and* $Lg = g$.

Proof. **It suffices to show:** $g'' = g$.
We have: $\forall j \in \mathbb{N}_0$, $L^j f = f^{(2j)}$.
Then: $f^{(2j)} \to g$ pointwise on I, as $j \to \infty$.
It follows that: f has PBED on I.
Then, by Theorem 42, we get: f has PBD on I.
Then, by Tao's Theorem (Theorem 30), we get: f has UBD on I.
Then: f has UBED on I. Choose $M \geqslant 0$ s.t. f has M-BED on I.
Then: $\forall j \in \mathbb{N}_0$, $|f^{(2j)}| \leqslant M$ on I.
For all $j \in \mathbb{N}_0$, let $f_j := L^j f$. Then: $\forall j \in \mathbb{N}_0$, $f_j = f^{(2j)}$.
Then: $f_j \to g$ pointwise on I, as $j \to \infty$.
Also, $\forall j \in \mathbb{N}_0$, $|f_j| \leqslant M$ on I.
Then, since $f_j \to g$ pointwise on I, as $j \to \infty$, by Theorem 51,
g is bounded and measurable on I and
$(f_j)_I^{\#} \to g_I^{\#}$ pointwise on I, as $j \to \infty$.
By Theorem 52, we get: $\forall j \in \mathbb{N}_0$, $(f_j)_I^{\#}$ is M-Lipschitz on I.
Let $c := (a + b)/2$. Then: $\forall j \in \mathbb{N}_0$, $(f_j)_I^{\#}(c) = 0$.
Let $M' := M \cdot (b - a)$. Then $M' \geqslant 0$.
Also, $\forall j \in \mathbb{N}_0$, $M' = |(f_j)_I^{\#}(c)| + M \cdot (b - a)$.

Then, by Theorem 47, we get: $\forall j \in \mathbb{N}_0,\ \ |(f_j)_I^{\#}| \leqslant M'$ on I.
Then, since $(f_j)_I^{\#} \to g_I^{\#}$ pointwise on I, as $j \to \infty$, by Theorem 51,
$g_I^{\#}$ is bounded and measurable on I and
$(f_j)_I^{\#\#} \to g_I^{\#\#}$ pointwise on I, as $j \to \infty$.
Recall: $f_j \to g$ pointwise on I, as $j \to \infty$.
Then: $(f_j'')_I^{\#\#} - f_j \to g_I^{\#\#} - g$ pointwise on I, as $j \to \infty$.
For all $j \in \mathbb{N}_0$, let $\lambda_j := (f_j'')_I^{\#\#} - f_j$. Let $\mu := g_I^{\#\#} - g$.
Then $\lambda_j \to \mu$ pointwise on I, as $j \to \infty$. Also, $g = g_I^{\#\#} - \mu$.
Since $f \in S = C^\infty(I, \mathbb{R})$ and
since $\forall j \in \mathbb{N}_0,\ \ f_j'' = (L^j f)'' = (f^{(2j)})'' = f^{(2j+2)}$, we conclude:
$\forall j \in \mathbb{N}_0,\ \ I \subseteq \mathbb{D}_{f_j}^{(2)}$ and f_j'' is continuous on I.
We have: $\forall j \in \mathbb{N}_0,\ \ f_j'' = Lf_j = LL^j f = L^{j+1} f = f_{j+1}$.
Then: $\forall j \in \mathbb{N}_0,\ \ |f_j''| = |f_{j+1}| \leqslant M$ on I.
Then: $\forall j \in \mathbb{N}_0,\ \ f_j''$ is bounded on I.
Then, by Theorem 56, we have: $\forall j \in \mathbb{N}_0,\ \ (f_j'')_I^{\#\#} - f_j$ is affine on I.
So, since $\forall j \in \mathbb{N}_0,\ \ \lambda_j = (f_j'')_I^{\#\#} - f_j$,
we get: $\forall j \in \mathbb{N}_0,\ \ \lambda_j$ is affine on I.
So, since $\lambda_j \to \mu$ pointwise on I, as $j \to \infty$,
by Theorem 45, we get: μ is affine on I.
So, by Theorem 46, we get: μ is Lipschitz on I.
Then, by Theorem 48, we get: μ is bounded and continuous on I.
Recall: $g_I^{\#}$ is bounded and measurable on I.
So, by Theorem 53, $g_I^{\#\#}$ is bounded and continuous on I.
Then, since $g = g_I^{\#\#} - \mu$, we get: g is bounded and continuous on I.
Then, by Theorem 55, we get: $(g_I^{\#\#})'' = g$.
Since μ is affine on I, we get: $\mu'' = 0$.
Then, by subtracting, we get: $(g_I^{\#\#} - \mu)'' = g$.
So, since $g = g_I^{\#\#} - \mu$, we get: $g'' = g$. □