Points of Density and Continuity in Probability

The main results in this note are:

Theorem 18, Theorem 22, Theorem 24.

DEFINITION 1. Let $\mathcal{S}$ *be a set of sets.*

Then: $\boxed{\bigcup \mathcal{S}} := \begin{cases} \varnothing, & \text{if } \mathcal{S} = \varnothing \\ \bigcup\limits_{S \in \mathcal{S}} S, & \text{if } \mathcal{S} \neq \varnothing. \end{cases}$

We make a similar convention that an empty sum is equal to 0.

DEFINITION 2. *We* **define** $\boxed{\#\varnothing} := 0$.
For any nonempty finite set S,
$\boxed{\#S}$ **denotes** *the number of elements in* S.
For any infinite set S, *we* **define** $\boxed{\#S} := \infty$.

DEFINITION 3. Let $\boxed{\mathbb{R}^*} := \{-\infty\} \bigcup \mathbb{R} \bigcup \{\infty\}$.
For all $a, b \in \mathbb{R}^*$, **let**

$$\boxed{(a;b)} := \{x \in \mathbb{R}^* \mid a < x < b\}, \qquad \boxed{[a;b)} := \{x \in \mathbb{R}^* \mid a \leqslant x < b\},$$
$$\boxed{(a;b]} := \{x \in \mathbb{R}^* \mid a < x \leqslant b\}, \qquad \boxed{[a;b]} := \{x \in \mathbb{R}^* \mid a \leqslant x \leqslant b\}.$$

DEFINITION 4. *For all* $x \in \mathbb{R}^2$, *for all* $r > 0$, **let**

$$\boxed{B_x^r} := \{ \, y \in \mathbb{R}^2 \text{ s.t. } |y - x| < r \, \}.$$

That is: B_x^r is the open disk about x of radius r.

Let $\boxed{\mathcal{B}} := \{ \, B_x^r \mid x \in \mathbb{R}^2, \ r > 0 \, \}$.
Let $\boxed{\mathcal{T}}$ denote the standard topology on $\mathbb{R}^2$,
so $\mathcal{T}$ is the set of open subsets of $\mathbb{R}^2$.
Then: $\forall U \in \mathcal{T}$, U is Lebesgue-measurable. Also, $\mathcal{B} \subseteq \mathcal{T} \backslash \{\varnothing\}$.

DEFINITION 5. Let $x \in \mathbb{R}^2$, $r > 0$, $C := B_x^r$.
Then: $\boxed{\operatorname{rad} C} := r$ *and* $\boxed{\operatorname{cent} C} := x$ *and*
$\forall s > 0$, $\boxed{s \cdot C} := B_x^{s \cdot r}$.

According to the next theorem, if two disks meet,
then the triple of the larger covers the smaller.

THEOREM 6. Let $F, G \in \mathcal{B}$.
Assume: $\operatorname{rad} F \leqslant \operatorname{rad} G$ *and* $F \bigcap G \neq \varnothing$. *Then:* $3 \cdot G \supseteq F$.

Proof. **Given** $a \in F$, **want:** $a \in 3 \cdot G$.
Since $F \bigcap G \neq \varnothing$, **choose** $p \in F \bigcap G$. Then: $p \in F$ and $p \in G$.
Let $x := \text{cent}\, F$, $y := \text{cent}\, G$, $r := \text{rad}\, F$, $s := \text{rad}\, G$.
Then, by hypothesis, we have: $r \leqslant s$.
Also, $F = B_x^r$ and $G = B_y^s$ and $3 \cdot G = B_y^{3s}$.
Want: $a \in B_y^{3s}$. **Want:** $|a - y| < 3s$.
Since $a \in F = B_x^r$, we get: $|a - x| < r$.
Since $p \in F = B_x^r$, we get: $|p - x| < r$.
Since $p \in G = B_y^s$, we get: $|p - y| < s$.
Since $r \leqslant s$, we get: $r + r + s \leqslant 3s$.
Then $|a - y| \leqslant |a - x| + |x - p| + |p - y| < r + r + s \leqslant 3s$. $\square$

Let $\boxed{\mathbb{N}} := \{1, 2, 3, \ldots\}$ be the set of positive integers.
We use "pw-dj" to abbreviate "pairwise-disjoint".
For any set $\mathcal{S}$ of sets, by $\boxed{\mathcal{S}\textbf{ is pw-dj}}$,
we mean: $\forall S, T \in \mathcal{S}$, $(\, S \neq T \,) \Rightarrow (\, S \bigcap T = \varnothing \,)$.
For any sequence $(S_1, S_2, \ldots)$ of sets, by $\boxed{(S_1, S_2, \ldots)\textbf{ is pw-dj}}$,
we mean: $\forall i, j \in \mathbb{N}$, $(\, i \neq j \,) \Rightarrow (\, S_i \bigcap S_j = \varnothing \,)$.
For any $\mathcal{C} \subseteq \mathcal{B}$, for any $s > 0$, **we define:**
$\boxed{s \cdot \mathcal{C}} := \{s \cdot C \,|\, C \in \mathcal{C}\}$.

THEOREM 7. Let $\mathcal{F} \subseteq \mathcal{B}$. *Assume $\mathcal{F}$ is finite.*
Then: $\exists$*pw-dj* $\mathcal{E} \subseteq \mathcal{F}$ *s.t.* $\bigcup(3 \cdot \mathcal{E}) \supseteq \bigcup \mathcal{F}$.

Proof. **Let** $n := \#\mathcal{F}$.
In case $n = 0$, **let** $\mathcal{E} := \varnothing$. We therefore assume $n \geqslant 1$.
By induction on n, we also assume: $\forall \mathcal{Q} \subseteq \mathcal{B}$,
$(\, \#\mathcal{Q} < n \,) \Rightarrow (\, \exists\text{pw-dj } \mathcal{P} \subseteq \mathcal{Q} \text{ s.t. } \bigcup(3 \cdot \mathcal{P}) \supseteq \bigcup \mathcal{Q} \,)$.
Let $R := \{\text{rad}\, F \,|\, F \in \mathcal{F}\}$. Then R is a finite subset of $\mathbb{R}$.
Let $r := \max R$. Then $r \in R$, so **choose** $G \in \mathcal{F}$ s.t. $\text{rad}\, G = r$.
Since $G \in \mathcal{F} \subseteq \mathcal{B} \subseteq \mathcal{T} \backslash \{\varnothing\}$, we get: $G \neq \varnothing$.
Let $\mathcal{Q} := \{F \in \mathcal{F} \,|\, F \bigcap G = \varnothing\}$. Then $\mathcal{Q} \subseteq \mathcal{F}$ and $G \notin \mathcal{Q}$.
Then $\mathcal{Q} \subseteq \mathcal{F} \backslash \{G\}$, so: $\#\mathcal{Q} \leqslant \#(\mathcal{F} \backslash \{G\})$.
Since $G \in \mathcal{F}$ and since $\mathcal{F}$ is finite, we get: $\#(\mathcal{F} \backslash \{G\}) < \#\mathcal{F}$.
Since $\#\mathcal{Q} \leqslant \#(\mathcal{F} \backslash \{G\}) < \#\mathcal{F} = n$, by the induction assumption,
choose a pw-dj $\mathcal{P} \subseteq \mathcal{Q}$ s.t. $\bigcup(3 \cdot \mathcal{P}) \supseteq \bigcup \mathcal{Q}$.
Since $\mathcal{P} \subseteq \mathcal{Q}$, by definition of $\mathcal{Q}$,
we get: $\forall P \in \mathcal{P}$, $P \bigcap G = \varnothing$.
So, since $\mathcal{P}$ is pw-dj, we get: $\mathcal{P} \bigcup \{G\}$ is pw-dj.

Since $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{F}$ and since $G \in \mathcal{F}$, we get: $\mathcal{P} \bigcup \{G\} \subseteq \mathcal{F}$.
Let $\mathcal{E} := \mathcal{P} \bigcup \{G\}$. Then: $\mathcal{E}$ is pw-dj and $\mathcal{E} \subseteq \mathcal{F}$.
It remains only to show: $\bigcup(3 \cdot \mathcal{E}) \supseteq \bigcup \mathcal{F}$.
Want: $\forall F \in \mathcal{F}$, $F \subseteq \bigcup(3 \cdot \mathcal{E})$.
Given $F \in \mathcal{F}$, **want:** $F \subseteq \bigcup(3 \cdot \mathcal{E})$.

Case 1: $F \in \mathcal{Q}$. *Proof in Case 1:*
Since $\mathcal{P} \subseteq \mathcal{P} \bigcup \{G\} = \mathcal{E}$, we get $3 \cdot \mathcal{P} \subseteq 3 \cdot \mathcal{E}$, so $\bigcup(3 \cdot \mathcal{P}) \subseteq \bigcup(3 \cdot \mathcal{E})$.
By the choice of $\mathcal{P}$, we have: $\bigcup(3 \cdot \mathcal{P}) \supseteq \bigcup \mathcal{Q}$.
Since $F \in \mathcal{Q}$, we get: $F \subseteq \bigcup \mathcal{Q}$.
Then: $F \subseteq \bigcup \mathcal{Q} \subseteq \bigcup(3 \cdot \mathcal{P}) \subseteq \bigcup(3 \cdot \mathcal{E})$.
End of proof in Case 1.

Case 2: $F \notin \mathcal{Q}$. *Proof in Case 2:* Recall: $F \in \mathcal{F}$.
So, by definition of R, we have: $\operatorname{rad} F \in R$. Then $\operatorname{rad} F \leqslant \max R$.
Since $F \in \mathcal{F}$ and $F \notin \mathcal{Q}$, by definition of $\mathcal{Q}$, we get: $F \bigcap G \neq \varnothing$.
So, since $\operatorname{rad} F \leqslant \max R = r = \operatorname{rad} G$,
by Theorem 6, we get: $3 \cdot G \supseteq F$.
Since $G \in \mathcal{P} \bigcup \{G\} = \mathcal{E}$, we get $3 \cdot G \in 3 \cdot \mathcal{E}$, so $3 \cdot G \subseteq \bigcup(3 \cdot \mathcal{E})$.
Then: $F \subseteq 3 \cdot G \subseteq \bigcup(3 \cdot \mathcal{E})$.
End of proof in Case 2. □

Let $\boxed{\lambda}$ denote Lebesgue-outer-measure on $\mathbb{R}^2$.

THEOREM 8.
Let $(A_1, A_2, \ldots)$ *be a sequence of Lebesgue-measurable subsets of* $\mathbb{R}^2$.
Then: *as* $k \to \infty$, $\lambda(A_1 \bigcup \cdots \bigcup A_k) \to \lambda(A_1 \bigcup A_2 \bigcup \cdots)$.

Proof. For all $k \in \mathbb{N}$, **let** $D_k := A_k \backslash (A_1 \bigcup \cdots \bigcup A_{k-1})$.
Then, $\forall k \in \mathbb{N}$, D_k is Lebesgue-measurable
and, $\forall k \in \mathbb{N}$, $D_1 \bigcup \cdots \bigcup D_k = A_1 \bigcup \cdots \bigcup A_k$
and $D_1 \bigcup D_2 \bigcup \cdots = A_1 \bigcup A_2 \bigcup \cdots$
and $(D_1, D_2, \ldots)$ is pw-dj.
Since $(D_1, D_2, \ldots)$ is pw-dj, by countable-additivity of λ, we get

$$\lambda(D_1 \bigcup D_2 \bigcup \cdots) = (\lambda(D_1)) + (\lambda(D_2)) + \cdots;$$

also, by finite-additivity of λ, we get

$$\forall k \in \mathbb{N}, \qquad \lambda(D_1 \bigcup \cdots \bigcup D_k) = (\lambda(D_1)) + \cdots + (\lambda(D_k)).$$

By definition of infinite-summation, we have

$$\text{as } k \to \infty, \qquad (\lambda(D_1)) + \cdots + (\lambda(D_k)) \to (\lambda(D_1)) + (\lambda(D_2)) + \cdots.$$

Then: as $k \to \infty$, $\lambda(D_1 \bigcup \cdots \bigcup D_k) \to \lambda(D_1 \bigcup D_2 \bigcup \cdots)$.
Then: as $k \to \infty$, $\lambda(A_1 \bigcup \cdots \bigcup A_k) \to \lambda(A_1 \bigcup A_2 \bigcup \cdots)$. $\square$

The next result says: for any collection of open disks,
if its union has finite Lebesgue-measure, then
$\exists$finite pw-dj subcollection that covers at least 10% of that union.

THEOREM 9. Let $\mathcal{A} \subseteq \mathcal{B}$. *Assume:* $\lambda(\bigcup \mathcal{A}) < \infty$.
Then: $\exists$*finite pw-dj* $\mathcal{E} \subseteq \mathcal{A}$ *s.t.* $\lambda(\bigcup \mathcal{E}) \geqslant 0.1 \cdot (\lambda(\bigcup \mathcal{A}))$.

Proof. In case $\lambda(\bigcup \mathcal{A}) = 0$, **let** $\mathcal{E} := \varnothing$.
We therefore assume $\lambda(\bigcup \mathcal{A}) \neq 0$. Then $\lambda(\bigcup \mathcal{A}) > 0$.
By hypothesis, $\lambda(\bigcup \mathcal{A}) < \infty$. **Let** $c := \lambda(\bigcup \mathcal{A})$.
Then $0 < c < \infty$. Then: $0.9 \cdot c < c$.
Since $\bigcup \mathcal{A}$ is Lindelöf, **choose** $A_1, A_2, \ldots \in \mathcal{A}$
s.t. $A_1 \bigcup A_2 \bigcup \cdots = \bigcup \mathcal{A}$.
Since $A_1, A_2, \ldots \in \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{T}$, we get:
$(A_1, A_2, \ldots)$ is a sequence of Lebesgue-measurable subsets of $\mathbb{R}^2$.
So, by Theorem 8, we have:
as $k \to \infty$, $\lambda(A_1 \bigcup \cdots \bigcup A_k) \to \lambda(A_1 \bigcup A_2 \bigcup \cdots)$.
So, since $0.9 \cdot c < c = \lambda(\bigcup \mathcal{A}) = \lambda(A_1 \bigcup A_2 \bigcup \cdots)$,
choose $k \in \mathbb{N}$ s.t. $\lambda(A_1 \bigcup \cdots \bigcup A_k) \geqslant 0.9 \cdot c$.
Let $\mathcal{F} := \{A_1, \ldots, A_k\}$. Then $\lambda(\bigcup \mathcal{F}) \geqslant 0.9 \cdot c$ and $\mathcal{F} \subseteq \mathcal{A}$.
Also, $\mathcal{F}$ is finite, so, since $\mathcal{F} \subseteq \mathcal{A} \subseteq \mathcal{B}$, by Theorem 7,
choose a pw-dj $\mathcal{E} \subseteq \mathcal{F}$ s.t. $\bigcup(3 \cdot \mathcal{E}) \supseteq \bigcup \mathcal{F}$.
Since $\mathcal{E} \subseteq \mathcal{F}$ and since $\mathcal{F}$ is finite, we get: $\mathcal{E}$ is finite.
Since $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{A}$, **it remains only to show:** $\lambda(\bigcup \mathcal{E}) \geqslant 0.1 \cdot (\lambda(\bigcup \mathcal{A}))$.
Since $c = \lambda(\bigcup \mathcal{A})$, **we want:** $\lambda(\bigcup \mathcal{E}) \geqslant 0.1 \cdot c$.
We have: $\forall B \in \mathcal{B}$, $\lambda(3 \cdot B) = 9 \cdot (\lambda(B))$.
Since $\bigcup \mathcal{F} \subseteq \bigcup (3 \cdot \mathcal{E})$, by monotonicity and subadditivity of λ,

$$\lambda(\bigcup \mathcal{F}) \leqslant \sum_{E \in \mathcal{E}} (\lambda(3 \cdot E)).$$

Since $\lambda(\bigcup \mathcal{F}) \leqslant \sum_{E \in \mathcal{E}} (\lambda(3 \cdot E)) = \sum_{E \in \mathcal{E}} (9 \cdot (\lambda(E))) = 9 \cdot \sum_{E \in \mathcal{E}} (\lambda(E))$,
we get: $(1/9) \cdot (\lambda(\bigcup \mathcal{F})) \leqslant \sum_{E \in \mathcal{E}} (\lambda(E))$.

Since $\mathcal{E} \subseteq \mathcal{B} \subseteq \mathcal{T}$, we get: $\forall E \in \mathcal{E}$, E is Lebesgue-measurable.
So, since $\mathcal{E}$ is finite and pw-dj, by finite-additivity of λ, we get:

$$\lambda(\bigcup \mathcal{E}) \ = \ \sum_{E \in \mathcal{E}} (\lambda(E)).$$

Since $\lambda(\bigcup \mathcal{F}) \geqslant 0.9 \cdot c$, we get: $\quad (1/9) \cdot (\lambda(\bigcup \mathcal{F})) \ \geqslant \ 0.1 \cdot c.$

Then $\quad \lambda(\bigcup \mathcal{E}) \ = \ \sum_{E \in \mathcal{E}} (\lambda(E)) \ \geqslant \ (1/9) \cdot (\lambda(\bigcup \mathcal{F})) \ \geqslant \ 0.1 \cdot c.$ □

Let A and B be sets.
By B is a **superset of** A, we will mean: $B \supseteq A$.

Let $\mathcal{B}$ be a set of sets and **let** A be a set.
By $\mathcal{B}$ is a **covering of** A, we will mean:
$\bigcup \mathcal{B}$ is a superset of A.

DEFINITION 10. Let $Q \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$.
By $\mathcal{V}$ *is a* **fine-covering** *of* Q, *we mean:*
$\forall x \in Q$, $\forall \delta > 0$, $\exists V \in \mathcal{V}$ *s.t.* $(\, x \in V\,) \,\&\, (\, \mathrm{rad}\, V < \delta\,)$.

NOTE: A fine-covering is a covering, *i.e.*: $\forall Q \subseteq \mathbb{R}^2$, $\forall \mathcal{V} \subseteq \mathcal{B}$,
if $\mathcal{V}$ is a fine-covering of Q, then $\bigcup \mathcal{V} \supseteq Q$.

Let Q be a set and **let** $\mathcal{P}$ be a set of sets. We'll say
$\mathcal{P}$ is **inside Q** if: $\bigcup \mathcal{P} \subseteq Q$.

According to the next theorem,
for any fine-covering $\mathcal{V} \subseteq \mathcal{B}$ of a set $Q \subseteq \mathbb{R}^2$,
for any open $W \subseteq \mathbb{R}^2$,
there is a subset of $\mathcal{V}$ that is
both inside W and a fine-covering of $Q \bigcap W$.

THEOREM 11. Let $Q, W \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$.
Assume: $W \in \mathcal{T}$ *and* $\mathcal{V}$ *is a fine-covering of* Q.
Let $\mathcal{V}' := \{V \in \mathcal{V} \,|\, V \subseteq W\}$. *Then:* $\mathcal{V}'$ *is a fine-covering of* $Q \bigcap W$.

Proof. **Given** $x \in Q \bigcap W$, $\delta > 0$,
want: $\exists V \in \mathcal{V}'$ s.t. $(\, x \in V\,) \,\&\, (\, \mathrm{rad}\, V < \delta\,)$.
Since $x \in Q \bigcap W \subseteq W$ and $W \in \mathcal{T}$, **choose** $\beta > 0$ s.t. $B_x^\beta \subseteq W$.
Let $\alpha := \min\{\beta/2, \delta\}$. Then $\alpha > 0$ and $\alpha \leqslant \beta/2$ and $\alpha \leqslant \delta$.
Since $x \in Q \bigcap W \subseteq Q$ and $\alpha > 0$ and $\mathcal{V}$ is a fine-covering of Q,
choose $V \in \mathcal{V}$ s.t. $(\, x \in V\,) \,\&\, (\, \mathrm{rad}\, V < \alpha\,)$.
Since $\mathrm{rad}\, V < \alpha \leqslant \delta$, **it remains only to show:** $V \in \mathcal{V}'$.

By definition of $\mathcal{V}'$, since $V \in \mathcal{V}$, **we wish to show:** $V \subseteq W$.
Given $v \in V$, **want:** $v \in W$.
Since $B_x^\beta \subseteq W$, **it suffices to show:** $v \in B_x^\beta$. **Want:** $|v - x| < \beta$.
Since $V \in \mathcal{V} \subseteq \mathcal{B}$, **choose** $c \in \mathbb{R}^2$ and $r > 0$ s.t. $V = B_c^r$.
Since $v, x \in V = B_c^r$, we get: $|v - c| < r$ and $|x - c| < r$.
Since $r = \operatorname{rad} B_c^r = \operatorname{rad} V < \alpha \leqslant \beta/2$, we get: $2r < \beta$.
Then: $|v - x| \leqslant |v - c| + |c - x| < r + r = 2r < \beta$. □

According to the next theorem,
for any fine-covering $\mathcal{V} \subseteq \mathcal{B}$ of a set $Q \subseteq \mathbb{R}^2$,
for any open $W \subseteq \mathbb{R}^2$ that is a superset of Q,
there is a subset of $\mathcal{V}$ that is
both inside W and a fine-covering of Q.

THEOREM 12. Let $W \subseteq \mathbb{R}^2$, $Q \subseteq W$, $\mathcal{V} \subseteq \mathcal{B}$.
Assume: $W \in \mathcal{T}$ *and* $\mathcal{V}$ *is a fine-covering of* Q.
Let $\mathcal{V}' := \{V \in \mathcal{V} \,|\, V \subseteq W\}$. *Then:* $\mathcal{V}'$ *is a fine-covering of* Q.

Proof. Since $Q \subseteq W$, we get: $Q \bigcap W = Q$.
So, by Theorem 11, we get: $\mathcal{V}'$ is a fine-covering of Q. □

According to the **Carathéodory-condition**,
$\forall Q \subseteq \mathbb{R}^2$, Q is Lebesgue-measurable iff
$\forall S \subseteq \mathbb{R}^2$, $\lambda(S) = [\,\lambda(S \bigcap Q)\,] + [\,\lambda(S \backslash Q)\,]$.
That is: Q is Lebesgue-measurable iff Q "splits all sets well".

According to the next theorem,
for any $Q \subseteq \mathbb{R}^2$ of finite Lebesgue-outer-measure,
for any fine-covering $\mathcal{V}$ of Q,
there is a finite pw-dj subset of $\mathcal{V}$ covering at least 1% of Q.

THEOREM 13. Let $Q \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$.
Assume: $\mathcal{V}$ *is a fine-covering of* Q. *Assume:* $\lambda(Q) < \infty$.
Then: $\exists$*finite pw-dj* $\mathcal{E} \subseteq \mathcal{V}$ *s.t.* $\lambda(Q \bigcap (\bigcup \mathcal{E})) \geqslant 0.01 \cdot (\lambda(Q))$.

Idea of proof: In case $\lambda(Q) = 0$, **let** $\mathcal{E} := \varnothing$, so assume $\lambda(Q) > 0$.
Let $\varepsilon := 0.1 \cdot (\lambda(Q))$. By outer-regularity of λ, **choose** $W \in \mathcal{T}$
s.t. $W \supseteq Q$ and $\lambda(W) \leqslant (\lambda(Q)) + \varepsilon$.
Then: W approximates Q in measure, to within ε.
By Theorem 12, **choose** a fine-covering $\mathcal{V}' \subseteq \mathcal{V}$, inside W, of Q.
Since $Q \subseteq \bigcup \mathcal{V}' \subseteq W$ and since W approximates Q in measure,

we conclude that: $\bigcup \mathcal{V}'$ also approximates Q in measure.
By Theorem 9, **choose** a finite pw-dj $\mathcal{E} \subseteq \mathcal{V}'$ s.t.
$\mathcal{E}$ covers at least 10% of $\bigcup \mathcal{V}'$.
There are details to check, but,
assuming our choice of $\varepsilon = 0.1 \cdot (\lambda(Q))$ is small enough, *i.e.*,
assuming $\bigcup \mathcal{V}'$ approximates Q sufficiently closely in measure,
then, because $\mathcal{E}$ covers at least 10% of $\bigcup \mathcal{V}'$,
it will follow that $\mathcal{E}$ covers at least 1% of Q. **QED**

Proof. In case $\lambda(Q) = 0$, **let** $\mathcal{E} := \varnothing$. We therefore assume $\lambda(Q) \neq 0$.
Then $\lambda(Q) > 0$. By hypothesis, $\lambda(Q) < \infty$. **Let** $b := \lambda(Q)$.
Then $0 < b < \infty$. Then: $1.1 \cdot b > b$.
Since $1.1 \cdot b > b = \lambda(Q)$, by outer-regularity of λ,
choose $W \in \mathcal{T}$ s.t. $W \supseteq Q$ and $\lambda(W) \leqslant 1.1 \cdot b$.
Let $\mathcal{V}' := \{V \in \mathcal{V} \,|\, V \subseteq W\}$. Then $\mathcal{V}' \subseteq \mathcal{V}$. Also, $\bigcup \mathcal{V}' \subseteq W$.
Let $V := \bigcup \mathcal{V}'$. Then: $V \subseteq W$.
So, by monotonicity of λ, we get: $\lambda(V) \leqslant \lambda(W)$.
Let $c := \lambda(V)$. Then: $c \leqslant \lambda(W)$.
Since $c \leqslant \lambda(W) \leqslant 1.1 \cdot b$, we get: $c \leqslant 1.1 \cdot b$.
So, since $b < \infty$, we get: $c < \infty$. **Let** $\mathcal{A} := \mathcal{V}'$.
Since $\mathcal{A} = \mathcal{V}' \subseteq \mathcal{V}$ and since $\mathcal{V} \subseteq \mathcal{B}$, we get: $\mathcal{A} \subseteq \mathcal{B}$.
So, since $\lambda(\bigcup \mathcal{A}) = \lambda(\bigcup \mathcal{V}') = \lambda(V) = c < \infty$, by Theorem 9,
choose a finite pw-dj $\mathcal{E} \subseteq \mathcal{A}$ s.t. $\lambda(\bigcup \mathcal{E}) \geqslant 0.1 \cdot (\lambda(\bigcup \mathcal{A}))$.
Then, since $\lambda(\bigcup \mathcal{A}) = \lambda(\bigcup \mathcal{V}') = \lambda(V) = c$, $\lambda(\bigcup \mathcal{E}) \geqslant 0.1 \cdot c$.
Since $\mathcal{E} \subseteq \mathcal{A} = \mathcal{V}'$, we get: $\mathcal{E} \subseteq \mathcal{V}'$.
So, since $\mathcal{V}' \subseteq \mathcal{V}$, we get: $\mathcal{E} \subseteq \mathcal{V}$.
It remains only to show: $\lambda(Q \bigcap (\bigcup \mathcal{E})) \geqslant 0.01 \cdot (\lambda(Q))$.
Since $b = \lambda(Q)$, **we want:** $\lambda(Q \bigcap (\bigcup \mathcal{E})) \geqslant 0.01 \cdot b$.
Let $E := \bigcup \mathcal{E}$. **Want:** $\lambda(Q \bigcap E) \geqslant 0.01 \cdot b$.
Let $x := \lambda(Q \bigcap E)$. **Want:** $x \geqslant 0.01 \cdot b$.
We have: $\forall B \in \mathcal{B}$, $\lambda(B) < \infty$.
So, since $\mathcal{E} \subseteq \mathcal{V} \subseteq \mathcal{B}$ and $\mathcal{E}$ is finite, we get: $\lambda(\bigcup \mathcal{E}) < \infty$.
So, since $E = \bigcup \mathcal{E}$, we get: $\lambda(E) < \infty$.
Let $y := \lambda(E)$. Then: $y < \infty$.
Since $x = \lambda(Q \bigcap E) \leqslant \lambda(E) < \infty$, we get: $x < \infty$.
Since $\mathcal{E} \subseteq \mathcal{V}'$, we get $\bigcup \mathcal{E} \subseteq \bigcup \mathcal{V}'$.
Since $E = \bigcup \mathcal{E} \subseteq \bigcup \mathcal{V}' = V$, we get: $V \bigcap E = E$.
Then: $\lambda(V \bigcap E) = \lambda(E)$.

Since $\mathcal{E} \subseteq \mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T}$ and since $\mathcal{T}$ is a topology, we get: $\bigcup \mathcal{E} \in \mathcal{T}$.
Since $E = \bigcup \mathcal{E} \in \mathcal{T}$, it follows that: E is Lebesgue-measurable.
So, by the Carathéodory-condition,
$\lambda(V) = [\,\lambda(V \backslash E)\,] + [\,\lambda(V \bigcap E)\,]$.
So, since $\lambda(V \bigcap E) = \lambda(E) < \infty$, we get:
$\lambda(V \backslash E) = [\,\lambda(V)\,] - [\,\lambda(V \bigcap E)\,]$.
So, since $c = \lambda(V)$ and $\lambda(V \bigcap E) = \lambda(E) = y$, we get:
$\lambda(V \backslash E) = c - y$.
Since E is Lebesgue-measurable, by the Carathéodory-condition,
$\lambda(Q) = [\,\lambda(Q \backslash E)\,] + [\,\lambda(Q \bigcap E)\,]$.
So, since $\lambda(Q \bigcap E) \leqslant \lambda(E) < \infty$, we get:
$\lambda(Q \backslash E) = [\,\lambda(Q)\,] - [\,\lambda(Q \bigcap E)\,]$.
So, since $b = \lambda(Q)$ and $\lambda(Q \bigcap E) = x$, we get:
$\lambda(Q \backslash E) = b - x$.
By Theorem 12, $\mathcal{V}'$ is a fine-covering of Q, so:
$\bigcup \mathcal{V}' \supseteq Q$.
Since $V = \bigcup \mathcal{V}' \supseteq Q$, we get: $V \backslash E \supseteq Q \backslash E$.
So, by monotonicity of λ, we get: $\lambda(V \backslash E) \geqslant \lambda(Q \backslash E)$.
So, since $\lambda(V \backslash E) = c - y$ and $\lambda(Q \backslash E) = b - x$, $c - y \geqslant b - x$.
Recall: $b < \infty$, $c < \infty$, $x < \infty$, $y < \infty$, $c \leqslant 1.1 \cdot b$.
Since $y = \lambda(E) = \lambda(\bigcup \mathcal{E}) \geqslant 0.1 \cdot c$,
we get: $c - y \leqslant 0.9 \cdot c$.
Since $c \leqslant 1.1 \cdot b$, we get: $0.9 \cdot c \leqslant 0.99 \cdot b$.
Since $b - x \leqslant c - y \leqslant 0.9 \cdot c \leqslant 0.99 \cdot b$,
we get: $x \geqslant 0.01 \cdot b$. $\square$

For any two sets A and B, **we define:** $\boxed{A \triangle B} := (A \backslash B) \bigcup (B \backslash A)$.
For any $A, B \subseteq \mathbb{R}^2$, by $\boxed{A \equiv B}$, we mean: $\lambda(A \triangle B) = 0$.
We will read "$\equiv$" as: "**is a.e.-equal to**".
For all sets A, B, we have:
$(\ A \subseteq B \bigcup (A \triangle B)\ )\ \&\ (\ B \subseteq A \bigcup (A \triangle B)\ )$.
So, by monotonicity and subadditivity of λ, we conclude:
$\forall A, B \subseteq \mathbb{R}^2$, $(\,A \equiv B\,) \Rightarrow (\,\lambda(A) = \lambda(B)\,)$.
For any sets A, B, Y, Z, we have:
$(A \bigcup Y) \triangle (B \bigcup Z) \subseteq (A \triangle B) \bigcup (Y \triangle Z)$ and
$(A \bigcap Y) \triangle (B \bigcap Z) \subseteq (A \triangle B) \bigcup (Y \triangle Z)$ and
$(A \backslash Y) \triangle (B \backslash Z) \subseteq (A \triangle B) \bigcup (Y \triangle Z)$.
So, $\forall A, B, Y, Z \subseteq \mathbb{R}^2$, if $A \equiv B$ and $Y \equiv Z$, then:

$$A \bigcup Y \equiv B \bigcup Z \quad \text{and} \quad A \bigcap Y \equiv B \bigcap Z \quad \text{and} \quad A \backslash Y \equiv B \backslash Z.$$

For all $S \subseteq \mathbb{R}^2$, **let** $\boxed{\overline{S}}$ denote the closure in $\mathbb{R}^2$ of S.

NOTE: $\forall x \in \mathbb{R}^2, \forall r > 0$, we have: $\lambda(B_x^r) = \pi r^2 = \lambda(\overline{B_x^r})$.

It follows that: $\forall B \in \mathcal{B}$, $B \equiv \overline{B}$.

The next result says that
if $Q \subseteq \mathbb{R}^2$ has finite Lebesgue-outer-measure, and
if $\mathcal{V} \subseteq \mathcal{B}$ is a fine-covering of Q, and
if, using a finite pw-dj $\mathcal{E} \subseteq \mathcal{V}$, we can cover some portion of Q,
then, using a bigger finite pw-dj collection $\mathcal{F} \subseteq \mathcal{V}$,
we can cover substantially more, by which we mean:
the UNcovered portion decreases by at least 1%.

THEOREM 14. Let $Q \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$, $\mathcal{E} \subseteq \mathcal{V}$.
Assume: $\mathcal{V}$ *is a fine-covering of* Q. *Assume:* $\lambda(Q) < \infty$.
Assume: $\mathcal{E}$ *is finite and pw-dj.*
Then: $\exists$*finite pw-dj* $\mathcal{F} \subseteq \mathcal{V}$ *s.t.* $\mathcal{E} \subseteq \mathcal{F}$ *and s.t.*

$$\lambda(Q \backslash (\bigcup \mathcal{F})) \leqslant 0.99 \cdot (\lambda(Q \backslash (\bigcup \mathcal{E}))).$$

Idea of Proof: **Let** $S := \bigcup \mathcal{E}$. Then: $\mathcal{E}$ is inside S.
Since $\mathcal{E}$ is a finite set of disks, we get: $\overline{S} \equiv S$.
Then $\mathbb{R}^2 \backslash \overline{S} \equiv \mathbb{R}^2 \backslash S$. **Let** $W := \mathbb{R}^2 \backslash \overline{S}$.
Then: $W \equiv \mathbb{R}^2 \backslash S$ and W is open in $\mathbb{R}^2$.
We have $Q \bigcap W \equiv Q \bigcap (\mathbb{R}^2 \backslash S) = Q \backslash S = Q \backslash (\bigcup \mathcal{E})$,
so $Q \bigcap W \equiv$ (the portion of Q that is uncovered by $\mathcal{E}$).
Using Theorem 11, **choose** $\mathcal{V}' \subseteq \mathcal{V}$ s.t.
$\mathcal{V}'$ is a fine-covering of $Q \bigcap W$ and $\mathcal{V}'$ is inside W.
Apply Theorem 13 to get a finite pw-dj subset $\mathcal{E}' \subseteq \mathcal{V}'$ which
covers at least 1% of $Q \bigcap W$, and, therefore,
covers at least 1% of (the portion of Q that is uncovered by $\mathcal{E}$).
Since $\mathcal{E}' \subseteq \mathcal{V}'$ and since $\mathcal{V}'$ is inside W and since $W = \mathbb{R}^2 \backslash \overline{S}$,
we conclude: $\mathcal{E}'$ is inside $\mathbb{R}^2 \backslash \overline{S}$.
On the other hand, recall: $\mathcal{E}$ is inside S. **Let** $\mathcal{F} := \mathcal{E} \bigcup \mathcal{E}'$. **QED**

Proof. **Let** $\overline{\mathcal{E}} := \{ \overline{E} \,|\, E \in \mathcal{E} \}$. We have: $\forall B \in \mathcal{B}$, $B \equiv \overline{B}$.
So, since $\mathcal{E} \subseteq \mathcal{V} \subseteq \mathcal{B}$, we get: $\forall E \in \mathcal{E}$, $E \equiv \overline{E}$.
So, since $\mathcal{E}$ is finite, we get: $\bigcup \mathcal{E} \equiv \bigcup \overline{\mathcal{E}}$.
Let $S := \bigcup \mathcal{E}$. Since $\mathcal{E}$ is finite, we get: $\overline{S} = \bigcup \overline{\mathcal{E}}$. Then $S \equiv \overline{S}$.
Let $W := \mathbb{R}^2 \backslash \overline{S}$. Since $\overline{S}$ is closed in $\mathbb{R}^2$, we get: $W \in \mathcal{T}$.

Let $\mathcal{V}' := \{V \in \mathcal{V} \,|\, V \subseteq W\}$. Then: $\mathcal{V}' \subseteq \mathcal{V}$ and $\bigcup \mathcal{V}' \subseteq W$.
Also, by Theorem 11, $\mathcal{V}'$ is a fine-covering of $Q \bigcap W$.
Let $Q' := Q \bigcap W$. Then $\mathcal{V}'$ is a fine-covering of Q'.
Since $Q' = Q \bigcap W \subseteq Q$, by monotonicity of λ, we get: $\lambda(Q') \leqslant \lambda(Q)$.
Since $\lambda(Q') \leqslant \lambda(Q) < \infty$, by Theorem 13,
choose a finite pw-dj $\mathcal{E}' \subseteq \mathcal{V}'$ s.t. $\lambda(Q' \bigcap (\bigcup \mathcal{E}')) \geqslant 0.01 \cdot (\lambda(Q'))$.
Since $\mathcal{E}' \subseteq \mathcal{V}'$, we get: $\bigcup \mathcal{E}' \subseteq \bigcup \mathcal{V}'$. Recall: $\bigcup \mathcal{V}' \subseteq W$.
Since $\overline{S} \supseteq S$, we get: $\mathbb{R}^2 \backslash \overline{S} \subseteq \mathbb{R}^2 \backslash S$. Recall: $S = \bigcup \mathcal{E}$.
Since $\bigcup \mathcal{E}' \subseteq \bigcup \mathcal{V}' \subseteq W = \mathbb{R}^2 \backslash \overline{S} \subseteq \mathbb{R}^2 \backslash S = \mathbb{R}^2 \backslash (\bigcup \mathcal{E})$,
we get: $(\bigcup \mathcal{E}) \bigcap (\bigcup \mathcal{E}') = \varnothing$.
Then: $\forall E \in \mathcal{E}$, $\forall E' \in \mathcal{E}'$, $E \bigcap E' = \varnothing$.
So, since $\mathcal{E}$ and $\mathcal{E}'$ are both pw-dj, we get: $\mathcal{E} \bigcup \mathcal{E}'$ is pw-dj.
Since $\mathcal{E}$ and $\mathcal{E}'$ are both finite, we conclude: $\mathcal{E} \bigcup \mathcal{E}'$ is finite.
By hypothesis, $\mathcal{E} \subseteq \mathcal{V}$, so, since $\mathcal{E}' \subseteq \mathcal{V}' \subseteq \mathcal{V}$, we get: $\mathcal{E} \bigcup \mathcal{E}' \subseteq \mathcal{V}$.
Let $\mathcal{F} := \mathcal{E} \bigcup \mathcal{E}'$. Then $\mathcal{F}$ is finite and pw-dj. Also, $\mathcal{F} \subseteq \mathcal{V}$.
Since $\mathcal{E} \subseteq \mathcal{E} \bigcup \mathcal{E}' = \mathcal{F}$,
it remains only to show: $\lambda(Q \backslash (\bigcup \mathcal{F})) \leqslant 0.99 \cdot (\lambda(Q \backslash (\bigcup \mathcal{E})))$.
Recall: $S = \bigcup \mathcal{E}$. **Let** $S' := \bigcup \mathcal{E}'$.
Then, since $\bigcup \mathcal{F} = \bigcup (\mathcal{E} \bigcup \mathcal{E}') = (\bigcup \mathcal{E}) \bigcup (\bigcup \mathcal{E}') = S \bigcup S'$,
we want to show: $\lambda(Q \backslash (S \bigcup S')) \leqslant 0.99 \cdot (\lambda(Q \backslash S))$.
By hypothesis, $\mathcal{V} \subseteq \mathcal{B}$. So, since $\mathcal{E}' \subseteq \mathcal{V}' \subseteq \mathcal{V}$, we get: $\mathcal{E}' \subseteq \mathcal{B}$.
Since $\mathcal{E}' \subseteq \mathcal{B} \subseteq \mathcal{T}$ and since $\mathcal{T}$ is a topology, we get: $\bigcup \mathcal{E}' \in \mathcal{T}$.
So, since $S' = \bigcup \mathcal{E}'$, we get: $S' \in \mathcal{T}$.
Then S' is Lebesgue-measurable, so, by the Carathéodory-condition,
we get: $\lambda(Q') = [\,\lambda(Q' \backslash S')\,] + [\,\lambda(Q' \bigcap S')\,]$.
Let $c := \lambda(Q')$, $a := \lambda(Q' \backslash S')$, $b := \lambda(Q' \bigcap S')$.
Then: $c = a + b$.
By choice of $\mathcal{E}'$, we have: $\lambda(Q' \bigcap (\bigcup \mathcal{E}')) \geqslant 0.01 \cdot (\lambda(Q'))$.
Then: $\lambda(Q' \bigcap S') \geqslant 0.01 \cdot (\lambda(Q'))$.
Then: $b \geqslant 0.01 \cdot c$.
Recall: $S \equiv \overline{S}$.
Then: $Q \backslash S \equiv Q \backslash \overline{S}$. Recall: $W = \mathbb{R}^2 \backslash \overline{S}$ and $Q' = Q \bigcap W$.
Since $Q \backslash S \equiv Q \backslash \overline{S} = Q \bigcap (\mathbb{R}^2 \backslash \overline{S}) = Q \bigcap W = Q'$, we get:
both $(Q \backslash S) \backslash S' \equiv Q' \backslash S'$ and $\lambda(Q \backslash S) = \lambda(Q')$.
Since $Q \backslash (S \bigcup S') = (Q \backslash S) \backslash S' \equiv Q' \backslash S'$, we get:
$\lambda(Q \backslash (S \bigcup S')) = \lambda(Q' \backslash S')$.
Since $\lambda(Q \backslash (S \bigcup S')) = \lambda(Q' \backslash S') = a$ and since $\lambda(Q \backslash S) = \lambda(Q') = c$,
we want to show: $a \leqslant 0.99 \cdot c$.

Recall: $c = a + b$ and $b \geqslant 0.01 \cdot c$.
Since $c = a + b \geqslant a + 0.01 \cdot c$,
we get: $0.99 \cdot c \geqslant a$. Then: $a \leqslant 0.99 \cdot c$. □

Let $A, B \subseteq \mathbb{R}^2$.
By B is an **a.e.-superset of A**, we will mean: $\lambda(A \backslash B) = 0$.

Let $A, B \subseteq \mathbb{R}^2$, $\varepsilon > 0$.
By B is an **ε-efficient-superset of A**, we will mean:
$A \subseteq B$ and $\lambda(B) \leqslant e^{\varepsilon} \cdot (\lambda(A))$.
By B is an **ε-efficient-a.e.-superset of A**, we will mean:
$\lambda(A \backslash B) = 0$ and $\lambda(B) \leqslant e^{\varepsilon} \cdot (\lambda(A))$.

Let $\mathcal{B}$ be a set of subsets of $\mathbb{R}^2$, $A \subseteq \mathbb{R}^2$.
By $\mathcal{B}$ is an **a.e.-covering of A**, we will mean:
$\bigcup \mathcal{B}$ is an a.e.-superset of A.

Let $\mathcal{B}$ be a set of subsets of $\mathbb{R}^2$, $A \subseteq \mathbb{R}^2$, $\varepsilon > 0$.
By $\mathcal{B}$ is an **ε-efficient-covering of A**, we will mean:
$\bigcup \mathcal{B}$ is an ε-efficient-superset of A.
By $\mathcal{B}$ is an **ε-efficient-a.e.-covering of A**, we will mean:
$\bigcup \mathcal{B}$ is an ε-efficient-a.e.-superset of A.

DEFINITION 15. Let $S \subseteq \mathbb{R}^2$. *By S is* **Vitali**, *we mean:*
$\forall \mathcal{V} \subseteq \mathcal{B}$, if $\mathcal{V}$ is a fine-covering of S,
then $\exists$countable pw-dj $\mathcal{D} \subseteq \mathcal{V}$ s.t. $\lambda(S \backslash (\bigcup \mathcal{D})) = 0$.

So, a Vitali set is one for which
any fine-covering admits a countable pw-dj a.e.-subcovering.
In Theorem 17, below, we will show: *any* subset of $\mathbb{R}^2$ is Vitali.

By an **a.e.-partition** of a set $S \subseteq \mathbb{R}^2$, we will mean:
a pw-dj set of subsets of S that is an a.e.-covering of S.

According to the next theorem, for any $S \subseteq \mathbb{R}^2$,
for any countable a.e.-partition of S into relatively-open subsets,
if each subset is Vitali, then S is Vitali.

THEOREM 16. Let $S \subseteq \mathbb{R}^2$, $W_1, W_2, \ldots \in \mathcal{T}$.
Assume: ($(W_1, W_2, \ldots)$ *is pw-dj*) & ($\lambda(S \backslash (W_1 \bigcup W_2 \bigcup \cdots)) = 0$).
Assume: $\forall n \in \mathbb{N}$, $S \bigcap W_n$ *is Vitali.* *Then:* S *is Vitali.*

WARNING: In the following proof, $\forall n \in \mathbb{N}$, $\bigcup \mathcal{D}_n = \bigcup_{D \in \mathcal{D}_n} D$.

By contrast, $\bigcup_{n=1}^{\infty} \mathcal{D}_n = \mathcal{D}_1 \bigcup \mathcal{D}_2 \bigcup \cdots$.

Care must be taken not to confuse $\bigcup \mathcal{D}_n$ with $\bigcup_{n=1}^{\infty} \mathcal{D}_n$.

Proof. **Given** $\mathcal{V} \subseteq \mathcal{B}$, assume $\mathcal{V}$ is a fine-covering of S,
want: $\exists$countable pw-dj $\mathcal{D} \subseteq \mathcal{V}$ s.t. $\lambda(S \backslash (\bigcup \mathcal{D})) = 0$.
For all $n \in \mathbb{N}$, **let** $\mathcal{V}_n := \{V \in \mathcal{V} \,|\, V \subseteq W_n\}$. Then: $\forall n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{V}$.
Also, by Theorem 11, $\forall n \in \mathbb{N}$, $\mathcal{V}_n$ is a fine-covering of $S \bigcap W_n$.
For all $n \in \mathbb{N}$, **let** $Q_n := S \bigcap W_n$.
Then: $\forall n \in \mathbb{N}$, $\mathcal{V}_n$ is a fine-covering of Q_n.
By hypothesis, we have: $\forall n \in \mathbb{N}$, Q_n is Vitali.
Then, $\forall n \in \mathbb{N}$, **choose** a countable pw-dj $\mathcal{D}_n \subseteq \mathcal{V}_n$
s.t. $\lambda(Q_n \backslash (\bigcup \mathcal{D}_n)) = 0$.
Let $\mathcal{D} := \mathcal{D}_1 \bigcup \mathcal{D}_2 \bigcup \cdots$.
Since, $\forall n \in \mathbb{N}$, $\mathcal{D}_n$ is countable, we get: $\mathcal{D}$ is countable.
Since, $\forall n \in \mathbb{N}$, $\mathcal{D}_n \subseteq \mathcal{V}_n \subseteq \mathcal{V}$, we get: $\mathcal{D} \subseteq \mathcal{V}$.
It remains to show: (1) $\mathcal{D}$ is pw-dj and (2) $\lambda(S \backslash (\bigcup \mathcal{D})) = 0$.

Proof of (1): **Given** $A, B \in \mathcal{D}$, assume $A \neq B$, **want:** $A \bigcap B = \varnothing$.
Since $A \in \mathcal{D} = \mathcal{D}_1 \bigcup \mathcal{D}_2 \bigcup \cdots$, **choose** $a \in \mathbb{N}$ s.t. $A \in \mathcal{D}_a$.
Since $B \in \mathcal{D} = \mathcal{D}_1 \bigcup \mathcal{D}_2 \bigcup \cdots$, **choose** $b \in \mathbb{N}$ s.t. $B \in \mathcal{D}_b$.
In case $a = b$, we have $A, B \in \mathcal{D}_a$, and so,
since $\mathcal{D}_a$ is pw-dj and since $A \neq B$, we get: $A \bigcap B = \varnothing$.
We therefore assume that $a \neq b$.
By hypothesis, $(W_1, W_2, \ldots)$ is pw-dj. Then: $W_a \bigcap W_b = \varnothing$.
Since $A \in \mathcal{D}_a \subseteq \mathcal{V}_a$, by definition of $\mathcal{V}_a$, we get: $A \subseteq W_a$.
Since $B \in \mathcal{D}_b \subseteq \mathcal{V}_b$, by definition of $\mathcal{V}_b$, we get: $B \subseteq W_b$.
Then $A \bigcap B \subseteq W_a \bigcap W_b = \varnothing$, so $A \bigcap B = \varnothing$.
End of proof of (1).

Proof of (2): **Let** $D := \bigcup \mathcal{D}$. **Want:** $\lambda(S \backslash D) = 0$.
Let $Q := Q_1 \bigcup Q_2 \bigcup \cdots$.

For all sets X, Y, Z, we have: $X\backslash Z \subseteq (X\backslash Y) \bigcup (Y\backslash Z)$.
Therefore, $S\backslash D \subseteq (S\backslash Q) \bigcup (Q\backslash D)$.
It therefore suffices to show: $\lambda(S\backslash Q) = 0 = \lambda(Q\backslash D)$.
By hypothesis, we have: $\lambda(S\backslash(W_1 \bigcup W_2 \bigcup \cdots)) = 0$.
Let $W := W_1 \bigcup W_2 \bigcup \cdots$. Then: $\lambda(S\backslash \quad W \quad) = 0$.
For all $n \in \mathbb{N}$, by definition of Q_n, we have: $S \bigcap W_n = Q_n$.
Since $S \bigcap W = (S \bigcap W_1) \bigcup (S \bigcap W_2) \bigcup \cdots = Q_1 \bigcup Q_2 \bigcup \cdots = Q$,
we get: $S\backslash(S \bigcap W) = S\backslash Q$.
For any sets X, Y, by definition of set-subtraction, we have:
$X\backslash Y = X\backslash(X \bigcap Y)$.
Since $S\backslash W = S\backslash(S \bigcap W) = S\backslash Q$, we get: $\lambda(S\backslash W) = \lambda(S\backslash Q)$.
Since $\lambda(S\backslash Q) = \lambda(S\backslash W) = 0$,
it remains only to show: $\lambda(Q\backslash D) = 0$.
Since $Q = Q_1 \bigcup Q_2 \bigcup \cdots$, we get: $Q\backslash D = (Q_1\backslash D) \bigcup (Q_2\backslash D) \bigcup \cdots$.
It therefore suffices to show: $\forall n \in \mathbb{N}$, $\lambda(Q_n\backslash D) = 0$.
Given $n \in \mathbb{N}$, **let** $P := Q_n$, **Want:** $\lambda(P \backslash D) = 0$.
By choice of $\mathcal{D}_n$, we have: $\lambda(Q_n\backslash(\bigcup \mathcal{D}_n)) = 0$.
Let $\mathcal{C} := \mathcal{D}_n$. Then: $\lambda(P \backslash(\bigcup \mathcal{C})) = 0$.
Since $\mathcal{D} = \mathcal{D}_1 \bigcup \mathcal{D}_2 \bigcup \cdots \supseteq \mathcal{D}_n = \mathcal{C}$, we get: $\bigcup \mathcal{D} \supseteq \bigcup \mathcal{C}$.
Since $D = \bigcup \mathcal{D} \supseteq \bigcup \mathcal{C}$, we get: $P\backslash D \subseteq P\backslash(\bigcup \mathcal{C})$.
So, since $\lambda(P\backslash(\bigcup \mathcal{C})) = 0$, we get: $\lambda(P\backslash D) = 0$.
End of proof of (2). □

THEOREM 17. Let $S \subseteq \mathbb{R}^2$. *Then: S is Vitali.*

Idea of Proof: Intersecting S with each set of
an a.e.-partition of $\mathbb{R}^2$ by open bounded subsets,
we get an a.e.-partition of S into relatively-open bounded subsets.
By Theorem 16, it suffices to show each realtively-open subset is Vitali.
Given one of these subsets, Q, and a fine-covering of Q,
we seek a countable pw-dj a.e.-subcovering of Q.
Since Q is bounded, we get: $\lambda(Q) < \infty$.
Starting with the empty set (which covers none of Q),
we use Theorem 14 repeatedly to find an increasing sequence of
finite pw-dj coverings of more and more of Q.
Taking the union of these countably-many finite partial coverings,
we arrive at a countable pw-dj a.e.-covering of Q. **QED**

Proof. **Let** $z := (0, 0)$. For all $j \in \mathbb{N}$, **let** $B_j := B_z^j$ and $D_j := \overline{B_j}$.
Let $D_0 := \varnothing$. For all $j \in \mathbb{N}$, **let** $W_j := B_j\backslash D_{j-1}$.

Then: $W_1, W_2 \cdots \in \mathcal{T}$. Also, $(W_1, W_2, \ldots)$ is pw-dj.
We have: $\forall j \in \mathbb{N}$, $\lambda(B_j) = \pi j^2 = \lambda(D_j)$.
It follows that: $\forall j \in \mathbb{N}$, $\lambda(D_j \backslash B_j) = 0$.
So, since $\mathbb{R}^2 \backslash (W_1 \bigcup W_2 \bigcup \cdots) \subseteq (D_1 \backslash B_1) \bigcup (D_2 \backslash B_2) \bigcup \cdots$,
we get: $\lambda(\ \mathbb{R}^2 \backslash (W_1 \bigcup W_2 \bigcup \cdots)\ ) = 0$.
So, since $\mathbb{R}^2 \backslash (W_1 \bigcup W_2 \bigcup \cdots) \supseteq S \backslash (W_1 \bigcup W_2 \bigcup \cdots)$
we get: $\lambda(\ S \backslash (W_1 \bigcup W_2 \bigcup \cdots)\ ) = 0$.
By Theorem 16, **it suffices to show:** $\forall n \in \mathbb{N}$, $S \bigcap W_n$ is Vitali.
Given $n \in \mathbb{N}$, **let** $Q := S \bigcap W_n$, **want:** Q is Vitali.
Given $\mathcal{V} \subseteq \mathcal{B}$, assume $\mathcal{V}$ is a fine-covering of Q,
want: $\exists$countable pw-dj $\mathcal{D} \subseteq \mathcal{V}$ s.t. $\lambda(Q \backslash (\bigcup \mathcal{D})) = 0$.
Since $Q = S \bigcap W_n \subseteq W_n = B_n \backslash D_{n-1} \subseteq B_n$
and since $\lambda(B_n) = \pi n^2 < \infty$,
by monotonicity of λ, we conclude: $\lambda(\ Q\ ) < \infty$.
Let $\mathcal{E}_0 := \varnothing$. Then $\mathcal{E}_0 \subseteq \mathcal{V}$ and $\mathcal{E}_0$ is finite and pw-dj.
By applying Theorem 14 repeatedly, **choose** $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots \subseteq \mathcal{V}$
s.t. $\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots$ and
s.t. $\forall j \in \mathbb{N}$, $\mathcal{E}_j$ is finite and pw-dj and
s.t. $\forall j \in \mathbb{N}$, $\lambda(Q \backslash (\bigcup \mathcal{E}_j)) \leqslant 0.99 \cdot (\lambda(Q \backslash (\bigcup \mathcal{E}_{j-1})))$.
Let $\mathcal{D} := \mathcal{E}_1 \bigcup \mathcal{E}_2 \bigcup \cdots$. Then $\mathcal{D} \subseteq \mathcal{V}$ and $\mathcal{D}$ is countable.
It remains to show: (1) $\mathcal{D}$ is pw-dj and (2) $\lambda(Q \backslash (\bigcup \mathcal{D})) = 0$.

Proof of (1): **Given** $E, F \in \mathcal{D}$, assume $E \neq F$, **want:** $E \bigcap F = \varnothing$.
Since $E \in \mathcal{D} = \mathcal{E}_1 \bigcup \mathcal{E}_2 \bigcup \cdots$, **choose** $p \in \mathbb{N}$ s.t. $E \in \mathcal{E}_p$.
Since $F \in \mathcal{D} = \mathcal{E}_1 \bigcup \mathcal{E}_2 \bigcup \cdots$, **choose** $q \in \mathbb{N}$ s.t. $F \in \mathcal{E}_q$.
Let $r := \max\{p, q\}$. Recall: $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots$. Then $E, F \in \mathcal{E}_r$.
So, since $\mathcal{E}_r$ is pw-dj and since $E \neq F$, we get: $E \bigcap F = \varnothing$.
End of proof of of (1).

Proof of (2): Recall: $\lambda(Q) < \infty$. **Let** $m := \lambda(Q)$.
Then: $0 \leqslant m < \infty$. Then: as $k \to \infty$, $(0.99)^k \cdot m \to 0$.
It therefore suffices to show: $\forall k \in \mathbb{N}$, $\lambda(Q \backslash (\bigcup \mathcal{D})) \leqslant (0.99)^k \cdot m$.
Given $k \in \mathbb{N}$, **let** $s := (0.99)^k$, **want:** $\lambda(Q \backslash (\bigcup \mathcal{D})) \leqslant s \cdot m$.
Since $\mathcal{E}_0 = \varnothing$, we get $\bigcup \mathcal{E}_0 = \varnothing$, so $Q \backslash (\bigcup \mathcal{E}_0) = Q$.
Since $\mathcal{D} = \mathcal{E}_1 \bigcup \mathcal{E}_2 \bigcup \cdots \supseteq \mathcal{E}_k$, we get:

$$\bigcup \mathcal{D} \supseteq \bigcup \mathcal{E}_k.$$

Then: $Q \backslash (\bigcup \mathcal{D}) \subseteq Q \backslash (\bigcup \mathcal{E}_k)$.
So, by monotonicity of λ, we get:

$$\lambda(Q\backslash(\bigcup\mathcal{D})) \leqslant \lambda(Q\backslash(\bigcup\mathcal{E}_k)).$$

Then:
$$\begin{aligned}\lambda(Q\backslash(\bigcup\mathcal{D})) \leqslant \lambda(Q\backslash(\bigcup\mathcal{E}_k)) &\leqslant (0.99)\ \cdot (\lambda(Q\backslash(\bigcup\mathcal{E}_{k-1}))) \\ &\leqslant (0.99)^2 \cdot (\lambda(Q\backslash(\bigcup\mathcal{E}_{k-2}))) \\ &\leqslant \cdots \\ &\leqslant (0.99)^k \cdot (\lambda(Q\backslash(\bigcup\mathcal{E}_0))) \\ &= \quad s \quad \cdot (\lambda(Q)) \ = \ s\cdot m.\end{aligned}$$

End of proof of (2). □

We make the convention that, $\forall c > 0, \ \ c\cdot\infty = \infty$.
Then: $\forall Q \subseteq \mathbb{R}^2, \forall\varepsilon\in\mathbb{R}, \ \ (\,\lambda(Q) = \infty\,) \Rightarrow (\,\lambda(\mathbb{R}^2) \leqslant e^\varepsilon\cdot(\lambda(Q))\,).$
So, using outer-regularity of λ, we can prove:

Let $Q \subseteq \mathbb{R}^2$, $\varepsilon > 0$. Assume: $\lambda(Q) > 0$.
Then: $\exists W \in \mathcal{T}$ s.t. W is an ε-efficient-superset of Q.
(NOTE: In case $\lambda(Q) = \infty$, **let** $W := \mathbb{R}^2$.)

According to the next theorem, for any $Q \subseteq \mathbb{R}^2$,
for any fine-covering of Q, for any $\varepsilon > 0$,
there is a countable pw-dj ε-efficient-a.e.-subcovering of Q.
The set Q need not be Lebesgue-measurable.

THEOREM 18. Let $Q \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$, $\varepsilon > 0$.
Assume: $\mathcal{V}$ *is a fine-covering of* Q.
Then: $\exists$*countable pw-dj* $\mathcal{C} \subseteq \mathcal{V}$ *s.t.*
$$(\ \lambda(Q\backslash(\bigcup\mathcal{C})) = 0\ ) \quad \& \quad (\ \lambda(\bigcup\mathcal{C}) \leqslant e^\varepsilon\cdot(\lambda(Q))\ ).$$

Proof. In case $\lambda(Q) = 0$, **let** $\mathcal{C} := \varnothing$. We therefore assume $\lambda(Q) > 0$.
By outer-regularity of λ, **choose** $W \in \mathcal{T}$ s.t.
both $W \supseteq Q$ and $\lambda(W) \leqslant e^\varepsilon\cdot(\lambda(Q))$.
Let $\mathcal{V}' := \{V \in \mathcal{V} \,|\, V \subseteq W\}$. Then: $\mathcal{V}' \subseteq \mathcal{V}$ and $\bigcup\mathcal{V}' \subseteq W$.
By Theorem 12, $\mathcal{V}'$ is a fine-covering of Q.
So, since, by Theorem 17, Q is Vitali,
choose a countable pw-dj $\mathcal{C} \subseteq \mathcal{V}'$ s.t. $\lambda(Q\backslash(\bigcup\mathcal{C})) = 0$.
Since $\mathcal{C} \subseteq \mathcal{V}' \subseteq \mathcal{V}$, **it remains only to show:** $\lambda(\bigcup\mathcal{C}) \leqslant e^\varepsilon\cdot(\lambda(Q))$.
Since $\mathcal{C} \subseteq \mathcal{V}'$, we get: $\bigcup\mathcal{C} \subseteq \bigcup\mathcal{V}'$.
Since $\bigcup\mathcal{C} \subseteq \bigcup\mathcal{V}' \subseteq W$, by monotonicity of λ, we get: $\lambda(\bigcup\mathcal{C}) \leqslant \lambda(W)$.
Then: $\lambda(\bigcup\mathcal{C}) \leqslant \lambda(W) \leqslant e^\varepsilon\cdot(\lambda(Q))$. □

DEFINITION 19. Let $Q \subseteq \mathbb{R}^2$, $\varepsilon > 0$.
Then: $\boxed{\mathcal{I}_Q^\varepsilon} := \{B \in \mathcal{B} \,|\, \lambda(B) > e^\varepsilon\cdot(\lambda(Q\bigcap B))\}.$

Then $\mathcal{I}_Q^\varepsilon$ is the set of all
disks B that are NOT ε-efficient in covering $Q \bigcap B$.
The letter "$\mathcal{I}$" stands for "inefficient".
By Theorem 18, every fine-covering has some ε-efficiency.
The next theorem is based on the contrapositive:
Since $\mathcal{I}_Q^\varepsilon$ has no ε-efficiency, it cannot be a fine-covering.

THEOREM 20. Let $Q \subseteq \mathbb{R}^2$, $\varepsilon > 0$. *Assume:* $\lambda(Q) > 0$.
Then: $\mathcal{I}_Q^\varepsilon$ *is not a fine-covering of* Q.

Idea of proof:
Assume, **for a contradiction**, that: $\mathcal{I}_Q^\varepsilon$ is a fine-covering of Q.
By Theorem 18, **choose**
a countable pw-dj ε-efficient-a.e.-subcovering, $\mathcal{C}$, of Q.
Since $\mathcal{C}$ is an a.e.-covering of Q, we get: $Q \bigcap (\bigcup \mathcal{C}) \equiv Q$.
Since $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon$, we get: each $C \in \mathcal{C}$ is ε-inefficient at covering $Q \bigcap C$.
Summing, we find that: $\mathcal{C}$ is ε-inefficient at covering $Q \bigcap (\bigcup \mathcal{C})$.
So, since $Q \bigcap (\bigcup \mathcal{C}) \equiv Q$, $\mathcal{C}$ is ε-inefficient at a.e.-covering Q.
This contradicts the choice of $\mathcal{C}$. **QED**

Proof. Assume $\mathcal{I}_Q^\varepsilon$ is a fine-covering of Q. **Want:** Contradiction.
By Theorem 18, **choose** a countable pw-dj $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon$ s.t.
$(\lambda(Q \backslash (\bigcup \mathcal{C})) = 0)$ & $(\lambda(\bigcup \mathcal{C}) \leqslant e^\varepsilon \cdot (\lambda(Q)))$.
Since $\lambda(Q \backslash (\bigcup \mathcal{C})) = 0 < \lambda(Q)$, we get: $Q \backslash (\bigcup \mathcal{C}) \neq Q$.
Then $\bigcup \mathcal{C} \neq \varnothing$. Then $\mathcal{C} \neq \varnothing$.
Since $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon \subseteq \mathcal{B} \subseteq \mathcal{T}$ and since $\mathcal{T}$ is a topology, we get: $\bigcup \mathcal{C} \in \mathcal{T}$.
Let $A := \bigcup \mathcal{C}$. Then $A \in \mathcal{T}$. Then A is Lebesgue-measurable.
So, by the Carathéodory-condition, we get:

$$\lambda(Q) = [\lambda(Q \bigcap A)] + [\lambda(Q \backslash A)].$$

So, since $\lambda(Q \backslash A) = \lambda(Q \backslash (\bigcup \mathcal{C})) = 0$,
we get: $\lambda(Q) = \lambda(Q \bigcap A)$.
Since $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon \subseteq \mathcal{B} \subseteq \mathcal{T}$, we conclude:

$$\forall C \in \mathcal{C}, \quad \mathcal{C} \text{ is Lebesgue-measurable.}$$

So, since $\mathcal{C}$ is countable and pw-dj,
by countable-additivity of λ, $\lambda(\bigcup \mathcal{C}) = \sum_{C \in \mathcal{C}} (\lambda(C))$.

Since $Q \bigcap A = Q \bigcap (\bigcup \mathcal{C}) = Q \bigcap (\bigcup_{C \in \mathcal{C}} C) = \bigcup_{C \in \mathcal{C}} (Q \bigcap C)$, by countable-subadditivity of λ, $\lambda(Q \bigcap A) \leqslant \sum_{C \in \mathcal{C}} (\lambda(Q \bigcap C))$.

So, since $\lambda(Q) = \lambda(Q \bigcap A)$, we get: $\lambda(Q) \leqslant \sum_{C \in \mathcal{C}} (\lambda(Q \bigcap C))$.

By choice of $\mathcal{C}$, $\lambda(\bigcup \mathcal{C}) \leqslant e^{\varepsilon} \cdot (\lambda(Q))$.

Since $\sum_{C \in \mathcal{C}} (\lambda(C)) = \lambda(\bigcup \mathcal{C}) \leqslant e^{\varepsilon} \cdot (\lambda(Q)) \leqslant e^{\varepsilon} \cdot \sum_{C \in \mathcal{C}} (\lambda(Q \bigcap C))$,

we get: $\sum_{C \in \mathcal{C}} (\lambda(C)) \leqslant e^{\varepsilon} \cdot \sum_{C \in \mathcal{C}} (\lambda(Q \bigcap C))$.

On the other hand, since $\mathcal{C} \subseteq \mathcal{I}_Q^{\varepsilon}$, by definition of $\mathcal{I}_Q^{\varepsilon}$, we get:

$\forall C \in \mathcal{C}$, $\lambda(C) > e^{\varepsilon} \cdot (\lambda(Q \bigcap C))$.

So, since $\mathcal{C} \neq \varnothing$, summing these inequalities gives:

$$\sum_{C \in \mathcal{C}} (\lambda(C)) > e^{\varepsilon} \cdot \sum_{C \in \mathcal{C}} (\lambda(Q \bigcap C)). \quad \text{Contradiction.} \quad \square$$

DEFINITION 21. *For every $X \subseteq \mathbb{R}^2$,* **we define:**

$$\boxed{\mathrm{DP}_X} := \left\{ x \in X \;\middle|\; \lim_{r \to 0^+} \frac{\lambda(X \bigcap B_x^r)}{\lambda(B_x^r)} = 1 \right\}.$$

Elements of DP_X are called "**X-density-points**".

According to the next theorem,
every subset of $\mathbb{R}^2$ is comprised a.e. of density-points.
The same result can be proved, similarly, in any Euclidean space.
Interestingly, the subset need not be Lebesgue-measurable.

THEOREM 22. Let $X \subseteq \mathbb{R}^2$. *Then:* $\lambda(X \backslash \mathrm{DP}_X) = 0$.

Sketch of proof:

For all $j \in \mathbb{N}$, **let** $S_j := \left\{ x \in X \;\middle|\; \liminf_{r \to 0^+} \frac{\lambda(X \bigcap B_x^r)}{\lambda(B_x^r)} \geqslant \frac{j}{j+1} \right\}$.

Then $\mathrm{DP}_X = S_1 \bigcap S_2 \bigcap \cdots$, so $X \backslash \mathrm{DP}_X = (X \backslash S_1) \bigcup (X \backslash S_2) \bigcup \cdots$.
It therefore suffices to show, **given** $j \in \mathbb{N}$, that $\lambda(X \backslash S_j) = 0$.
Let $Q := X \backslash S_j$ and assume, **for a contradiction**, that $\lambda(Q) > 0$.
Let $\varepsilon := \ln((j+1)/j)$. Then $e^{-\varepsilon} = j/(j+1)$ and $\varepsilon > 0$.
Since $Q \subseteq X$, by monotonicity of λ, we get:

$\forall x \in \mathbb{R}^2$, $\forall r > 0$, $\lambda(Q \bigcap B_x^r) \leqslant \lambda(X \bigcap B_x^r)$.

For all $x \in Q$, since $x \notin S_j$, we get: $\liminf_{r \to 0^+} \frac{\lambda(X \bigcap B_x^r)}{\lambda(B_x^r)} < \frac{j}{j+1}$.

For all $x \in Q$, we have

$$\liminf_{r \to 0^+} \frac{\lambda(Q \bigcap B_x^r)}{\lambda(B_x^r)} \leqslant \liminf_{r \to 0^+} \frac{\lambda(X \bigcap B_x^r)}{\lambda(B_x^r)} < \frac{j}{j+1} = e^{-\varepsilon},$$

so, for some sequence of positive reals $r_1, r_2, \ldots \to 0$, we have

$$\forall i \in \mathbb{N}, \quad \frac{\lambda(Q \bigcap B_x^{r_i})}{\lambda(B_x^{r_i})} < e^{-\varepsilon},$$

and so $\quad \forall i \in \mathbb{N}, \quad \lambda(B_x^{r_i}) > e^{\varepsilon} \cdot (\lambda(Q \bigcap B_x^{r_i})),$

and so $\quad \forall i \in \mathbb{N}, \quad B_x^{r_i} \in \mathcal{I}_Q^{\varepsilon}.$

Then $\mathcal{I}_Q^{\varepsilon}$ covers each point of Q by balls of arbitrarily small radii.
Then $\mathcal{I}_Q^{\varepsilon}$ is a fine-covering of Q, contradicting Theorem 20. **QED**

Proof. **We wish to show:** for λ-a.e. $x \in X$, $x \in \mathrm{DP}_X$.
Define $F : \mathbb{R}^2 \times (0; \infty) \to [0; 1]$ by:

$$\forall x \in \mathbb{R}^2, \quad \forall r > 0, \qquad F(x, r) = \frac{\lambda(X \bigcap B_x^r)}{\lambda(B_x^r)}.$$

We wish to show: for λ-a.e. $x \in X$, $\lim_{r \to 0^+} (F(x, r)) = 1$.
Define $\phi, \psi : X \to [0; 1]$ by: $\forall x \in X$,

$$\phi(x) = \liminf_{r \to 0^+} (F(x, r)) \quad \text{and} \quad \psi(x) = \limsup_{r \to 0^+} (F(x, r)).$$

We wish to show: for λ-a.e. $x \in X$, $\phi(x) = 1 = \psi(x)$.
We have: $\forall x \in X$, $\phi(x) \leqslant \psi(x) \leqslant 1$.
Therefore, **it suffices to show:** for λ-a.e. $x \in X$, $\phi(x) \geqslant 1$.
Let $P := \{x \in X \mid \phi(x) < 1\}$. **Want:** $\lambda(P) = 0$.
For all $j \in \mathbb{N}$, **let** $P_j := \{ x \in X \mid \phi(x) < j/(j+1) \}$.
Since $P = P_1 \bigcup P_2 \bigcup \cdots$, **it suffices to show:** $\forall j \in \mathbb{N}$, $\lambda(P_j) = 0$.
Given $j \in \mathbb{N}$, **let** $Q := P_j$, **want:** $\lambda(Q) = 0$.
Assume $\lambda(Q) > 0$, **want:** contradiction.
Let $\varepsilon := \ln((j+1)/j)$. Then: $e^{-\varepsilon} = j/(j+1)$.
So, since $Q = P_j = \{x \in X \mid \phi(x) < j/(j+1)\}$,
we get: $Q = \{x \in X \mid \phi(x) < e^{-\varepsilon}\}$. Note that $Q \subseteq X$.
Since $(j+1)/j > 1$ and since $\varepsilon = \ln((j+1)/j)$, we get: $\varepsilon > 0$.
So, by Theorem 20, $\mathcal{I}_Q^{\varepsilon}$ is not a fine-covering of Q. **Let** $\mathcal{W} := \mathcal{I}_Q^{\varepsilon}$.
Then $\mathcal{W}$ is not a fine-covering of Q, so **choose** $x \in Q$ and $\delta > 0$ s.t.

$$\forall W \in \mathcal{W}, \qquad (x \in W) \Rightarrow (\mathrm{rad}\, W \geqslant \delta).$$

Since $x \in Q$, we get: $\phi(x) < e^{-\varepsilon}$.
Since $\liminf_{r \to 0^+} (F(x, r)) = \phi(x) < e^{-\varepsilon}$,

choose $r \in (0;\delta)$ s.t. $F(x,r) < e^{-\varepsilon}$. **Let** $W := B_x^r$.
Since $r \in (0;\delta)$, we have $r > 0$, so: $\pi r^2 > 0$.
So, since $\lambda(W) = \lambda(B_x^r) = \pi r^2$, we get: $\lambda(W) > 0$.
Since $Q \subseteq X$, we get: $Q \bigcap W \subseteq X \bigcap W$.
So, by monotonicity of λ, we get: $\lambda(Q \bigcap W) \leqslant \lambda(X \bigcap W)$.
Since $\dfrac{\lambda(Q \bigcap W)}{\lambda(W)} \leqslant \dfrac{\lambda(X \bigcap W)}{\lambda(W)} = \dfrac{\lambda(X \bigcap B_x^r)}{\lambda(B_x^r)} = F(x,r) < e^{-\varepsilon}$,
we get $\lambda(Q \bigcap W) < e^{-\varepsilon} \cdot (\lambda(W))$,
so $e^{\varepsilon} \cdot (\lambda(Q \bigcap W)) < \lambda(W)$,
so $\lambda(W) > e^{\varepsilon} \cdot (\lambda(Q \bigcap W))$,
so, since $W = B_x^r \in \mathcal{B}$, by definition of $\mathcal{I}_Q^\varepsilon$, we conclude: $W \in \mathcal{I}_Q^\varepsilon$.
Since $W \in \mathcal{I}_Q^\varepsilon = \mathcal{W}$ and since $x \in B_x^r = W$, by choice of x and δ,
we get: $\operatorname{rad} W \geqslant \delta$.
On the other hand, since $\operatorname{rad} W = \operatorname{rad} B_x^r = r \in (0;\delta)$,
we get: $\operatorname{rad} W < \delta$. Contradiction. □

For any function f, **let** $\boxed{\mathbb{D}_f}$ denote the domain of f.
For any function f, for any set S, **we define:**

$$\boxed{f^*S} := \{x \in \mathbb{D}_f \mid f(x) \in S\}.$$

Note: $\forall$function f, $\forall$set S, we have: $f^*S \subseteq \mathbb{D}_f$.

DEFINITION 23. Let $X \subseteq \mathbb{R}^2$, **let** $f : X \to \mathbb{R}$ *and* **let** $x \in X$.
Then, for all $\varepsilon > 0$, *for all* $r > 0$, **we define:**

$$\boxed{A_x^r(f,\varepsilon)} := \{\, u \in X \bigcap B_x^r \text{ s.t. } |(f(u)) - (f(x))| < \varepsilon \,\}.$$

We say $\boxed{f \textbf{ is CiOP at } x}$ *if:* $\forall \varepsilon > 0$, $\displaystyle\lim_{r \to 0^+} \frac{\lambda(A_x^r(f,\varepsilon))}{\lambda(B_x^r)} = 1$.

Here, "CiOP" stands for: "continuous-in-outer-probability".
Every function, measurable or not, is CiOP a.e.:

THEOREM 24. Let $X \subseteq \mathbb{R}^2$, $f : X \to \mathbb{R}$.
Then: for λ*-a.e.* $x \in X$, f *is CiOP at* x.

Here, we assume that the domain of f is a subset of $\mathbb{R}^2$
and that the image of f is a subset of $\mathbb{R}$,
but the result could be proved for any two Euclidean spaces.
Interestingly, neither X nor f need be Lebesgue-measurable.

Proof. **Let** $Y_1, Y_2, \ldots$ be a countable base for the topology on $\mathbb{R}$.
For all $j \in \mathbb{N}$, **let** $X_j := f^*Y_j$.

By Theorem 22, we have: $\forall j \in \mathbb{N}, \quad \lambda(X_j \backslash \mathrm{DP}_{X_j}) = 0.$
For all $j \in \mathbb{N}$, **let** $D_j := \mathrm{DP}_{X_j}.$
Then: $\forall j \in \mathbb{N}, \quad \lambda(X_j \backslash \ D_j \ ) = 0.$
For all $j \in \mathbb{N}$, **let** $Z_j := X_j \backslash \ D_j.$
Then: $\forall j \in \mathbb{N}, \quad \lambda(Z_j) = 0.$
Let $Z := Z_1 \bigcup Z_2 \bigcup \cdots.$ Then: $\lambda(\ Z\ ) = 0.$
It therefore suffices to show: $\forall x \in X \backslash Z$, f is CiOP at x.

Given $x \in X \backslash Z$, **given** $\varepsilon > 0$, **want:** $\displaystyle\lim_{r \to 0^+} \frac{\lambda(A_x^r(f,\varepsilon))}{\lambda(B_x^r)} = 1.$

Let $y := f(x)$. We have: $y \in (y - \varepsilon; y + \varepsilon).$
So, since $Y_1, Y_2, \ldots$ is a base for the topology on $\mathbb{R}$,
choose $j \in \mathbb{N}$ s.t. $y \in Y_j \subseteq (y - \varepsilon; y + \varepsilon).$
Since $f(x) = y \in Y_j$, we get: $x \in f^* Y_j.$
Since $x \in X \backslash Z$, we get: $x \in X$ and $x \notin Z.$
Since $x \notin Z = Z_1 \bigcup Z_2 \bigcup \cdots \supseteq Z_j$, we get: $x \notin Z_j.$
So, since $x \in f^* Y_j = X_j$, we get: $x \in X_j \backslash Z_j.$
Since $D_j = \mathrm{DP}_{X_j} \subseteq X_j$ and $Z_j = X_j \backslash D_j$, we get: $X_j \backslash Z_j = D_j.$

Since $x \in X_j \backslash Z_j = D_j = \mathrm{DP}_{X_j}$, we get: $\displaystyle\lim_{r \to 0^+} \frac{\lambda(X_j \bigcap B_x^r)}{\lambda(B_x^r)} = 1.$

So, by the Squeeze Theorem, **it suffices to show:**

$$\forall r > 0, \qquad \frac{\lambda(X_j \bigcap B_x^r)}{\lambda(B_x^r)} \leqslant \frac{\lambda(A_x^r(f,\varepsilon))}{\lambda(B_x^r)} \leqslant 1.$$

Given $r > 0$, **want:** $\lambda(X_j \bigcap B_x^r) \leqslant \lambda(A_x^r(f,\varepsilon)) \leqslant \lambda(B_x^r).$
By monotonicity of λ,
it suffices to show: $X_j \bigcap B_x^r \subseteq A_x^r(f,\varepsilon) \subseteq B_x^r.$
By definition of $A_x^r(f,\varepsilon)$, $A_x^r(f,\varepsilon) \subseteq X \bigcap B_x^r.$
Then: $A_x^r(f,\varepsilon) \subseteq B_x^r.$
It remains to show: $X_j \bigcap B_x^r \subseteq A_x^r(f,\varepsilon).$
Given $u \in X_j \bigcap B_x^r$, **want:** $u \in A_x^r(f,\varepsilon).$
Since $u \in X_j \bigcap B_x^r$, we get: $u \in X_j$ and $u \in B_x^r.$
Since $u \in X_j = f^* Y_j \subseteq \mathbb{D}_f = X$ and $u \in B_x^r$, we get: $u \in X \bigcap B_x^r.$
So, by definition of $A_x^r(f,\varepsilon)$, **we want:** $|(f(u)) - (f(x))| < \varepsilon.$
Since $u \in X_j = f^* Y_j$, we get: $f(u) \in Y_j.$
By the choice of j, we have: $Y_j \subseteq (y - \varepsilon; y + \varepsilon).$
Since $f(u) \in Y_j \subseteq (y - \varepsilon; y + \varepsilon)$, we get: $|(f(u)) - \ y \ | < \varepsilon.$
By definition of y, $y = f(x)$. Then: $|(f(u)) - (f(x))| < \varepsilon.$ $\square$