Notes for Math 8301, Manifolds and Topology, Fall 2004
The Fundamental Group and Covering Spaces

Definition. $B_\delta(x) = (x - \delta, x + \delta) \subseteq \mathbb{R}$.

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at** x if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$, *i.e.*, $B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$,

Note: For $\phi : A \to B$ and $B_0 \subseteq B$, we define $\phi^{-1}(B_0) = \{a \in A \,|\, \phi(a) \in B_0\}$.

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if, for all $x \in \mathbb{R}$, we have: f is continuous at x.

Question: How do we define continuity for $f : \mathbb{R}^m \to \mathbb{R}^n$? How about $f : S^n \to \mathbb{R}^m$, where $S^n := \{x \in \mathbb{R}^{n+1} \,|\, d(x, 0) = 1\}$? How about $f : S^n \to S^m$? Let's try to formulate a general definition of continuity that will include all these definitions as special cases.

Definition. An N-**space** consists of
(1) a set X and
(2) a function $x \mapsto \mathcal{N}_x : X \to \{\text{subsets of } \{\text{subsets of } X\}\}$
such that
(A) for all $x \in X$, we have: $\mathcal{N}_x \neq \emptyset$; and
(B) for all $x \in X$, for all $N \in \mathcal{N}_x$, we have: $x \in N$.

Example. $X = \mathbb{R}$, $\mathcal{N}_x = \{B_\delta(x) \,|\, \delta > 0\}$.

Definition. For all $x \in X$, an **"N-neighborhood"** of x is an element of $\mathcal{N}_x$.

Definition. Let X and Y be N-spaces and let $f : X \to Y$ be a function. For a point $x \in X$, we say that $f : X \to Y$ is N-**continuous at** x if, for any N-neighborhood V of $f(x)$, there exists an N-neighborhood U of x such that $f(U) \subseteq V$, *i.e.*, $U \subseteq f^{-1}(V)$. We say that f is N-**continuous** if f is N-continuous at every point $x \in X$.

Definition. A category consists of
(1) a class $\mathcal{C}$ of **objects**;
(2) for all $C, C' \in \mathcal{C}$, a set $\mathrm{Hom}(C, C')$ of **morphisms** or **arrows** from C to C';
(3) for all $C, C', C'' \in \mathcal{C}$, a **composition** function

$$\mathrm{Hom}(C, C') \times \mathrm{Hom}(C', C'') \to \mathrm{Hom}(C, C''); \qquad \text{and}$$

(4) for all $C \in \mathcal{C}$, an **identity** arrow $\mathrm{id}_C \in \mathrm{Hom}(C, C)$
such that
(A) for all $C, C' \in \mathcal{C}$, for all $f \in \mathrm{Hom}(C, C')$, we have $f \circ \mathrm{id}_C = \mathrm{id}_{C'} \circ f = f$;
(B) for all $C, C', C'', C''' \in \mathcal{C}$, for all $f \in \mathrm{Hom}(C, C')$, for all $g \in \mathrm{Hom}(C', C'')$, for all $h \in \mathrm{Hom}(C'', C''')$, we have $(f \circ g) \circ h = f \circ (g \circ h)$;
and probably other properties.

EXERCISE 1A: Look up, and write out the defintion of a category.

Example. $\mathcal{NS} := \{N\text{-spaces}\}$. That is, $\mathcal{NS}$ is the category of N-spaces, together with N-continuous maps, together with the usual composition of maps, together with the usual identity maps.

EXERCISE 1B:

(1) For all $X \in \mathcal{NS}$, show that $x \mapsto x : X \to X$ is N-continuous.
(2) For all $X, Y, Z \in \mathcal{NS}$, for all N-continuous $f : X \to Y$, for all N-continuous $g : Y \to Z$, show that $g \circ f : X \to Z$ is N-continuous.

Let $\{\text{Sets}\}$ denote the category of sets (and functions). For any $X \in \mathcal{NS}$, let X_{set} be the underlying set of X. For $X, Y \in \mathcal{NS}$, for $f \in \text{Hom}(X, Y)$, let $f_{\text{set}} : X_{\text{set}} \to Y_{\text{set}}$ be the underlying set function of f. Then $X \mapsto X_{\text{set}} : \mathcal{NS} \to \{\text{Sets}\}$ is an example of a **functor**. (Really, we should say $X \mapsto X_{\text{set}}$ together with $f \mapsto f_{\text{set}}$ forms a functor.) Note that, for $X, Y, Z \in \mathcal{NS}$, for $f \in \text{Hom}(X, Y)$, for $g \in \text{Hom}(Y, Z)$, we have $(g \circ f)_{\text{set}} = g_{\text{set}} \circ f_{\text{set}}$. This is one of the basic properties of a functor.

EXERCISE 1C: Look up, and write down the definition of a functor from one category to another. Be sure to note that some functors are covariant, while others are contravariant.

Example. Let $\{\text{Rings}\}$ denote the category of (not necessarily commutative) rings with unit, together with unit-preserving ring homomorphisms. Let $\{\text{Groups}\}$ denote the category of groups, with group homomorphisms. Then $(R, \cdot, +) \mapsto (R, +)$ is a functor from $\{\text{Rings}\}$ to $\{\text{Groups}\}$ (and even takes values in the subcategory of Abelian groups). Then $(R, \cdot, +) \mapsto (\{\text{units in } R\}, \cdot)$ is another functor from $\{\text{Rings}\}$ to $\{\text{Groups}\}$.

Definition. Let $\mathcal{C}$ be any category. Then there is an identity functor $\text{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ defined by $\text{Id}_{\mathcal{C}}(C) = C$. (Note: This defines the functor on objects, and it is left to you you to guess what the functor does on arrows.) (Note: Well, okay, I'll tell you this time – it leaves each arrow fixed, *i.e.*, for all $f : C \to C'$ in $\mathcal{C}$, we define $\text{Id}_{\mathcal{C}}(f) = f$. In general, though, it is an expectation that you should be able to figure this kind of thing out, but always feel free to ask if it's unclear.)

Fix, until END OF DISCUSSION #1, some $X \in \mathcal{NS}$. Let's say we've identified some points of X that are "happy".

Let $0 \in X$. If we say **"points close to x are happy"** or **"points sufficiently close to x are happy"** or **"all points sufficiently close to x are happy"**, we mean that there is an N-neighborhood U of x such that, for all $u \in U$, we have: u is happy. Now suppose that y is sufficiently close to x that it is guaranteed to be happy. That is, suppose that $y \in U$. *Question:* Does it automatically follow that all points sufficiently close to y are guaranteed to be happy by being in U? In other words, can we conclude that there is a neighborhoos V of y such that $V \subseteq U$.

Answer: No. For example, let $X_{\text{set}} = \mathbb{R}$, and for all $x \in X$, let $\mathcal{N}_x := \{[x-\delta, x+\delta] \mid \delta > 0\}$. Then $X = (X_{\text{set}}, x \mapsto \mathcal{N}_x)$ is an N-space.

Suppose that $[-1, 1]$ is exactly the set of happy points. Let $x := 0, y := -1$. Then $U = [-1, 1]$ is an N-neighborhood of x and $y \in U$, but there is no N-neighborhood V of y such that $V \subseteq U$.

So even though y is happy, we have no "room to maneuver" because the kinds of N-neigborhoods in X are, somehow, very *unforgiving*, in the sense that, if you have a point that is in an N-neighborhood, and you then perturb it ever so slightly, you may find yourself outside the N-neighborhood.

Definition. We say that an N-space X is **forgiving** if, for all $x \in X$, for any N-neighborhood U of x, for any $y \in U$, there is an N-neighborhood V of y such that $V \subseteq U$.

END OF DISCUSSION #1

Fix, until END OF DISCUSSION #2, some $X \in \mathcal{NS}$. Let's say we've identified some points of X that are "happy" and some point that are "dull". So X breaks up into happy dull points, unhappy dull points, happy sharp points and unhappy sharp points.

Let $x \in X$ and assume that all points sufficiently close to x are happy. That is, there is an N-neighborhood U of x such that every element of U is happy. Assume, furthermore that all points sufficiently close to x are dull. That is, there is an N-neighborhood V of x such that every element of V is happy. Does it automatically follow that there is a neighborhood W of x such that $W \subseteq U \cup V$, which guarantees that all points sufficiently close to x are both happy and dull?

The answer is no. It's not an assigned problem, but you should think about constructing a counterexample.

Definition. An N-space X is **intersection-stable** if, for all $x \in X$, for any N-neighborhoods U and V of x, there is an N-neighborhood W of x such that $W \subseteq U \cap V$.

END OF DISCUSSION #2

Definition. An N-space is said to be **topological** if it is both forgiving and intersection-stable.

Definition. Let $\mathcal{TNS}$ denote the category of topological N-spaces (with N-continuous maps).

Definition. Let X be a set. A **topology on** X is a subset $\mathcal{O}$ of {subsets of X} such that

(A) $\emptyset, X \in \mathcal{O}$;
(B) for all $\mathcal{S} \subseteq \mathcal{O}$, we have $\cup\mathcal{S} \in \mathcal{O}$; and
(C) for all finite $\mathcal{F} \subseteq \mathcal{O}$, we have $\cap\mathcal{F} \in \mathcal{O}$.

Definition. A **topological space** consists of

(1) a set X; and
(2) a topology on X.

Definition. An **open subset** of X is an element of $\mathcal{O}$. A subset of X is said to be **open** or **open in** X if it is an open subset of X.

Definition. Let X and Y be topological spaces and let $f : X \to Y$ be a function defined on their underlying sets. We say that f is **continuous** if, for any open $V \subseteq Y$, we have that $f^{-1}(V)$ is open in X.

What motivates the definition of topological space and of this definition of continuity?

Definition. Let $\mathcal{TS}$ denote the category of topological spaces, with continuous maps.

Definition. Define a functor $\mathcal{F} : \mathcal{TNS} \to \mathcal{TS}$ by

$$\mathcal{F}((X, x \mapsto \mathcal{N}_x)) = (X, \{\cup\mathcal{S} \,|\, \mathcal{S} \subseteq \cup_{x\in X}\mathcal{N}_x\}).$$

Here, I leave it to you to guess what $\mathcal{F}$ does on arrows. Note that {open sets in $\mathcal{F}X$} is exactly the closure of {N-neighborhoods of points of X} under union.

Define a functor $\mathcal{G} : \mathcal{TS} \to \mathcal{TNS}$ by

$$\mathcal{G}((Y, \mathcal{O})) = (Y, y \mapsto \{U \in \mathcal{O} \,|\, y \in U\}).$$

EXERCISE 1D: Show, for all $X \in \mathcal{TNS}$, that $\mathcal{F}X \in \mathcal{TS}$. (You must show that the open subsets of $\mathcal{F}X$ are closed under finite intersection and arbitrary union, and that both $\emptyset$ and X are open in $\mathcal{F}X$.)

EXERCISE 1E: Show, for all $Y \in \mathcal{TS}$, that $\mathcal{G}Y \in \mathcal{TNS}$.

EXERCISE 1F: Show, for all $Y \in \mathcal{TS}$, that $\mathcal{FG}Y = Y$. (You must show that a subset of Y is open in $\mathcal{FG}Y$ iff it is open in Y.)

The preceding exercise asks you to prove that $\mathcal{FG} = \mathrm{Id}_{\mathcal{TS}}$.

We now show that $\mathcal{GF}X$ is not necessarily equal to X. That is, we show that $\mathcal{FG} \neq \mathrm{Id}_{\mathcal{TNS}}$:

Example. Let $X := (\mathbb{R}, x \mapsto \{(x-\delta, x+\delta) \,|\, \delta > 0\})$. Then the set of open subsets of $\mathcal{F}X$ is exactly the closure of $\{(x-\delta, x+\delta) \,|\, x \in \mathbb{R}, \delta > 0\}$ under union. In particular, $(-1,1) \cup (2,4)$ is open in $\mathcal{F}X$. Then $(-1,1) \cup (2,4)$ is an N-neighborhood of 0 in $\mathcal{GF}X$. However it is *not* an N-neighborhood of 0 in X.

Therefore $\mathcal{GF}X \neq X$.

Definition. An arrow (or morphism) $f : C \to C'$ in a category $\mathcal{C}$ is an **isomorphism** if there exists an arrow $g : C' \to C$ such that $f \circ g = \mathrm{id}_{C'}$ and such that $g \circ f = \mathrm{id}_C$.

That is, an isomorphism is simply an invertible arrow.

Definition. In the category $\mathcal{TS}$, an isomorphism will be called a **homeomorphism**. In the category $\mathcal{TNS}$, an isomorphism will be called an N-**homeomorphism**.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A **natural transformation** $\tau : \mathcal{A} \to \mathcal{B}$ associates to each object $C \in \mathcal{C}$ an arrow $\tau_C : \mathcal{A}C \to \mathcal{B}C$, provided that this association has the property that: for any arrow $f : C \to C'$ in $\mathcal{C}$, we have $(\mathcal{B}f) \circ \tau_C = \tau_{C'} \circ (\mathcal{A}f)$.

Note that, for their to be a natural transformation from $\mathcal{A}$ to $\mathcal{B}$, the domains of $\mathcal{A}$ and $\mathcal{B}$ must agree. Moreover, their targets must agree as well.

Let $\mathcal{A}$ and $\mathcal{B}$ be functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. It may be helpful to picture $\mathcal{AC}$ as a subcategory inside of $\mathcal{D}$. In some sense, $\mathcal{A}$ is a "parametric" subcategory of $\mathcal{D}$, where the objects are parametrized by objects in $\mathcal{C}$. Similarly, $\mathcal{BC}$ is a parametric subcategory of $\mathcal{D}$. With this intution, a transformation $\tau : \mathcal{A} \to \mathcal{B}$ may be thought of as a parametric family of arrows in $\mathcal{D}$, one for each object in $\mathcal{C}$. For each $C \in \mathcal{C}$, the corresponding arrow runs from $\mathcal{A}C$ to $\mathcal{B}C$.

Moreover, for each arrow in $\mathcal{C}$, we now get a diagram in $\mathcal{D}$ (with four objects and four arrows). If these diagrams all commute, then we say that the transformation is "natural".

Example. Let $\mathcal{V}$ be the category of finite dimensional vector spaces, and linear transformations. Let $\mathcal{I} : \mathcal{V} \to \mathcal{V}$ be the (covariant) identity functor. Let $\mathcal{D} : \mathcal{V} \to \mathcal{V}$ be the (contravariant) functor defined by $\mathcal{D}(V) = V^* := \mathrm{Hom}(V, \mathbb{R})$. Let $\mathcal{DD} : \mathcal{V} \to \mathcal{V}$ be the (covariant) functor $\mathcal{DD} := \mathcal{D} \circ \mathcal{D}$ obtained by composing $\mathcal{D}$ with itself. Then, for all $V \in \mathcal{V}$, we have $\mathcal{DD}(V) = V^{**}$. For all $V \in \mathcal{V}$, let $\tau_V : V \to V^{**}$ be defined by $(\tau_V v)(l) = l(v)$. (Note: The map $\tau_V v : V^* \to \mathbb{R}$ is sometimes called the "evaluational at v".) Then $\tau : \mathcal{I} \to \mathcal{DD}$ is a natural tranformation, as Exercise 2B will verify. By contrast, the next exercise shows that there is no natural transformation from $\mathcal{I}$ to $\mathcal{D}$, even though, for all $V \in \mathcal{V}$, the vector spaces V and V^* are isomorphic, since they have the same dimension.

EXERCISE 2A: Let $V := \mathbb{R}^2$ and let $f : V \to V^*$ be an isomorphism. For any linear map $g : V \to V$, let $g^* : V^* \to V^*$ be defined by $g^*(l) = l \circ g$. Show that there exists an isomorphism $g : V \to V$ such that $f \neq g^* \circ f \circ g$.

EXERCISE 2B: As in the example above, for any real vector space V, let $\tau_V : V \to V^{**}$ be defined by $(\tau_V v)(l) = l(v)$. Let W and X be real vector spaces and let $g : W \to X$ be a linear transformation. Define $g^* : X^* \to W^*$ by $g^*(l) = l \circ g$ and define $g^{**} : W^{**} \to X^{**}$ by $g^{**}(l) = l \circ g^*$. Show that $g^{**} \circ \tau_W = \tau_X \circ g$.

Definition. Let $\mathcal{C}$ be a category. Then the **arrow category of** $\mathcal{C}$, $\mathrm{Arr}(\mathcal{C})$, is a category whose objects are the arrows in $\mathcal{C}$, and for which, for all pairs of arrows $f : C_1 \to C_2$, $f' : C_1' \to C_2'$ in $\mathcal{C}$, we define $\mathrm{Hom}(f, f')$ to be the collection of all those

$$(g_1, g_2) \in \mathrm{Hom}(C_1, C_1') \times \mathrm{Hom}(C_2, \times C_2')$$

such that $g_2 \circ f = f' \circ g_1$.

We leave it as an exercise to guess what the definitions of compositions and identity arrows in $\mathrm{Arr}(\mathcal{C})$.

Definition. Given a category $\mathcal{C}$, we define two functors $\mathrm{Dom}_{\mathcal{C}}, \mathrm{Tar}_{\mathcal{C}} : \mathrm{Arr}(\mathcal{C}) \to \mathcal{C}$ by: for all $f : D \to T$ in $\mathcal{C}$, we set $\mathrm{Dom}_{\mathcal{C}}(f) = D$ and $\mathrm{Tar}_{\mathcal{C}}(f) = T$.

With this terminology in place, if $\mathcal{A}, \mathcal{B} : \mathcal{C} \to \mathcal{D}$ are two functors, then a natural transformation $\mu : \mathcal{A} \to \mathcal{B}$ is equivalent to a functor $\mathcal{M} : \mathcal{C} \to \mathrm{Arr}(\mathcal{D})$ satisfying both $\mathrm{Dom}_{\mathcal{D}} \circ \mathcal{M} = \mathcal{A}$ and $\mathrm{Tar}_{\mathcal{D}} \circ \mathcal{M} = \mathcal{B}$.

Now recall the functors $\mathcal{F} : \mathcal{TNS} \to \mathcal{TS}$ and $\mathcal{G} : \mathcal{TS} \to \mathcal{TNS}$ defined above. Recall that $\mathcal{FG}$ is the identity functor on $\mathcal{TS}$, but that $\mathcal{GF}$ is not the identity functor on $\mathcal{TNS}$. We will make clear next a sense in which $\mathcal{FG}$ is "close" to the identity on $\mathcal{TNS}$.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $\mathcal{A} : \mathcal{C} \to \mathcal{D}$ be a functor. Define a natural transformation $\iota^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ by $\iota^{\mathcal{A}}_C := \mathrm{id}_{\mathcal{A}C} : \mathcal{A}C \to \mathcal{A}C$. This natural transformation is called **the identity on** $\mathcal{A}$.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathcal{P}, \mathcal{Q}, \mathcal{R} : \mathcal{C} \to \mathcal{D}$ be functors and let $\alpha : \mathcal{P} \to \mathcal{Q}$ and $\beta : \mathcal{Q} \to \mathcal{R}$ be natural transformations. We define a natural transformation $\beta\alpha : \mathcal{P} \to \mathcal{R}$ by $(\beta\alpha)_C = \beta_C \circ \alpha_C$.

Definition. We say that two functors $\mathcal{A}, \mathcal{B} : \mathcal{C} \to \mathcal{D}$ are **equivalent** (and write $\mathcal{A} \sim \mathcal{B}$) if there exist natural transformations $\tau : \mathcal{A} \to \mathcal{B}$ and $\mu : \mathcal{B} \to \mathcal{A}$ such that $\mu\tau = \iota^{\mathcal{A}}$ and $\tau\mu = \iota^{\mathcal{B}}$.

Intuitively, if two functors are equivalent, then they have the same "information" in them. If I'm an expert on one of the functors and you're an expert on the other, then we'll probably spend our days quibbling over notation, but will be well aware that we both know the same stuff.

EXERCISE 2C: Let $\mathcal{A}, \mathcal{B} : \mathcal{C} \to \mathcal{D}$ be functors. Let $\tau : \mathcal{A} \to \mathcal{B}$ be a natural transformation. Suppose, for all $C \in \mathcal{C}$, that $\tau_C : \mathcal{A}C \to \mathcal{B}C$ is an isomorphism. Show that there is a unique natural transformation $\mu : \mathcal{B} \to \mathcal{A}$ such that $\mu\tau = \iota^{\mathcal{A}}$ and $\tau\mu = \iota^{\mathcal{B}}$.

EXERCISE 2D: Let $\mathcal{A}, \mathcal{B} : \mathcal{C} \to \mathcal{D}$ be functors. Let $\tau : \mathcal{A} \to \mathcal{B}$ and $\mu : \mathcal{B} \to \mathcal{A}$ be natural transformations such that $\mu\tau = \iota^{\mathcal{A}}$ and $\tau\mu = \iota^{\mathcal{B}}$. Show, for all $C \in \mathcal{C}$, that $\tau^C : \mathcal{A}C \to \mathcal{B}C$ is an isomorphism.

EXERCISE 2E: Recall the functors $\mathcal{F} : \mathcal{TNS} \to \mathcal{TS}$ and $\mathcal{G} : \mathcal{TS} \to \mathcal{TNS}$ defined above. Recall that $\mathcal{FG} = \mathrm{Id}_{\mathcal{TS}}$ but that $\mathcal{GF} \neq \mathrm{Id}_{\mathcal{TNS}}$. Prove that $\mathcal{GF} \sim \mathrm{Id}_{\mathcal{TNS}}$.

Definition. A category $\mathcal{C}$ is said to be **equivalent** to a category $\mathcal{D}$ (write $\mathcal{C} \sim \mathcal{D}$) if there are functors $\mathcal{P} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{Q} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{PQ} \sim \mathrm{Id}_{\mathcal{D}}$ and $\mathcal{QP} \sim \mathrm{Id}_{\mathcal{C}}$.

Intuitively, if two categories are equivalent, then they have the same "information" in them. If I'm an expert on one of the categories and you're an expert on the other, then we'll probably spend our days quibbling over notation, but will be well aware that we both know the same stuff.

The preceding exercise asks you to demonstrate that $\mathcal{TNS}$ is equivalent to $\mathcal{TS}$. Which is better, $\mathcal{TNS}$ or $\mathcal{TS}$? In some sense, neither, since they're equivalent.

On the other hand, with a little thought, you will readily see that $\mathcal{F} : \mathcal{TNS} \to \mathcal{TS}$ is not "one-to-one". To almost any topological space Y, there are several topological N-spaces whose images under $\mathcal{F}$ are all equal to Y. For example, $(\mathbb{R}, x \mapsto \{(x-\delta, x+\delta)\})$ and $(\mathbb{R}, x \mapsto \{(x-2\delta, x+\delta)\})$ have the same image under $\mathcal{F}$.

So, in some sense, $\mathcal{TNS}$ is redundant, with more copies of each space than one really needs. From this perspective, $\mathcal{TS}$ is better than $\mathcal{TNS}$.

Finally, a few broad words about mathematics. Typically a working mathematician develops expertise in a category, *e.g.*, the category $\mathcal{TS}$ of topological spaces. A fundamental question in topology is to determine, given two topological spaces, whether they are homeomorphic, but, more generally, whatever your "category of choice" may be, you'll want to be able to form many examples of objects in it, and, in many cases to show that two such examples are not isomorphic in the category.

If one can form a functor $\mathcal{F}$ that goes from your category of choice $\mathcal{C}$ to some other category $\mathcal{D}$, and if tools are available for distinguishing objects in $\mathcal{D}$, then, given $C, C' \in \mathcal{C}$, if you are able to show that $\mathcal{F}C$ and $\mathcal{F}C'$ are not isomorphic in $\mathcal{D}$, then you can immediately conclude that C and C' are not isomorphic in $\mathcal{C}$. (See the next exercise.)

For example, there is a functor called "first homology", and denoted H_1 which runs from topological spaces to Abelian groups. Given two topological spaces, if one is able to show that their first homology groups are not isomorphic, then the spaces are not homeomorphic.

This kind of argument will be central to much of this course!

EXERCISE 2F: Show that functors carry isomorphisms to isomorphisms. That is, show that, if $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a functor, if $f : C \to C'$ is an isomorphism in $\mathcal{C}$, then $\mathcal{F}f : \mathcal{F}C \to \mathcal{F}C'$ is an isomorphism in $\mathcal{D}$.

In the next definition, we make the convention that $\cup\emptyset = \emptyset$.

Definition. Let X be a set and let $\mathcal{S} \subseteq \{\text{subsets of } X\}$. The **union closure of $\mathcal{S}$** is $\{\cup\mathcal{A} \,|\, \mathcal{A} \subseteq \mathcal{S}\}$. The **finite intersection closure in X of $\mathcal{S}$** is $\{X\} \cup \{\cap\mathcal{A} \,|\, \mathcal{A} \subseteq \mathcal{S}, 0 < |\mathcal{A}| < \infty\}$.

Definition. Let X be a set and let $\mathcal{O}$ be a topology on X. Let $\mathcal{B} \subseteq \{\text{subsets of } X\}$. We say that $\mathcal{B}$ is a **basis** for $\mathcal{O}$ if the union closure of $\mathcal{B}$ is $\mathcal{O}$. We say that $\mathcal{B}$ is a **subbasis** for $\mathcal{O}$ if the finite intersection closure in X of $\mathcal{B}$ is a basis for $\mathcal{O}$.

Note that any basis for a topology is a subbasis.

EXERCISE 2G: Let X be a set and let $\mathcal{S} \subseteq \{\text{subsets of } X\}$. Show that $\mathcal{S}$ is a subbasis of a topology on X. (That is, show that the union closure of the finite intersection closure in X of $\mathcal{S}$ is a topology on X.)

EXERCISE 2H: Let X be a set and let $\mathcal{B} \subseteq \{\text{subsets of } X\}$. Show that $\mathcal{B}$ is a basis of a topology on X if and only if, both of the following conditions hold:

(1) $\cup\mathcal{B} = X$; and
(2) for all $B, B' \in \mathcal{B}$, for all $x \in B \cap B'$, there exists $B'' \in \mathcal{B}$ such that $x \in B'' \subseteq B \cap B'$.

Definition. Let $\mathcal{O}$ and $\mathcal{O}'$ be topologies on a set X. We say that $\mathcal{O}$ is **finer** (or **stronger**) than $\mathcal{O}'$ if $\mathcal{O} \supseteq \mathcal{O}'$. We say that $\mathcal{O}$ is **coarser** (or **weaker**) than $\mathcal{O}'$ if $\mathcal{O} \subseteq \mathcal{O}'$.

Remark. Let X and I be sets and, for all $i \in I$, let $\mathcal{O}_i$ be a topology on X. Then $\displaystyle\bigcap_{i \in I} \mathcal{O}_i$ is a topology on X.

The following is a restatement of the preceding remark.

Remark. Let $\Xi \subseteq \{\text{topologies on } X\}$. Then $\cap\Xi$ is a topology on X.

The preceding two remarks are asserting that, given a collection of topologies on a set, there exists a unique topology which is finest among:

those topologies coarser than every topology in the collection.

Question: How about the other way around? That is, given a collection of topologies on a set, does there there exist a unique topology which is coarsest among:

those topologies finer than every topology in the collection.

Answer: We decided in class that the answer is yes. This can probably be argued using Zorn's lemma, but more directly, one can simply take the union of the topologies in the collection and then take the intersection of all the topologies containing that union. By the preceding two remarks, that intersection is again a topology. Yet another way to say the same thing is that you take the union of the topologies in the collection, then take the finite intersection closure of that union, then take the union closure of *that*.

Definition. Let X be a set and let

$$\mathcal{S} \subseteq \{\text{subsets of } X\}.$$

Then the **topology generated by** $\mathcal{S}$ is the coarsest topology among those topologies containing $\mathcal{S}$.

Note that, if $\mathcal{B}$ is a basis or subbasis of a topology $\mathcal{O}$ on a set X, then $\mathcal{O}$ is the topology generated by $\mathcal{B}$.

Definition. Let X and Y be sets, let $f : X \to Y$ be a function and let

$$\mathcal{S} \subseteq \{\text{subsets of } Y\}.$$

Then we define $f^*\mathcal{S} := \{f^{-1}(S) \,|\, S \in \mathcal{S}\}$.

EXERCISE 2I: Let X and Y be sets, let $f : X \to Y$ be a function and let $\mathcal{O}$ be a topology on Y. Show that $f^*\mathcal{O}$ is a topology on X.

Remark. Let X and Y be sets, let $f : X \to Y$ be a function and let $\mathcal{O}$ be a topology on Y. Then $f^*\mathcal{O}$ is the coarsest topology making f continuous.

Definition. Let Y be a topological space and let $X \subseteq Y$. Then the **inherited topology on** X (or **relative topology on** X or **subspace topology on** X) is:

$$\{X \cap U \,|\, U \text{ is open in } Y\}.$$

Remark. Let Y be a topological space and let $X \subseteq Y$. Let $i : X \to Y$ be the inclusion, defined by $i(x) = x$. Let $\mathcal{O} := \{\text{open subsets of } Y\}$. Then $i^*\mathcal{O}$ is the inherited topology on X. It is the coarsest topology making i continuous.

Definition. Let I be a set and, for all $i \in I$, let X_i be a topological space. Let $X := \prod_{i\in I} X_i$. For all $i \in I$, let $p_i : X \to X_i$ be the ith coordinate projection map. For all $i \in I$, let $\mathcal{O}_i$ be the set of all open subsets of X_i. Then the **product topology** on X is the topology generated by $\bigcup_{i\in I} p_i^*\mathcal{O}_i$.

Note that the product topology is the coarsest among those that make all the projection maps p_i continuous.

In general, whenever a product of topological spaces is formed, we will give it the product topology, unless otherwise specified. In particular, when X and Y are topological spaces, $X \times Y$ is a topological space as well.

Definition. Let X be a topological space, let Y be a set and let $f : X \to Y$ be a function. The **quotient topology** on Y is the finest making f continuous. (See Exercise 3A below.)

Now let's start making some topological spaces!

Definition. Let $n \geq 0$ be an integer. The **standard topology** on $\mathbb{R}^n$ is the topology generated by $\{B_r(x) \,|\, x \in \mathbb{R}^n, r > 0\}$.

Definition. Let $n \geq 0$ be an integer. We define $S^n := \{x \in \mathbb{R}^{n+1} \,|\, d(x,0) = 1\}$. The **standard topology** on S^n is the topology inherited from $\mathbb{R}^{n+1}$.

Definition. If X is a topological space, and if $x \in X$, then an **open neighborhood of x in X** is an open subset U of X such that $x \in U$. If X is a topological space, and if $S \subseteq X$, then an **open neighborhood of S in X** is an open subset U of X such that $S \subseteq U$.

Definition. To say that a topological space X is **Hausdorff** means: for all $x, x' \in X$, if $x \neq x'$, then there are open neighborhoods U of x and U' of x' in X such that $U \cap U' = \emptyset$.

Definition. Let S be a subset of a topological space X. We say that S is **closed in** X if $X \backslash S$ is open in X.

Note that {closed subsets of X} is closed under finite union and arbitrary intersection.

Definition. Let X be a topological space and assume, for all $x \in X$, that $\{x\}$ is closed in X. We say that X is **regular** if the following condition holds: For all $x_0 \in X$, for any closed subset C_0 of X, if $x_0 \neq C_0$, then there exist open nieghborhoods U and V of x_0 and C_0 in X such that $U \cap V = \emptyset$. We say that X is **normal** if the following condition holds: For any closed subsets C, C' of X, if $C \cap C' = \emptyset$, then there exist open nieghborhoods U and U' of C and C' in X such that $U \cap U' = \emptyset$.

Definition. Let X be a topological space and let $S \subseteq X$. The **closure** of S in X is the intersection of all closed subsets of X that contain S, and is denoted $\overline{S}$ or $\mathrm{Cl}_X(S)$. The **interior** of S in X is the union of all open subsets of X contained in S, and is denoted S° or $\mathrm{Int}_X(S)$. The **boundary** of S in X is $\overline{S} \backslash S^\circ$, and is denoted ∂S or $\mathrm{Bd}_X(S)$.

Since an intersection of closed sets is closed and a union of open sets is open, it follows that the closure of S is the smallest closed set containing S, while the interior of S is the largest open set contained in S.

Let S be a subset of a topological space X. We say that S is **dense** in X if $\mathrm{Cl}_X(S) = X$.

Definition. We say that a topological space is **discrete** if all of its subsets are open. We say that a topological space X is **indiscrete** if its only open sets are $\emptyset$ and X.

Definition. Let X be a Hausdorff topological space. We say X is **compact** if, for any $\mathcal{S} \subseteq \{\text{open subsets of } X\}$, we have:

(∗) if $\cup\mathcal{S} = X$, then there is a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that $\cup\mathcal{F} = X$.

That is, a Hausdorff topological space is said to be compact if every open cover has a finite subcover, in the following terminology:

Definition. Let X be a topological space. An **open cover** of X is a subset $\mathcal{U}$ of $\{\text{open subsets of } X\}$ such that $\cup\mathcal{U} = X$. Given two covers $\mathcal{U}, \mathcal{V}$ of X, we say that $\mathcal{U}$ is a **subcover** of $\mathcal{V}$ if $\mathcal{U} \subseteq \mathcal{V}$.

Question: Is S^2 homeomorphic to $\mathbb{R}^2$?

Fact. S^2 is compact, whereas $\mathbb{R}^2$ is not.

Answer to preceding question: No.

Definition. Let X be a Hausdorff topological space. Then X is said to be **locally compact** if, for all $x \in X$, there exists an open neighborhood U of x in X such that $\overline{U}$ is compact.

Example. $\mathbb{R}$ is locally compact, and any finite product of locally compact topological spaces is again locally compact. Consequently, for all integers $n \geq 1$, $\mathbb{R}^n$ is locally compact. However infinite products of locally compact topological spaces are generally *not* locally compact, *e.g.*, $\prod\limits_{i=1}^{\infty} \mathbb{R}$ is not locally compact.

Next Question: Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?

EXERCISE 2J: For all $p, q \in \mathbb{R}^2$, show that $\mathbb{R}^2\backslash\{p\}$ is homeomorphic to $\mathbb{R}^2\backslash\{q\}$.

EXERCISE 2K: Let X and Y be topological spaces and let $x \in X$. Assume that X is homeomorphic to Y. Show that there exists $y \in Y$ such that $X\backslash\{x\}$ is homeomorphic to $Y\backslash\{y\}$.

Definition. Let S be a subset of a topological space X. We say that S is **clopen in** X if S is both closed an open in X.

Definition. We say that a topological space X is **connected** if it has no clopen sets other than $\emptyset$ and X. A topological space is **disconnected** if it is not connected.

Fact. $\mathbb{R}\backslash\{0\}$ is disconnected, while $\mathbb{R}^2\backslash\{(0,0)\}$ is connected.

Remark. $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Proof: Assume, for a contradiction that $\mathbb{R}$ and $\mathbb{R}^2$ are homeomorphic. By Exercise 2K, choose $y \in \mathbb{R}^2\backslash\{(0,0)\}$ such that $\mathbb{R}\backslash\{0\}$ is homeomorphic to $\mathbb{R}^2\backslash\{y\}$. By Exercise 2J, $\mathbb{R}^2\backslash\{y\}$ is homeomorphic to $\mathbb{R}^2\backslash\{(0,0)\}$. Then $\mathbb{R}\backslash\{0\}$ is homeomorphic to $\mathbb{R}^2\backslash\{(0,0)\}$. However, by the fact above, $\mathbb{R}\backslash\{(0\}$ is disconnected, while $\mathbb{R}^2\backslash\{(0,0)\}$ is connected. **QED**

In the above proof, we used, without comment, the fact that if two topological spaces are homeomorphic and if one is connected, than the other is as well. That is, "connectedness is a homeomorphism invariant". In fact, for a property of topological spaces to be useful, it is typically a homeomorphism invariant, and we will leave it as an implicit exercise to verify that the properties we may introduce below are, in fact, homeomorphism invariants. Typically, such arguments are straightforward.

Next Question: Is $\mathbb{R}^2$ homeomorphic to $\mathbb{R}^3$?

Note that both $\mathbb{R}^2\backslash\{(0,0)\}$ and $\mathbb{R}^3\backslash\{(0,0,0)\}$ are connected. Nevertheless, some variant of the argument given above will work to show that the answer to the above question is "no".

Definition. A **pointed topological space** consists of

(1) a topological space X; and
(2) a point $x \in X$.

We call x the **basepoint** of the pointed topological space (X, x).

Definition. The category of pointed topological spaces (with basepoint preserving continuous maps) is denoted $\mathcal{PTS}$. The forgetful functor $(X, x) \mapsto X\mathcal{PTS} \to \mathcal{TS}$ is denoted by $\mathcal{FB}$. An isomorphism in $\mathcal{PTS}$ is called a **basepoint-preserving homeomorphism**.

The next Fact is nontrivial, but we will assume it for now, and prove it in the next few lectures.

Fact. There is a functor $\pi_1 : \mathcal{PTS} \to \{\text{groups}\}$ such that:

(1) for all $x \in X := \mathbb{R}^2\backslash\{(0,0)\}$, we have that $\pi_1(X, x)$ is isomorphic to the additive group of integers; and
(2) for all $y \in Y := \mathbb{R}^3\backslash\{(0,0,0)\}$, we have that $\pi_1(Y, y)$ is isomorphic to the trivial group.

Remark. For all topological spaces X, Y, for all $x \in X$, if X is homeomorphic to Y then there exists $y \in Y$ such that (X, x) is basepoint-preserving homeomorphic to (Y, y).

Remark. Let $X := \mathbb{R}^2\backslash\{(0,0)\}$ and $Y := \mathbb{R}^3\backslash\{(0,0,0)\}$. Then X and Y are not homeomorphic.

Proof: Say for a contradiction that X is homeomorphic to Y. Choose $x \in X$. By the second to last Remark, choose $y \in Y$ such that (X, x) is basepoint-preserving homeomorphic to (Y, y). Let $\pi_1 : \mathcal{PTS} \to \{\text{groups}\}$ be the functor described in the preceding Fact.

Then, by Exercise 2F, $\pi_1(X, x)$ is isomorpic to $\pi_1(Y, y)$, in the category of groups. We say that $\pi_1(X, x)$ is the **fundamental group of** X (with respect to x). Then, by the preceding fact, the additive group of integers is isomorphic to the trivial group, contradiction. **QED**

Corollary. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.

Given the preceding Remark, the preceding Corollary is argued exactly as in the proof that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

Our goal now becomes to define and study the functor $\pi_1 : \mathcal{PTS} \to \{\text{groups}\}$ of the preceding Fact.

Definition. A space X is **locally compact** if every point of X has an open neighborhood whose closure in X is compact.

Definition. Let X and Y be topological spaces and assume that X locally compact. Recall that $\mathrm{Hom}(X, Y)$ is the set of continuous maps from X to Y. Let $\mathcal{K}$ denote the set of all compact subsets of X. Let $\mathcal{O}$ denote the set of all open subsets of Y. For all $K \in \mathcal{K}$, for all $U \in \mathcal{O}$, let let $\mathcal{W}(K, U) := \{f \in \mathrm{Hom}(X, Y) \,|\, f(K) \subseteq U\}$. The topology on $\mathrm{Hom}(X, Y)$ generated by the subbasis $\{\mathcal{W}(K, U) \,|\, K \in \mathcal{K}, U \in \mathcal{O}\}$ is called the **compact-open** topology on $\mathrm{Hom}(X, Y)$. We let $C(X, Y)$ denote the topological space whose underlying set is $\mathrm{Hom}(X, Y)$ and whose topology is the compact-open topology.

One has the general feeling that a space is locally compact if it is "almost" finite dimensional. In particular, if I is a set and, for all $i \in I$, we have a Hausdorff topological space X_i, then $\prod_{i \in I} X_i$ is locally compact iff $\{i \in I \,|\, X_i \text{ is noncompact}\}$ is finite. The topological space $C(\mathbb{R}, \mathbb{R})$ does not feel anywhere near finite dimensional, so one would guess that it is not locally compact, and that is, in fact true:

Remark. $C(\mathbb{R}, \mathbb{R})$ is not locally compact.

Sketch of proof: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$. Then $f \in C(\mathbb{R}, \mathbb{R})$. Let U be an open neighborhood of f in $C(\mathbb{R}, \mathbb{R})$. Assume, for a contradiction, that $\overline{U}$ is compact.

Choose $\epsilon > 0$ and a compact subset $K \subseteq \mathbb{R}$ such that

$$U_0 := \{g \in C(\mathbb{R}, \mathbb{R}) \,|\, |g| < \epsilon \text{ on } K\} \subseteq U.$$

(It is an unassigned exercise to prove existence of such an ϵ and K.) For all integers $j \geq 1$, let $g_j : \mathbb{R} \to \mathbb{R}$ be defined by $g_j(x) = [\epsilon/2][\sin(jx)]$; then $g_j \in U_0$, so $g_j \in U$. We leave it as an unassigned exercise to show that g_j has no convergent subsequence in $C(\mathbb{R}, \mathbb{R})$. After all, to what could these functions possibly converge? (Recall that, in any compact Hausdorff topological space, any sequence has a convergent subsequence.) **QED**

Fact. Let X and Y be locally compact topological spaces. For all $f \in C(X, C(Y, Z))$, define $\alpha_f \in C(X \times Y, Z)$ by $\alpha_f(x, y) = (f(x))(y)$. Then $f \mapsto \alpha_f : C(X, C(Y, Z)) \to C(X \times Y, Z)$ is a homeomorphism.

If X is a locally compact topological space and if (Y, d) is a metric space, then the **d-uniform on compacta topology** on $\mathrm{Hom}(X, Y)$ is the topology generated by the basis $\{f \in \mathrm{Hom}(X, Y) \,|\, \forall k \in K, d(f(k), f_0(k)) < \epsilon\}$, where f_0 ranges over $\mathrm{Hom}(X, Y)$, where K ranges over compact subsets of X and where ϵ ranges over the positive real numbers. It is a fact that, for the topology on Y generated by the basis

$$\{\text{open balls with respect to the metric } d\},$$

the compact-open topology on $\mathrm{Hom}(X, Y)$ agrees with the d-uniform on compacta topology on $\mathrm{Hom}(X, Y)$.

Definition. Let X be a topological space and let $I := [0,1]$. Let $x, x' \in X$. A **path in X from x to x'** is an element $\gamma \in C(I, X)$ such that both $\gamma(0) = x$ and $\gamma(1) = x'$. A **loop in X at x** is a path from x to x.

Definition. Let X be a topological space and let $I := [0,1]$. Let $x_0, x_1 \in X$ and let

$$P_{x_0}^{x_1} := \{\gamma \in C(I, X) \mid \gamma(0) = x_0, \gamma(1) = x_1\}$$

be the topological space of paths in X from x_0 to x_1. (Give $P_{x_0}^{x_1}$ the inherited topology from $C(I, X)$.) Let $\gamma_0, \gamma_1 \in P_{x_0}^{x_1}$. An **endpoint fixed homotopy from γ_0 to γ_1** is a path in $P_{x_0}^{x_1}$ from γ_0 to γ_1. We say that γ_0 and γ_1 are **endpoint fixed homotopic** if there is an endpoint fixed homotopy from γ_0 to γ_1.

Following the preceding Fact, we may equivalently define an endpoint fixed homotopy from γ_0 to γ_1 to be a continuous $H : I \times I \to X$ such that
(1) for all $s \in I$, we have $H(s, 0) = x_0$;
(2) for all $s \in I$, we have $H(s, 1) = x_1$;
(3) for all $t \in I$, we have $H(0, t) = \gamma_0(t)$; and
(4) for all $t \in I$, we have $H(1, t) = \gamma_1(t)$.
We have some point set topology exercises:

EXERCISE 3A: Let X be a topological space, let Y be a set and let $f : X \to Y$ be a function. Let $\mathcal{O} := \{U \subseteq Y \mid f^{-1}(U) \text{ is open in } X\}$. Show that $\mathcal{O}$ is the finest topology on Y making f continuous.

EXERCISE 3B: Let X be a topological space, let $S \subseteq X$ and let $x_0 \in X$. Show that $x_0 \in \partial S$ iff the following condition holds: For all open neighborhoods U of x_0 in X, we have $U \cap S \neq \emptyset \neq U \backslash S$.

EXERCISE 3C: Let X be a topological space and let $S \subseteq X$. Show that $X \backslash (S^\circ) = \overline{X \backslash S}$.

EXERCISE 3D: Let X be a topological space and let U be an open subset of X. Show that $(\partial U)^\circ = \emptyset$. That is, show that $\mathrm{Int}_X(\mathrm{Bd}_X(U)) = \emptyset$.

Definition. Let (X, x) be a pointed topological space. Define an equivalence relation $\sim$ on $P_x^x(X)$ by: For all $\gamma, \gamma' \in P_x^x(X)$, $\gamma \sim \gamma'$ means that γ and γ' are endpoint fixed homotopic, *i.e.*, that $P_\gamma^{\gamma'}(P_x^x(X)) \neq \emptyset$. We define $\pi_1(X, x) := (P_x^x(X))/\sim$.

Note that this defines a functor $\pi_1 : \{\text{pointed topological spaces}\} \to \{\text{sets}\}$. Eventually we'll redefine π_1 as a functor $\pi_1 : \{\text{pointed topological spaces}\} \to \{\text{groups}\}$, and the current π_1 will be the compostion with that redefined π_1, followed by the forgetful functor $\{\text{sets}\} \to \{\text{groups}\}$. For the moment, however, this definition will suffice for our purposes.

Recall that, in any topological space Z, if we define an equivalence relation $\cong$ on Z by $z \cong z'$ iff $P_z^{z'}(Z) \neq \emptyset$, then the equivalence classes of $\cong$ are called **path components of Z**. With that terminology, $\pi_1(X, x)$ is simply the set of path components in $P_x^x(X)$.

EXERCISE 3E: Given an arrow $f : (X, x) \to (Y, y)$ in the category

$$\{\text{pointed topological spaces}\},$$

define an arrow $\pi_1(f) : \pi_1(X,x) \to \pi_1(Y,y)$ in {sets}.

As is typical with covariant functors, we use f_* as an abbreviation for $\pi_1(f)$. (For *contra*variant functors, f^* is typical as an abbreviation for the result of applying the functor to an arrow f.)

Recall that we aim to prove that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic. We already observed that it suffices to show that $X := \mathbb{R}^2\backslash\{(0,0)\}$ and $Y := \mathbb{R}^3\backslash\{(0,0,0)\}$ are not homeomorphic. It now suffices to show, for all $x \in X$ and all $y \in Y$, that $|\pi_1(X,x)| \geq 2$, while $|\pi_1(Y,y)| = 1$.

Definition. We say that a topological space X is path connected if X has only one path component, *i.e.*, if the following condition holds: for all $x, x' \in X$, we have $P_x^{x'}(X) \neq \emptyset$.

We will argue, in Exercise 3G, that if X is path connected, then, for any $x, x' \in X$, we have that $\pi_1(X,x)$ and $\pi_1(X,x')$ are bijective, *i.e.*, isomorphic in the category {sets}. Eventually this will be refined to saying that $\pi_1(X,x)$ and $\pi_1(X,x')$ are isomorphic in the category {groups}.

Definition. Let $x, x', x'' \in X \in \mathcal{TS}$. For all $\gamma \in P_x^{x'}(X)$, for all $\gamma' \in P_{x'}^{x''}(X)$, we define $\gamma\|\gamma' \in P_x^{x''}(X)$ by

$$(\gamma\|\gamma')(t) = \begin{cases} \gamma(2t), & \text{if } t \in [0,1/2]; \\ \gamma'(2t-1), & \text{if } t \in [1/2,1]. \end{cases}$$

The path $\gamma\|\gamma'$ is called the **concatenation** of γ with γ'.

Definition. Let $x, x' \in X \in \mathcal{TS}$. For all $\gamma \in P_x^{x'}(X)$, we let $[\gamma]$ denote the endpoint fixed homotopy class of γ, *i.e.*, we define

$$[\gamma] := \{\delta \in P_x^{x'}(X) \,|\, P_\gamma^\delta(P_x^{x'}(X)) \neq \emptyset\}.$$

EXERCISE 3F: Let $a, b, c, d \in X \in \mathcal{TS}$. Let $\gamma \in P_a^b(X)$, $\delta \in P_b^c(X)$ and $\epsilon \in P_c^d(X)$. Show that $[\gamma\|(\delta\|\epsilon)] = [(\gamma\|\delta)\|\epsilon]$.

Exercise 3F asserts that concatenation is "associative up to homotopy". Note that it is not associative, *i.e.*, note that it frequently happens that $\gamma\|(\delta\|\epsilon) \neq (\gamma\|\delta)\|\epsilon$.

Definition. Let $a, b \in X \in \mathcal{TS}$. For all $\gamma \in P_a^b(X)$, we define $\overleftarrow{\gamma} \in P_b^a(X)$ by $(\overleftarrow{\gamma})(t) = \gamma(1-t)$.

EXERCISE 3G: Let $a, b \in X \in \mathcal{TS}$ and let $\gamma \in P_a^b(X)$. Show that

$$[\delta] \mapsto [\overleftarrow{\gamma}\|\delta\|\gamma] : \pi_1(X,a) \to \pi_1(X,b)$$

is a *well-defined* bijection.

Part of the the preceding exercise is to show that, if $[\delta] = [\delta']$, then $[\overleftarrow{\gamma}\|\delta\|\gamma] = [\overleftarrow{\gamma}\|\delta'\|\gamma]$. This is what is mean by proving "well-definedness".

One summarizes Exercise 3G by saying that "π_1 is essentially independent of the basepoint". This will even be true after redefining π_1 in a group theoretic way.

We can define a category of "pathed topological spaces", each object of which consists of a topological space X, together with a path $[0,1] \to X$. Let

$$\mathcal{A}, \mathcal{B} : \{\text{pathed topological spaces}\} \to \{\text{sets}\}$$

be defined by

$$\mathcal{A}(X,\gamma) = \pi_1(X, \gamma(0)), \qquad\qquad \mathcal{B}(X,\gamma) = \pi_1(X, \gamma(1)).$$

Then the argument of Exercise 3G shows that $\mathcal{A}$ and $\mathcal{B}$ are equivalent functors, and this is the key point of that exercise. So, for $x_0, x_1 \in X$, the bijection $\pi_1(X, x_0) \to \pi_1(X, x_1)$ is "natural up to choosing a path connecting the basepoints x_0 and x_1".

Note that, since $X := \mathbb{R}^2 \backslash \{(0,0)\}$ and $Y := \mathbb{R}^3 \backslash \{(0,0,0)\}$ are both path-connected, it now suffices to show that there exist $x \in X$ and $y \in Y$ such that $|\pi_1(X,x)| \geq 2$ and such that $|\pi_1(Y,y)| = 1$. (Before, we needed to show this for all $x \in X$, $y \in Y$, so the problem is now formally easier.)

Definition. Let $I := [0,1]$. Let $X, Y \in \mathcal{TS}$ and let $f, g : X \to Y$ be continuous. A **homotopy** from f to g is a continuous map $H : I \times X \to Y$ such that $H(0,\cdot) = f$ and $H(1,\cdot) = g$. We say that f is **homotopic** to g, if there exists a homotopy from f to g.

EXERCISE 3H: Let $X, Y \in \mathcal{TS}$, let $f, g : X \to Y$ be continuous and let $x \in X$. Assume that f is homotopic to g. As usual, let $f_* := \pi_1(f) : \pi_1(X,x) \to \pi_1(Y, f(x))$ and let $g_* := \pi_1(g) : \pi_1(X,x) \to \pi_1(Y, g(x))$. Prove that there is a bijection

$$b : \pi_1(Y, f(x)) \to \pi_1(Y, g(x))$$

such that $b \circ f_* = g_*$. (Hint: If H is a homotopy from f to g, then $H(\cdot, x)$ is a path from $f(x)$ to $g(x)$. The needed bijection then comes from Exercise 3G.)

One summarizes Exercise 3H by the buzzphrase: "Homotopic maps induce the same map on π_1." However, to be technically precise one needs to remember that the two maps on π_1, namely f_* and g_*, yield different basepoints on Y, namely $f(x)$ and $g(x)$. A precise interpretation of this buzzphrase must take this into account.

Definition. Let $X, Y \in \mathcal{TS}$. We say that X is **homotopy equivalent** to Y (or that X and Y **have the same homotopy type**) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that both of the following conditions hold:

(1) $g \circ f : X \to X$ is homotopic to $\mathrm{id}_X : X \to X$; and
(2) $f \circ g : Y \to Y$ is homotopic to $\mathrm{id}_Y : Y \to Y$.

EXERCISE 3I: Let $X, Y \in \mathcal{TS}$ and assume that both X and Y are path-connected. Assume that X is homotopy equivalent to Y. Show, for any $x \in X$ and any $y \in Y$, that $\pi_1(X,x)$ is bijective to $\pi_1(Y,y)$, *i.e.*, that $\pi_1(X,x)$ is isomorphic to $\pi_1(Y,y)$ in the category $\{\text{sets}\}$.

Once again, the preceding exercise will strengthen to saying that $\pi_1(X,x)$ and $\pi_1(Y,y)$ are isomorphic in the category $\{\text{groups}\}$, once we've redefined the functor π_1.

EXERCISE 3J: Let $n \geq 2$ be an integer. Show that $S^{n-1} := \{x \in \mathbb{R}^n \,|\, d(0,x) = 1\}$ is homotopy equivalent to $Q := \mathbb{R}^n \backslash \{0\}$. (Hint: Use the maps $p \mapsto p : S^{n-1} \to Q$ and $q \mapsto q/(d(0,q)) : Q \to S^{n-1}$.

Given the last two exercises, our goal is now to show that there exist $a \in S^1$ and $b \in S^2$ such that $|\pi_1(S^1, a)| \geq 2$ and such that $|\pi_1(S^2, b)| = 1$.

Definition. Let X be a topological space and $x \in X$. We say that X is **x-avoidable** if, for all $a, b \in X \backslash \{x\}$, for all $\gamma \in P_a^b(X)$, there exists $\gamma' \in P_a^b(X \backslash \{x\})$ such that γ is endpoint fixed homotopic to γ'.

EXERCISE 4A: Let $n \geq 2$ be an integer. Let U be an open, convex, nonempty subset of $\mathbb{R}^n$. Let X be a topological space and assume that X is homeomorphic to U. Show, for all $x \in X$, that X is x-avoidable. (Hint: Start by showing, for all $a, b \in U$, for all $\gamma, \gamma' \in P_a^b(U)$, that γ is endpoint fixed homotopic to γ'.)

Lemma. Let X be a topological space and let $x \in X$. Assume that $\{x\}$ is a closed subset of X. Let U be an open neighborhood of x in X. Assume that U is x-avoidable. Then X is x-avoidable.

Proof: Let $a, b \in X \backslash \{x\}$ and let $\gamma \in P_a^b(X)$. We wish to show, for some $\gamma \in P_a^b(X \backslash \{x\})$, that γ is endpoint fixed homotopic to γ'.

Let $V := X \backslash \{x\}$. Then $\{U, V\}$ is an open cover of X. Let $\delta > 0$ be a Lebesgue number for the open cover $\{\gamma^{-1}(U), \gamma^{-1}(V)\}$ of $[0,1]$. Choose $s_0, \ldots, s_n \in [0,1]$ such that $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ and such that, for all integers $i \in [1, n]$, we have $s_i - s_{i-1} < \delta/2$. Then, by definition of Lebesgue number, for all integers $i \in [1, n]$, we have either $\gamma([s_{i-1}, s_i]) \subseteq U$ or $\gamma([s_{i-1}, s_i]) \subseteq V$. For all integers $i \in [1, n]$, let $a_i := \gamma(s_{i-1})$ and $b_i := \gamma(s_i)$, and define $\gamma_i \in P_{a_i}^{b_i}(X)$ by $\gamma_i(t) = \gamma(s_{i-1}(1-t) + s_i t)$; then $\gamma_i \in P_{a_i}^{b_i}(U)$ or $\gamma_i \in P_{a_i}^{b_i}(V)$. Moreover, γ is endpoint fixed homotopic to $\gamma_1 \| \cdots \| \gamma_n$. (You may parenthesize this however you find convient.) For all integers $i \in [1, n]$, choose $\gamma_i' \in P_{a_i}^{b_i}(X \backslash \{x\})$ such that γ_i is endpoint fixed homotopic to γ_i'. (Note that if $\gamma_i \in P_{a_i}^{b_i}(V)$, then, as $V = X \backslash \{x\}$, we may simply set $\gamma_i' := \gamma_i$. On the other hand, if $\gamma_i \in P_{a_i}^{b_i}(U)$, then we use that U is x-avoidable.) Let $\gamma' := \gamma_1' \| \cdots \| \gamma_n'$. (Again, parenthesize as you wish.) Then γ is endpoint fixed homotopic to γ' and $\gamma' \in P_a^b(X \backslash \{x\})$. **QED**

EXERCISE 4B: Show, for all $p \in S^2$, that $S^2 \backslash \{p\}$ is homeomorphic to $\mathbb{R}^2$.

Remark. For all $q \in S^2$, S^2 is q-avoidable.

Proof: Fix $p \in S^2 \backslash \{q\}$. Note that $\{p\}$ is closed in S^2. Let $U := S^2 \backslash \{p\}$. By Exercise 4A and Exercise 4B, we see that U is q-avoidable. Then, by the preceding lemma, we conclude that S^2 is q-avoidable, as well. **QED**

Corollary. Let $q, y \in S^2$ and assume that $q \neq y$. Then the inclusion map $(S^2 \backslash \{q\}, y) \to (S^2, y)$ induces a surjection $\pi_1(S^2 \backslash \{q\}, y) \to \pi_1(S^2, y)$.

We can finally prove:

Fact. For all $y \in S^2$, we have $|\pi_1(S^2, y)| = 1$.

Proof: We have $\pi_1(S^2, y) \neq \emptyset$ (why?), so $|\pi_1(S^2, y)| \geq 1$. It suffices to show that $|\pi_1(S^2, y)| \leq 1$.

Fix $q \in S^2 \backslash \{y\}$. By the preceding corollary, we see that $|\pi_1(S^2, y)| \leq |\pi_1(S^2 \backslash \{q\}, y)|$ so it suffices to show that $|\pi_1(S^2 \backslash \{q\}, y)| = 1$.

By Exercise 4B, it suffices to prove, for all $z \in \mathbb{R}^2$, that $|\pi_1(\mathbb{R}^2, z)| = 1$. Let $Z := \{z\}$. Then $|\pi_1(Z, z)| = 1$, so, by Exercise 3I, it suffices to show that $\mathbb{R}^2$ is homotopy equivalent to Z.

Let $i : Z \to \mathbb{R}^2$ be the inclusion map and define $j : \mathbb{R}^2 \to Z$ by $j(x) = z$. Then $i \circ j$ is the identity on Z. Moreover, by convexity of $\mathbb{R}^2$, we see that any two maps $\mathbb{R}^2 \to \mathbb{R}^2$ are homotopic, so $j \circ i$ is homotopic to the identity on $\mathbb{R}^2$. **QED**

We now aim to show, for some $x \in S^1$, that $|\pi_1(X, x)| \geq 2$.

Let $x := (1, 0) \in S^1 \subseteq \mathbb{R}^2$. Define $\gamma, \gamma' \in P_x^x(S^1)$ by $\gamma(t) = x$ and $\gamma'(t) = (\cos(2\pi t), \sin(2\pi t))$. We wish to show that γ is *not* endpoint fixed homotopic to γ'.

Definition. Let X, Y and A be topological spaces. Let $\pi : X \to Y$ be continuous. Let $f : A \to Y$ be continuous. A **(full) π-lift of f** is a continuous map $\widehat{f} : A \to X$ such that $\pi \circ \widehat{f} = f$. Let $A_0 \subseteq A$. A **(partial) π-lift of f on A_0** is a continuous map $\widehat{f_0} : A_0 \to X$ such that $\pi \circ \widehat{f_0} = f|A_0$.

Definition. Let X, Y and A be topological spaces. Let $\pi : X \to Y$ be continuous. Let $A_0 \subseteq A$. We say that π **has unique** (A_0, A) **lifting** if, for all continuous $f : A \to Y$, for any partial π-lift $\widehat{f_0}$ of f on A_0, there is a unique full π-lift $\widehat{f}$ of f such that $\widehat{f}|A_0 = \widehat{f_0}$.

One may think of $\widehat{f_0}$ as "initial data", and the problem is to find a "solution" $\widehat{f}$ of the "lifting problem" given by (A, f, π), but the solution must fit the given initial data.

Definition. Let X, Y and A be topological spaces. Let $\pi : X \to Y$ be continuous. Let $A_0 \subseteq A$. We say that π **has local unique** (A_0, A) **lifting** if, for all $y \in Y$, there exists an open neighborhood V of y in Y such that $\pi|[\pi^{-1}(V)] : \pi^{-1}(V) \to V$ has unique (A_0, A) lifting.

A **topological pair** is a pair (A_0, A) such that A is a topological space and A_0 is a subset of A. These are the objects of a category. (Here, an arrow $(A_0, A) \to (B_0, B)$ is a continuous map $f : A \to B$ such that $f(A_0) \subseteq B_0$.) If (A_0, A) and (B_0, B) are isomorphic in the category of topological pairs (*i.e.*, if there is a homeomorphism $A \to B$ which carries A_0 to B_0), then

(1) unique (A_0, A) lifting is equivalent to unique (B_0, B) lifting; and

(2) local unique (A_0, A) lifting is equivalent to local unique (B_0, B) lifting.

Unassigned exercise: Show that $(\{0\}, [0, 1])$ is isomorphic, in the category of topological pairs, to $(\{5\}, [5, 5.3])$.

Definition. Let X, X' and Y be topological spaces and let $\pi : X \to Y$ and $\pi' : X' \to Y$ be continuous. We will say that π and π' are **Y-homeomorphic** if there is a homeomorphism $h : X \to X'$ such that $\pi' \circ h = \pi$.

Fix a topological space Y. A **topological space over** Y consists of a topological space X and a continuous map $X \to Y$. Then topological spaces over Y form the objects of a category. (What are the arrows?) Then π and π' are Y-homeomorphic iff we have: (X, π) and (X', π) are isomorphic in the category of topological spaces over Y.

Remark. Let A, X, X' and Y be topological spaces and let $\pi : X \to Y$ and $\pi' : X' \to Y$ be Y-homeomorphic continuous maps. Let $A_0 \subseteq A$. Then

(1) π has unique (A_0, A) lifting iff π' has unique (A_0, A) lifting; and

(2) π has locally unique (A_0, A) lifting iff π' has locally unique (A_0, A) lifting.

Definition. Let X, X', Y and Y' be topological spaces and let $\pi : X \to Y$ and $\pi' : X' \to Y'$ be continuous. We say that π and π' are **homeomorphic** if there are homeomorphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that $\pi' \circ \alpha = \beta \circ \pi$.

Note that π and π' are homeomorphic iff they are isomorphic objects in the category $\mathrm{Arr}(\mathcal{TS})$.

The next remark generalizes the preceding one.

Remark. Let A, X, X', Y and Y' be topological spaces and let $\pi : X \to Y$ and $\pi' : X' \to Y'$ be homeomorphic continuous maps. Let $A_0 \subseteq A$. Then

(1) π has unique (A_0, A) lifting iff π' has unique (A_0, A) lifting; and

(2) π has locally unique (A_0, A) lifting iff π' has locally unique (A_0, A) lifting.

Definition. Let X and Y be topological spaces and let $\pi : X \to Y$ be continuous. We say that π is a **trivial covering map** if there exists a discrete topological space D such that π is Y-homeomorphic to $(y, d) \mapsto y : Y \times D \to Y$.

Unassigned exercise: Show that $\pi : X \to Y$ is a trivial covering map iff there exist topological spaces D and Y' such that D is discrete and such that π is homeomorphic to $(y, d) \mapsto y : Y' \times D \to Y'$.

Suggestion for the next two exercises: Find a general result about lifting for trivial covering maps that will imply both of the next two exercises, and you can do both at once.

EXERCISE 4C: Let X and Y be topological spaces. Let $\pi : X \to Y$ be a trivial covering map. Show that π has unique $(\{0\}, [0, 1])$ lifting.

EXERCISE 4D: Let X and Y be topological spaces. Let $\pi : X \to Y$ be a trivial covering map. Let $I := [0, 1]$ and let $L := (I \times \{0\}) \cup (\{0\} \times I)$. Show that π has unique (L, I^2) lifting.

Unassigned exercise: Show that any trivial covering map has unique $((1, 0), S^1)$ lifting.

Definition. Let X and Y be topological spaces. Let $\pi : X \to Y$ be continuous. We say that π is a **covering map** if, for all $y \in Y$, there exists an open neighborhood V of y in Y such that $\pi|[\pi^{-1}(V)] : \pi^{-1}(V) \to V$ is a trivial covering map.

EXERCISE 4E: Show that $t \mapsto (\cos(2\pi t), \sin(2\pi t)) : \mathbb{R} \to S^1$ is a covering map.

Remark. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. Let $I := [0, 1]$ and let $L := (I \times \{0\}) \cup (\{0\} \times I)$. Then π has both local unique $(\{0\}, I)$ lifting and local unique (L, I^2) lifting.

Proof: This follows immediately from Exercise 4C and Exercise 4D. **QED**

Theorem. Let $I := [0, 1]$. Then local unique $(\{0\}, I)$ lifting implies unique $(\{0\}, I)$ lifting.

This is an example of a "local to global" result, which puts it among a very important class of theorems in mathematics.

Proof: We will prove existence, and leave uniqueness as an unassigned exercise.

Let $\pi : X \to Y$ have local unique $(\{0\}, I)$ lifting. Let $f : I \to Y$ be continuous, and let $x_0 \in \pi^{-1}(f(0))$. We wish to show that there is a continuous map $\widehat{f} : I \to X$ such that both $\pi \circ \widehat{f} = f$ and $\widehat{f}(0) = x_0$.

Fix an open cover $\mathcal{V}$ of Y such that, for all $V \in \mathcal{V}$, we have that $\pi|[\pi^{-1}(V)] : \pi^{-1}(V) \to V$ of any element of the open cover has unique $(\{0\}, I)$ lifting.

Let δ be a Lebesgue number for the f-pullback $\{f^{-1}(V) \,|\, V \in \mathcal{V}\}$ of $\mathcal{V}$ to I. Choose a positive integer n such that $1/n < \delta/2$. For all integers $i \in [1, n]$, let $a_i := (i-1)/n$, let $b_i := i/n$, let $I_i := [a_i, b_i]$, and let $f_i := f|I_i$.

Let $i \in [1, n]$ be an integer. Note that, in the category of topological pairs, defined above, the topological pair $(\{0\}, I)$ is isomorphic to the topological pair $(\{a_i\}, [a_i, b_i])$, so that unique $(\{0\}, I)$ lifting is equivalent to unique $(\{a_i\}, [a_i, b_i])$ lifting. By definition of Lebesgue number, there is some $V \in \mathcal{V}$ such that $f(I_i) \subseteq V$. Consequently, for all $x \in \pi^{-1}(f(a_i))$, there exists a unique $\hat{\phi} : [a_i, b_i] \to X$ such that both $\pi \circ \hat{\phi} = f_i$ and $\phi(a_i) = x$.

Using this, choose $\widehat{f_1} : [a_1, b_1] \to X$ such that both $\pi \circ \widehat{f_1} = f_1$ and $\widehat{f_1}(a_1) = x_0$. Then let $x_1 := \widehat{f_1}(b_1)$.

Then choose $\widehat{f_2} : [a_2, b_2] \to X$ such that both $\pi \circ \widehat{f_2} = f_2$ and $\widehat{f_2}(a_2) = x_1$. Then let $x_2 := \widehat{f_1}(b_2)$.

Next, choose $\widehat{f_3} : [a_3, b_3] \to X$ such that both $\pi \circ \widehat{f_3} = f_3$ and $\widehat{f_3}(a_3) = x_2$. Then let $x_3 := \widehat{f_1}(b_3)$.

Continuing on, we obtain $\widehat{f_1}, \ldots, \widehat{f_n}$. Finally, define $\widehat{f} : I \to X$ by: for all integers $i \in [1, n]$, $\widehat{f}|I_i = \widehat{f_i}$. **QED**

Theorem. Let $I := [0, 1]$ and let $L := (I \times \{0\}) \cup (\{0\} \times I)$. Then local unique (L, I^2) lifting implies unique (L, I^2) lifting.

This is another local to global result. It is proved in a similar way to the theorem preceding it, but instead of dividing I into small subintervals, we divide I^2 into small squares, and then, to begin, solve the lifting problem on the bottom row of squares, left to right. Then we solve the lifting problem on the next row of squares up, again moving from left to right. Continuing, we get a solution on all squares, and then piece the results together to get a lift on all of I^2.

Question: Let A be a topological space and let A_0 be a subset of A. Is it automatically true that local unique (A_0, A) lifting implies unique (A_0, A) lifting?

Proposition. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. Let $I := [0,1]$, let $I^2 := I \times I$ and let $L := (I \times \{0\}) \cup (\{0\} \times I)$. Then π has both unique $(\{0\}, I)$ lifting and unique (L, I^2) lifting.

Proof: This follows from the last two theorems, together with the last remark. **QED**

EXERCISE 4F: Let $I := [0,1]$. Let X and Y be topological spaces and let $\pi : X \to Y$ be a covering map. Let $\gamma : I \to Y$ be constant and let $\widehat{\gamma} : I \to X$ be a π-lift of γ. Show that $\widehat{\gamma}$ is also constant. (Hint: Use the fact that, by the preceding proposition, π has unique $(\{0\}, [0,1])$ lifting.)

Theorem. Let $I := [0,1]$. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. Let $\gamma, \gamma' : I \to Y$ be continuous and assume that γ and γ' are endpoint fixed homotopic. Let $\widehat{\gamma}, \widehat{\gamma}' : I \to X$ be π-lifts of γ, γ', respectively. Assume that $\widehat{\gamma}(0) = \widehat{\gamma}'(0)$. Then $\widehat{\gamma}(1) = \widehat{\gamma}'(1)$, and $\widehat{\gamma}$ is endpoint fixed homotopic to $\widehat{\gamma}'$.

Proof: Let $I^2 := I \times I$ and $L := (I \times \{0\}) \cup (\{0\} \times I)$. By the preceding proposition, π has unique $(\{0\}, I)$ lifting and also has unique (L, I^2) lifting. Let $x_0 := \widehat{\gamma}(0) = \widehat{\gamma}'(0)$. Since γ and γ' are endpoint fixed homotopic, let $h : I^2 \to Y$ be a continuous map such that:

(1) $h(0, \cdot) = \gamma$;
(2) $h(1, \cdot) = \gamma'$;
(3) $h(\cdot, 0)$ is constant; and
(4) $h(\cdot, 1)$ is constant.

We have $\pi(x_0) = \pi(\widehat{\gamma}(0)) = \gamma(0) = h(0,0)$. Then, by (3), for all $s \in I$, we have $h(s,0) = \pi(x_0)$. By unique (L, I^2) lifting, choose a π-lift $\widehat{h} : I^2 \to X$ of $h : I^2 \to Y$ such that

(5) $\widehat{h}(0, \cdot) = \widehat{\gamma}$; and
(6) for all $s \in I$, we have $\widehat{h}(s, 0) = x_0$.

By (6), $\widehat{h}(1,0) = x_0$. By (2), $\widehat{h}(1, \cdot)$ is a π-lift of γ'. So, since $\widehat{\gamma}'$ is another π-lifts of γ', and since $\widehat{h}(1,0) = x_0 = \widehat{\gamma}'(0)$, it follows, from unique $(\{0\}, I)$ lifting, that $\widehat{h}(1, \cdot) = \widehat{\gamma}'$. In particular, we have $\widehat{h}(1,1) = \widehat{\gamma}'(1)$.

By (4), $\widehat{h}(\cdot, 1)$ is a π-lift of a constant, so, by Exercise 4F, we conclude that $\widehat{h}(\cdot, 1)$ is constant. In particular, we have $\widehat{h}(0,1) = \widehat{h}(1,1)$.

By (5), $\widehat{h}(0,1) = \widehat{\gamma}(1)$. Then $\widehat{\gamma}(1) = \widehat{h}(0,1) = \widehat{h}(1,1) = \widehat{\gamma}'(1)$. Moreover, $\widehat{h}$ is an endpoint fixed homotopy from $\widehat{\gamma}$ to $\widehat{\gamma}'$ in X. **QED**

Alternate Proof: Let $I^2 := I \times I$ and $L := (I \times \{0\}) \cup (\{0\} \times I)$. Let $U := L \cup (\{1\} \times I)$. By the preceding proposition, π has unique $(\{0\}, I)$ lifting and also has unique (L, I^2) lifting. It is an unassigned exercise to show that (L, I^2) is isomorphic in the category of topological pairs to (U, I^2). Then π also has unique (U, I^2) lifting.

Let $x_0 := \widehat{\gamma}(0) = \widehat{\gamma}'(0)$. Since γ and γ' are endpoint fixed homotopic, let $h : I^2 \to Y$ be a continuous map such that:

(1) $h(0, \cdot) = \gamma$;
(2) $h(1, \cdot) = \gamma'$;
(3) $h(\cdot, 0)$ is constant; and
(4) $h(\cdot, 1)$ is constant.

We have $\pi(x_0) = \pi(\widehat{\gamma}(0)) = \gamma(0) = h(0,0)$. Then, by (3), for all $s \in I$, we have $h(s,0) = \pi(x_0)$. By unique (U, I^2) lifting, choose a π-lift $\widehat{h} : I^2 \to X$ of $h : I^2 \to Y$ such that

(5) $\widehat{h}(0, \cdot) = \widehat{\gamma}$;

(6) $\widehat{h}(1, \cdot) = \widehat{\gamma}'$; and

(7) for all $s \in I$, we have $\widehat{h}(s, 0) = x_0$.

By (4), $\widehat{h}(\cdot, 1)$ is a π-lift of a constant, so, by Exercise 4F, we conclude that $\widehat{h}(\cdot, 1)$ is constant. In particular, we have $\widehat{h}(0,1) = \widehat{h}(1,1)$. Then, using (5) and (6), we get $\widehat{\gamma}(1) = \widehat{h}(0,1) = \widehat{h}(1,1) = \widehat{\gamma}'(1)$. **QED**

Recall that we asked the following question:

Question: Let A be a topological space and let A_0 be a subset of A. Is it automatically true that local unique (A_0, A) lifting implies unique (A_0, A) lifting?

We can now answer that question in the negative:

EXERCISE 5A: Define $\pi_0 : \mathbb{R} \to S^1$ by $\pi_0(t) = (\cos(2\pi t), \sin(2\pi t))$. Let $x := (1,0) \in S^1$. Show that π_0 has local unique $(\{x\}, S^1)$ lifting, but does not have unique $(\{x\}, S^1)$ lifting.

EXERCISE 5B: Let X be a topological space and let $x, x', x'' \in X$. Let $\gamma, \gamma' \in P_x^{x'}(X)$ and let $\delta, \delta' \in P_{x'}^{x''}(X)$. Assume that γ is endpoint fixed homotopic to γ'. Assume that δ is endpoint fixed homotopic to δ'. Show that $\gamma \| \delta$ is endpoint fixed homotompic to $\gamma' \| \delta'$.

Recall that, for all $\gamma \in P_x^x(X)$, we have defined $[\gamma]$ to be the endpoint fixed homotopy class of γ, *i.e.*, the set of all elements of $P_x^x(X)$ which are endpoint fixed homotopic to γ. Recall that $\pi_1(X, x) := \{[\gamma] \,|\, \gamma \in P_x^x(X)\}$ is the set of all endpoint fixed homotopy classes of elements of $P_x^x(X)$.

Definition. Let X be a topological space and let $x \in X$. For all $[\gamma], [\delta] \in \pi_1(X, x)$, we define $[\gamma][\delta] := [\gamma \| \delta]$.

Exercise 5B tells us that this multiplication on $\pi_1(X, x)$ is well-defined, *i.e.*, that if $[\gamma] = [\gamma']$ and if $[\delta] = [\delta']$, then $[\gamma \| \delta] = [\gamma' \| \delta']$.

EXERCISE 5C: Let X be a topological space and let $x \in X$. Let $\Gamma := \pi_1(X, x)$. Show that under the multiplication defined above, Γ is a group. That is, show all of the following:

(1) For all $a, b, c \in \Gamma$, we have $(ab)c = a(bc)$.

(2) There exists a unique $e \in \Gamma$ such that, for all $a \in \Gamma$, we have $ea = ae = a$.

(3) For all $a \in \Gamma$, there exists $b \in \Gamma$ such that $ab = ba = e$.

In Exercise 3E, given an arrow $f : (X, x) \to (Y, y)$ in {pointed topological spaces}, you were asked to define $f_* = \pi_1(f) : \pi_1(X, x) \to \pi_1(Y, y)$. The definition remains the same: $(\pi_1(f))([\gamma]) = [f \circ \gamma]$. It is now an unassigned exercise to show that this map $\pi_1(f)$ is a homomorphism of groups, *i.e.*, is an arrow in the category of groups.

Recall that we defined the functor π_1 : {pointed topological spaces} $\to$ {sets}. We have now redefined it as a functor π_1 : {pointed topological spaces} $\to$ {groups}. From now on, if we want to think of π_1 as a functor into {sets}, then we will write π_1^{set}.

Define functors α, ω : {pathed topological spaces} $\to$ {pointed topological spaces} by $\alpha(X,\gamma) = (X,\gamma(0))$ and $\omega(X,\gamma) = (X,\gamma(1))$. The argument of Exercise 3G now shows that the functors $\pi_1 \circ \alpha$: {pathed topological spaces} $\to$ {groups} and $\pi_1 \circ \omega$: {pathed topological spaces} $\to$ {groups} are equivalent functors. From this we conclude:

Corollary. Let X be a path-connected topological space. Let $x, x' \in X$. Then the groups $\pi_1(X,x)$ and $\pi_1(X,x')$ are isomorphic.

The argument of Exercise 3H shows:

Lemma. Let X and Y be topological spaces, let $x \in X$ and let $f, g : X \to Y$ be continuous maps. Assume that f and g are homotopic. Let $\widetilde{f}_* : \pi_1(X,x) \to \pi_1(Y,f(x))$ be the group homomorphism induced by f and let $\widetilde{g}_* : \pi_1(X,x) \to \pi_1(Y,g(x))$ be the group homomorphism induced by g. Then there is a group isomorphism $\sigma : \pi_1(Y,f(x)) \to \pi_1(Y,g(x))$ such that $\sigma \circ \widetilde{f}_* = \widetilde{g}_*$.

Corollary. Let X and Y be topological spaces. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Assume that $g \circ f$ is homomotopic to the identity $\mathrm{id}_X : X \to X$. Let $x \in X$. Let $\widetilde{f}_* : \pi_1(X,x) \to \pi_1(Y,f(x))$ be the group homomorphism induced by f. Let $\widetilde{g}_* : \pi_1(X,f(x)) \to \pi_1(Y,g(f(x)))$ be the group homomorphism induced by g. Then there is a group isomorphism $\sigma : \pi_1(X,x) \to \pi_1(X,g(f(x)))$ such that $\widetilde{f}_* \circ \widetilde{g}_* = \sigma$.

Proof: This follows from the preceding lemma, with Y replaced by X, f replaced by $g \circ f$ and g replaced by by id_X. **QED**

Corollary. Let X and Y be topological spaces. Assume that X and Y are homotopy equivalent. Then there exist $x_0 \in X$ and $y_0 \in Y$ such that $\pi_1(X,x_0)$ is isomorphic to $\pi_1(Y,y_0)$.

Proof: Let $x \in X$. Define $y := f(x)$, $x' := g(y)$ and $y' := f(x')$. Let $\widetilde{f}_* : \pi_1(X,x) \to \pi_1(Y,y)$ be induced by f. Let $\widetilde{g}_* : \pi_1(Y,y) \to \pi_1(X,x')$ be induced by g. Let $f'_* : \pi_1(X,x') \to \pi_1(Y,y')$ be induced by f. We will show that $\widetilde{g}_*$ is an isomorphism.

By the preceding corollary, choose group isomorphisms $\sigma : \pi_1(X,x) \to \pi_1(X,x')$ and $\tau : \pi_1(Y,y) \to \pi_1(Y,y')$ such that $\widetilde{g}_* \circ \widetilde{f}_* = \sigma$ and such that $f'_* \circ \widetilde{g}_* = \tau$.

Then $\widetilde{g}_* \circ (\widetilde{f}_* \circ \sigma^{-1})$ is the identity on $\pi_1(X,x')$. Moreover, $(\tau^{-1} \circ f'_*) \circ \widetilde{g}_*$ is the identity on $\pi_1(Y,y)$. We are now done, by applying the next remark, with $\mathcal{C}$ replaced by {groups}, with g replaced by $\widetilde{g}_*$, with a replaced by $\tau^{-1} \circ f'_*$ and with b replaced by $\widetilde{f}_* \circ \sigma^{-1}$. **QED**

Remark. Let $\mathcal{C}$ be a category and let $g : C \to C'$ be an arrow in $\mathcal{C}$. Assume that there exist arrows $a, b : C' \to C$ in $\mathcal{C}$ such that $ag = \mathrm{id}_C$ and $gb = \mathrm{id}_{C'}$. Then g is an isomorphism.

We are saying that, if an arrow in a category has a left inverse and a right inverse, then it has a two-sided inverse. In fact, we'll show that the two one-sided inverses are equal:

Proof: We have $a = a(\mathrm{id}_{C'}) = a(gb) = (ag)b = (\mathrm{id}_C)b = b$. Then $ag = \mathrm{id}_C$ and $ga = \mathrm{id}_{C'}$, so a is an inverse for g, so g is an isomorphism. **QED**

We will soon be spending quite a bit of time developing further our ability to compute π_1 on (pointed) topological spaces, but we break off from that momentarily to make some general comments.

We have now shown that $\mathbb{R}^1$ and $\mathbb{R}^2$ are not homeomorphic and that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic. Eventually we'll develop enough algebraic topological tools to show that, if $m, n \geq 0$ are integers then: $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ iff $m = n$. There are, in fact, a number of possible approaches to this, but they mostly rely on a good understanding of homology theory, which is a major topic in the last half of this course.

Eventually, we'll develop enough tools to show both of the following:

(•) For any $C \subseteq \mathbb{R}^2$ if C is homeomorphic to S^1, then $\mathbb{R}^2 \backslash C$ is not connected. (This is sometimes called the "Jordan Curve Theorem".)

(•) For any $C \subseteq \mathbb{R}^3$, if C is homeomorphic to S^1, then $\mathbb{R}^3 \backslash C$ is connected.

This gives an alternative way to show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$, and is somehow in the same spirit as our proof that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

We'll also be able to show:

(•) For any $L \subseteq \mathbb{R}^2$ if L is closed in $\mathbb{R}^2$ and if L is homeomorphic to $\mathbb{R}^1$, then $\mathbb{R}^2 \backslash L$ is not connected.

(•) For any $L \subseteq \mathbb{R}^3$, if L is closed in $\mathbb{R}^3$ and if L is homeomorphic to $\mathbb{R}^1$, then $\mathbb{R}^3 \backslash L$ is connected.

This also shows that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.

We'll also be able to show:

(•) For any $S \subseteq \mathbb{R}^3$ if S is homeomorphic to S^2, then $\mathbb{R}^2 \backslash S$ is not connected.

(•) For any $S \subseteq \mathbb{R}^4$, if S is homeomorphic to S^2, then $\mathbb{R}^4 \backslash S$ is connected.

This gives an alternative way to show that $\mathbb{R}^3$ is not homeomorphic to $\mathbb{R}^4$.

Definition. A topological space X is said to be **simply connected** if it is path-connected, and if, for all $x \in X$, we have that $|\pi_1(X, x)| = 1$, *i.e.*, if we have that $\pi_1(X, x)$ is the trivial group.

We'll also be able to show:

(•) For any $C \subseteq \mathbb{R}^3$ if C is homeomorphic to S^1, then $\mathbb{R}^3 \backslash C$ is not simply connected.

Here's a challenge: Show

(•) For any $C \subseteq \mathbb{R}^4$ if C is homeomorphic to S^1, then $\mathbb{R}^4 \backslash C$ is simply connected.

Given the last two results, we get yet another proof that $\mathbb{R}^3$ is not homeomorphic to $\mathbb{R}^4$.

To be efficient about it, much of this work needs to wait until the development of homology theory, which comes later. For now, we return our focus to π_1.

Our next goal is to show that $\pi_1(S^1)$ is isomorphic to the additive group $\mathbb{Z}$.

Sketch of Proof: Let $I := [0, 1]$ and let $x := (1, 0) \in S^1$. Define $\pi_0 : \mathbb{R} \to S^1$ by $\pi_0(t) = (\cos(2\pi t), \sin(2\pi t))$. Using that π_0 has unique $(\{0\}, I)$ lifting, for all $\gamma \in P_x^x(S^1)$, let $\widehat{\gamma} : I \to \mathbb{R}$ be the unique π_0-lift of γ such that $\widehat{\gamma}(0) = 0$. We now define a map $\Phi : \pi_1(S^1, x) \to \mathbb{Z}$ by $\Phi([\gamma]) = \widehat{\gamma}(1)$. We leave it as an unassigned exercise to show that Φ is well-defined. (Hint: Use the theorem following Exercise 4F.) We also leave it as an unassigned exercise to show that Φ is a group isomorphism. **QED**

After filling in the details, one has the interesting situation where covering space theory has been used to compute π_1 of a pointed topological space.

It is our intention to generalize the above proof, in the hopes that we may be able to use covering space theory as a general tool for computing π_1.

Definition. A topological group is

(1) a group G; and

(2) a Hausdorff topology on G

such that $(x, y) \mapsto xy : G \times G \to G$ and $x \mapsto x^{-1} : G \to G$ are both continuous.

The arrows in the category of topological groups are continuous homomorphisms.

First example of a topological group is the additive group $\mathbb{R}$. Another example is the multiplicative group of matrices $\mathrm{SL}(2, \mathbb{R}) := \{g \in \mathbb{R}^{2\times 2} \mid \det(g) \neq 0\}$. (This is given the inherited topology from the vector space topology on the set $\mathbb{R}^{2\times 2}$ of 2×2 matrices with real entries.

EXERCISE 5D: Let G be a topological group, let 1 denote the identity in G and let V be an open neighborhood of 1 in G. Show that there exists an open neighborhood U of 1 in G such that $UU^{-1} \subseteq V$, *i.e.*, such that, for all $a, b \in U$, we have $ab^{-1} \in V$.

Note that we can replace UU^{-1} in the above exercise by various similar expresssions, and the result remains true, with much the same proof. For example, we could use UU, often denoted U^2. Or we could use $U^{-1}U$ or $U^2U^{-1}U^{-2}UU^{-1}$. Or any product of postive and negative powers of U.

In this course "action" will always mean "left action" unless otherwise specified.

Definition. Let G be a topological group acting on a topological space X. We say that the action is **continuous** if the action map $(g, x) \mapsto gx : G \times X \to X$ is continuous.

This is equivalent to saying that the "secondary" action map $(g, x) \mapsto (g, gx) : G \times X \to G \times X$ is continuous. (The use of the word "secondary" there is my choice. Either of these two maps is, at times, called the action map, but I want to distinguish between them to avoid confusion.)

A simple example of a continuous action is obtained by taking a topological space G and letting G act on itself by multiplication. That is, the action map is just multiplication $(g, x) \mapsto gx : G \times G \to G$.

Another equally simple example is as follows: Given a topological group G and a topological space Y, let G act on $G \times Y$ via the action $g(a, y) = (ga, y)$.

Definition. Let G be a topological group. A **topological G-space** is a topological space X, together with a continuous G-action on X.

Recall that if a group G acts on a set X then a subset $X_0 \subseteq X$ is said to be **G-invariant** if $GX_0 \subseteq X_0$, *i.e.*, if, for all $g \in G$, for all $x \in X_0$, we have $gx \in X_0$.

In the category of topological G-spaces, the arrows are continuous G-equivariant maps. That is, if X and Y are topological G-spaces, then an arrow from X to Y is a continuous map $f : X \to Y$ such that, for all $g \in G$, for all $x \in X$, we have $f(gx) = g(f(x))$.

Note that, if X is a topological G-space and if $X_0 \subseteq X$ is nonempty and G-invariant, then X_0 is a topological G-space (under the inherited topology and restricted G-action) and the inclusion $X_0 \to X$ is an arrow in the category of topological G-spaces.

I remark again that, if G is a topological group, and if Y is a topological space, then $G \times Y$ is a topological G-space, where the action is given by $g(a, y) = (ga, y)$.

Definition. Let G be a topological group and X a topological G-space. We say that X is **trivially principal** if there is a topological space Y such that X is isomorphic, in the category of topological G-spaces, to $G \times Y$. We say that X is **principal** if, for all $x \in X$, there is a G-invariant open neighborhood X_0 of x in X such that X_0 is trivially principal.

So one might use "locally trivially principal" as a synonym for "principal".

If G is a topological group acting continuously on a topological space X, then we'll say that the G action is **principal** if X is a principal G-space. Similarly, we'll say that the G action is **trivially principal** if X is a trivially principal G-space. By a **principal G-space**, we mean a topological G-space that is principal. By a **trivially principal G-space**, we mean a topological G-space that is trivially principal.

Let the additive group $\mathbb{Z}$ act on $\mathbb{R}$ by: the action of n on t yields $n + t$. It is not hard to show that, with this action, $\mathbb{R}$ becomes a principal $\mathbb{Z}$-space which is not trivially principal.

A **discrete group** is a topological group whose topology is discrete (which means that every subset of the group is open). Note that there is a forgetful functor $\{\text{discrete groups}\} \to \{\text{groups}\}$. Conversely, given any group, we can give it the discrete topology, making it into a discrete group. This gives a functor $\{\text{groups}\} \to \{\text{discrete groups}\}$. These two functors are inverses of one another, and so the category $\{\text{groups}\}$ is isomorphic to the category $\{\text{discrete groups}\}$. That is, expertise in group theory is exactly the same as expertise in discrete group theory.

Recall some terminology and notation from group theory. Given a group G acting on a set X, we let $G\backslash X := \{Gx \mid x \in X\}$ be the set of orbits of G on X. The canonical map $X \to G\backslash X$ is the map $x \mapsto Gx$.

When G is a topological group and X is a topological G-space, then $G\backslash X$ automatically has a topology, namely, the quotient topology, *i.e.*, the finest topology making the canonical map $X \to G\backslash X$ continuous.

EXERCISE 5E: Let Γ be a discrete group, and let X be a principal Γ-space. Show that the canonical map $X \to \Gamma\backslash X$ is a covering map.

Theorem. Let Γ be a discrete group and let X be a principal Γ-space. Assume that X is simply connected. Then, for all $y \in \Gamma\backslash X$, we have that the groups $\pi_1(\Gamma\backslash X, y)$ and Γ are isomorphic.

Sketch of Proof: Let $Y := \Gamma\backslash X$. We wish to show that $\pi_1(Y, y)$ is isomoprhic to Γ.

Let $I := [0, 1]$. Let $\pi : X \to \Gamma\backslash X$ be the canonical map. Then π is surjective. Choose $x \in X$ such that $\pi(x) = y$. By Exercise 5E, Γ is a covering map, and therefor has unique $(\{0\}, I)$ lifting. For all $\alpha \in P_y^y(Y)$, let $\widehat{\alpha} : I \to \mathbb{R}$ be the unique π-lift of α such that $\widehat{\alpha}(0) = x$. Note that $\pi(\widehat{\alpha}(1)) = \alpha(1) = y$ and that $\pi(\widehat{\alpha}(0)) = \pi(x) = y$, so there is a unique $\gamma_\alpha \in \Gamma$ such that $\widehat{\alpha}(1) = \gamma_\alpha(\widehat{\alpha}(0))$. (Unassigned exercise: Explain existence and uniqueness in the last sentence.) We now define a map $\Phi : \pi_1(Y, y) \to \Gamma$ by $\Phi([\alpha]) = \gamma_{\widehat{\alpha}}$. We leave it as an unassigned exercise to show that Φ is well-defined. (Hint: Use the theorem following Exercise 4F.) We also leave it as an unassigned exercise to show that Φ is a group isomorphism. (Note: Simple connectedness of X is needed in the proof that Φ is injective.) **QED**

A more colloquial way of saying the above theorem is: If a discrete group Γ acts principally on a simply connected topological space X, then $\pi_1(\Gamma\backslash X)$ is isomorphic to Γ. Note that, since $\Gamma\backslash X$ is path-connected, the fundamental group does not depend on the basepoint, up to isomorphism.

Question: Is the preceding theorem true if we drop the assumption that X be simply connected?

Answer: No. Let $\Gamma := \mathbb{Z}/2\mathbb{Z}$ act on $X := \mathbb{R}/(2\mathbb{Z})$ by: the action of 1 on $t + (2\mathbb{Z})$ yields $t + 1 + (2/Z)$. Let $y \in Y := \Gamma\backslash X$. We have $Y = \mathbb{R}/Z$, so $\pi_1(Y, y) = \mathbb{Z}$. So $\pi_1(Y, y)$ is not isomorphic to Γ.

Question: Is the preceding theorem true if we drop the assumption that the Γ-action on X is principal?

We will answer this last question below.

For any group G, we will let 1_G denote the identity element of G.

EXERCISE 5F: Let Γ be a discrete group acting continuously on a topological space X. Show that the Γ-action on X is principal iff: for every $x \in X$, there exists an open neighborhood U of x in X such that, for all $\gamma \in \Gamma\backslash\{1_\Gamma\}$, we have $(\gamma U) \cap U = \emptyset$.

EXERCISE 5G: Let G be a topological group and let Γ be a discrete subgroup of G. Let Γ act on G by left multiplication, *i.e.*, the result of γ acting on g is the product γg. Show that this Γ-action on G is principal. (Hint: Use Exercise 5F and Exercise 5D.)

Let the additive group $\mathbb{Z}$ act on $\mathbb{R}$ by the rule: n acting on t yields $n + t$. By Exercise 5F (adapted for additive groups and with G replaced by $\mathbb{R}$), we see that this $\mathbb{Z}$-action on $\mathbb{R}$ is principal. The quotient topological space $\mathbb{Z}\backslash\mathbb{R}$ is the same as the topological space $\mathbb{R}/\mathbb{Z}$ of cosets of $\mathbb{Z}$ in $\mathbb{R}$. By the preceding theorem, we conclude that $\pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$.

The next three exercises, taken togther, show that $\mathbb{R}/\mathbb{Z}$ is homeomoprhic to S^1, so we obtain another proof that $\pi_1(S^1) \cong \mathbb{Z}$.

EXERCISE 5H: Show that $\mathbb{R}/\mathbb{Z}$ is compact.

EXERCISE 5I: Show that there is a continous bijection $\mathbb{R}/\mathbb{Z} \to S^1$.

EXERCISE 5J: Show that, if X and Y are topological spaces, if X is compact, and if $f : X \to Y$ is a continuous bijection, then $f : X \to Y$ is a homeomorphism.

Definition. Let G be a group. Then a G-**set** is a set X, together with a G-action on X.

The arrows in the category of G-sets are G-equivariant functions. Note that, for every group homomorphism $\phi : G \to H$, we get a functor $\mathcal{F}_\phi\{\text{H}-\text{sets}\} \to \{\text{G}-\text{sets}\}$, given by: For any H-set X, $\mathcal{F}_\phi(X)$ is the G-set whose underlying set is X and where the G-action on X is given by: g acting on x yields $(f(g))x$. (Question: What is the effect of $\mathcal{F}_\phi$ on arrows?)

Some notation: When a group G acts on a set X and when $x \in X$, we define $\mathrm{Stab}_G(x) = \{g \in G \,|\, gx = x\}$. This subgroup of G is called the **stabilizer of x in G**.

A G-set X is **transitive** if $|G\backslash X| = 1$, *i.e.*, if there is exactly one orbit in X. For any subgroup G_0 of G, we let G act on G/G_0 by g acting on $g'G_0$ yields $gg'G_0$. This action is transitive. It is not hard to show that if X is a transitive G-space, if $x \in X$ and if $G_0 := \mathrm{Stab}_G(x)$, then X is isomorphic to G/G_0 in the category $\{G - \text{sets}\}$. This can be thought of as a "classification of transitive G-sets".

Note that any G-set breaks up into orbits, and the G-action on each orbit is transitive. Thus any G-set is isomorphic (in the category $\{G - \text{sets}\}$) to a disjoint union of transitive G-sets. Combined with the earlier classification of transitive G-sets, we have no classified *all* G-sets, up to isomorphism.

Definition. Let G be a group acting on a set X. The G-action is said to be **free** if there is a set Y such that, in the category of G-sets, X is isomorphic to $G \times Y$, where the G-action on $G \times Y$ is given by $g(a, y) = (ga, y)$.

Fact. Let G be a group acting on a set X. Then the G-action on X is free iff, for all $x \in X$, we have $\mathrm{Stab}_G(x) = \{1_G\}$.

More colloquially put, "an action is free iff all stabilizers are trivial". Or: "an action is free iff nothing stabilizes anything, except for the identity, which stabilizes everything".

EXERCISE 5K: Show that any principal action is free. That is, if G is a topological group acting continuously and principally on a topological space X, show that the G-action on X is free.

Question: Is the converse true? That is, if G is a topological group acting continuously and freely on a topological space X, does it follow that G-action on X is principal?

Answer: No. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and let $\mathbb{Z}$ act on S^1 by: the action of 1 on $(\cos(2\pi t), \sin(2\pi t))$ yields $(\cos(2\pi(t + \alpha)), \sin(2\pi(t + \alpha)))$. This action is sometimes called the "irrational rotation" action on the circle. We leave it as an unassigned exercise to show that this action is free and that every orbit is dense. Since orbits are dense, it follows from Exercise 5F that the Γ-action on X is not principal.

We are now in a position to answer an earlier question.

Question: Is the preceding theorem true if we drop the assumption that the Γ-action on X is principal?

Answer: Let $\Gamma := Z$ act on $X := S^1$ by irrational rotation, as above. Let $Y := \Gamma\backslash X$ and let $y \in Y$. Because every orbit is dense, it follows that Y is indiscrete, and therefore that $\pi_1(Y, y)$ is trivial. Then $\pi_1(Y, y)$ is not isomorphic to Γ.

I don't know a counterexample in which the quotient space is Hausdorff. That is, I don't know the answer to the following question:

Question: Let X be a simply connected topological space and let Γ be a discrete group. Let Γ act freely on X. Assume that $Y := \Gamma\backslash X$ is Hausdorff. Let $y \in Y$. Does it follow that $\pi_1(Y, y)$ is isomorphic to Γ?

As we have seen, principal implies free, but free does not always imply principal. However, the next exercise gives one situation in which free does imply principal.

EXERCISE 5L: Let Γ be a finite discrete group acting continuously and freely on a Hausdorff topological space X. Then show that the Γ-action on X is principal. (Hint: You may use, without proof, the fact that, for any integer $n \geq 2$, for any n distinct points in a Hausdorff topological space, one may choose open neighborhoods of the n points which are pairwise disjoint. Now use Exercise 5F.)

Let $\Gamma := \mathbb{Z}/2\mathbb{Z}$. Then Γ is the additive cyclic group with two elements. We label the identity element as 0 and the other as 1. Give Γ the discrete topology, so that Γ is a finite discrete group. Let $n \geq 1$ be an integer. Let Γ act on S^n by: 1 acting on p yields $-p$. We define $\mathbb{R}P^n := \Gamma\backslash S^n$.

By the preceding exercise, the Γ-action on S^n, being free, must be principal. We already proved that S^2 is simply connected, and the same argument works to show that, for all $n \geq 2$, S^n is simply connected. We can appeal to the preceding theorem to show that, if $n \geq 2$, then $\pi_1(\mathbb{R}P^n) \cong \Gamma = \mathbb{Z}/2\mathbb{Z}$.

Definition. Let Γ be a group and let X be a Γ-set. Let $x \in X$. The **orbit map at** x is the map $\gamma \mapsto \gamma x : \Gamma \to X$.

Let $f : \Gamma \to X$ be the orbit map at x, defined by $f(\gamma) = \gamma x$. Note that the image $f(\Gamma)$ is the orbit Γx of x. Note that $f^{-1}(x) = \mathrm{Stab}_\Gamma(x)$.

EXERCISE 6A: Let Γ be a group acting on a set X. Show that the Γ-action on X is free iff every orbit map is injective.

We now return back to the last theorem, and give a more detailed proof.

Theorem. Let Γ be a discrete group and let X be a principal Γ-space. Assume that X is simply connected. Then, for all $y \in \Gamma\backslash X$, we have that the groups $\pi_1(\Gamma\backslash X, y)$ and Γ are isomorphic.

Proof: Let $Y := \Gamma\backslash X$. Let $\pi : X \to Y$ be the canonical map. Fix $x \in \pi^{-1}(y)$. Then $\Gamma x = \pi^{-1}(y)$.

Let $f : \Gamma \to X$ be the orbit map at x, defined by $f(\gamma) = \gamma x$. By Exercise 6A, we see that f is injective. Moreover, $f(\Gamma) = \Gamma x = \pi^{-1}(y)$. Then $f : \Gamma \to \pi^{-1}(y)$ is a bijection. Let $f^{-1} : \pi^{-1}(y) \to \Gamma$ be its inverse. (Intuitively, because the Γ-action on X is free, every Γ-orbit "looks like" Γ; moreover the orbits are exactly the fibers of the canonical map. So $\pi^{-1}(y)$, being a fiber of the canonical map, must "look like" Γ, and f^{-1} is the map that makes this precise.)

For any $\alpha \in P_y^y(Y)$, let $[\alpha]$ denote the endpoint fixed homotopy class of α. Then $\pi_1(Y,y) = \{[\alpha] \,|\, \alpha \in P_y^y(Y)\}$. Similarly, for any $\alpha \in P_x^x(X)$, let $[\alpha]$ denote the endpoint fixed homotopy class of x, so that $\pi_1(X,x) = \{[\alpha] \,|\, \alpha \in P_x^x(X)\}$.

By Exercise 5E, we see that $\pi : X \to Y$ is a covering map. Then, by an earlier theorem, π has unique $(\{0\}, I)$ lifting. For every $\alpha \in P_y^y(Y)$, let $\widehat{\alpha} : I \to X$ be the unique π-lift of α such that $\widehat{\alpha}(0) = x$. By an earlier theorem, for all $\alpha, \beta \in P_y^y(Y)$, if α is endpoint fixed homotopic to β, then $\widehat{\alpha}(1) = \widehat{\beta}(1)$. For all $[\alpha] \in \pi_1(Y,y)$, let $t_{[\alpha]} := \widehat{\alpha}(1)$; then $\pi(t_{[\alpha]}) = \pi(\widehat{\alpha}(1)) = \alpha(1) = y$, so $t_{[\alpha]} \in \pi^{-1}(y)$.

Define $\psi : \pi_1(Y,y) \to \Gamma$ by $\psi([\alpha]) = f^{-1}(t_{[\alpha]})$. We will show that this map is a homomorphism of groups and that it is injective. We will leave the proof of surjectivity as an unassigned exercise.

Proof that ψ is a homomorphism: Fix $\alpha, \beta \in P_y^y(Y)$. We wish to show that

$$(\psi([\alpha]))(\psi([\beta])) = \psi([\alpha][\beta]).$$

Let $\gamma := \psi([\alpha])$, let $\delta := \psi([\beta])$. As $[\alpha][\beta] = [\alpha \| \beta]$, we must show that $\gamma\delta = (\psi([\alpha \| \beta])$.

We have $\widehat{\alpha}(1) = t_{[\alpha]} = f(\psi([\alpha])) = f(\gamma) = \gamma x$. Similarly, $\widehat{\beta}(1) = \delta x$.

Define $\gamma\widehat{\beta} : I \to X$ by $(\gamma\widehat{\beta})(t) = \gamma(\widehat{\beta}(t))$. Then $(\gamma\widehat{\beta})(0) = \gamma(\widehat{\beta}(0)) = \gamma x = \widehat{\alpha}(1)$, so $\widehat{\alpha} \| (\gamma\widehat{\beta})$ is defined. We have $\pi \circ (\gamma\widehat{\beta}) = \pi \circ \widehat{\beta} = \beta$, so $\pi \circ (\widehat{\alpha} \| (\gamma\widehat{\beta})) = \alpha \| \beta$. Moreover, $(\widehat{\alpha} \| (\gamma\widehat{\beta}))(0) = \widehat{\alpha}(0) = x$. Then $\widehat{\alpha \| \beta} = \widehat{\alpha} \| (\gamma\widehat{\beta})$. Then $t_{[\alpha \| \beta]} = \widehat{\alpha \| \beta}(1) = (\widehat{\alpha} \| (\gamma\widehat{\beta}))(1) = (\gamma\widehat{\beta})(1) = \gamma(\widehat{\beta}(1)) = \gamma(\delta x) = \gamma\delta x = f(\gamma\delta)$. Then $\psi([\alpha \| \beta]) = f^{-1}(t_{[\alpha \| \beta]}) = \gamma\delta$, *End of proof that ψ is a homomorphism.*

Proof that ψ is injective: Let $c_x : I \to X$ be defined by $c_x(t) = x$ and let $c_y : I \to Y$ be defined by $c_y(t) = y$. Then $[c_x]$ is the identity element of $\pi_1(X,x)$ and $[c_y]$ is the identity element of $\pi_1(Y,y)$. Let $\alpha \in P_y^y(Y)$ and assume that $\psi([\alpha]) = 1_\Gamma$. We wish to show that $[\alpha] = [c_y]$.

We have $\widehat{\alpha}(0) = x$, by definition of $\widehat{\alpha}$. We have $\widehat{\alpha}(1) = t_{[\alpha]} = f(\psi([\alpha])) = f(1_\Gamma) = x$. Then $\widehat{\alpha} \in P_x^x(X)$. However, X is simply connected, so $\pi_1(X,x)$ is trivial. Then $[\widehat{\alpha}] = [c_x]$. That is, $\widehat{\alpha}$ is endpoint fixed homotopic to c_x. Then $\pi \circ \widehat{\alpha}$ is endpoint fixed homotopic to $\pi \circ c_x$. That is, α is endpoint fixed homotopic to c_y. That is, $[\alpha] = [c_y]$. *End of proof that ψ is injective.* **QED**

So, given a path-connected topological space Y we have a strategy for computing its fundamental group: Find a principal action of a discrete group Γ on a simply connected topological space whose quotient is (homeomorphic to) Y. If we can do this, then the preceding result immediately gives us that $\pi_1(Y) \cong \Gamma$.

This strategy worked for computing $\pi_1(S^1)$ and $\pi_1(\mathbb{R}P^2)$.

A reasonable question is: Does it *always* work? That is:

Question: Given a path-connected topological space Y, can we always find a simply connected topological space X, a discrete group Γ and a principal action of Γ on X such that $\Gamma \backslash X$ is homeomorphic to Y?

We explore this question next.

Definition. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. A **deck transformation** of π is an isomorphism $(X, \pi) \to (X, \pi)$ in the category

$$\{\text{topological spaces over } Y\}.$$

That is, it is a homeomorphism $h : X \to X$ such that $\pi \circ h = \pi$.

EXERCISE 6B: Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. Let $\Gamma := \{\text{deck transformations of } \pi\}$. Show that Γ is a group under composition.

We will always consider $\{\text{deck transformations of } \pi\}$ to be a topological group: It is a group under composition, and it is given the discrete topology.

EXERCISE 6C: Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. Let $\Gamma := \{\text{deck transformations of } \pi\}$. Show, for all $y \in Y$, that $\pi^{-1}(y)$ is Γ-invariant.

We give Γ the discrete topology. Note that Γ, by construction, acts continuously on X. Eventually we will show that Γ acts principally on X.

Definition. Let X be a topological space. A **topological space under** X consists of

(1) a topological space Y; and
(2) a continuous map $\pi : X \to Y$.

What are the arrows in the category $\{\text{topological spaces under } X\}$?

Note that, if X is a topological space, if Γ is a group and if Γ acts continuously on X, then there is a canonical map $c : X \to \Gamma \backslash X$, and therefore $(\Gamma \backslash X, c)$ is a topological space under X. We will often just say that $\Gamma \backslash X$ is a topological space under X, the canonical map c being understood.

Definition. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. Let $\Gamma := \{\text{deck transformations of } \pi\}$. We say that π is **regular** or **a regular covering map** if $\Gamma \backslash X$ is isomorphic to (Y, π) in the category $\{\text{topological spaces under } X\}$.

Are all covering maps regular? More starkly, if $\pi : X \to Y$ is a covering map and π is not a homeomorphism, does it always follow that there is a deck transformation of π that is not equal to the identity map $\mathrm{id}_X : X \to X$? The answer to both questions is no, and we will get to this, but a full theory of covering spaces is needed to address these questions properly.

For now, (2) in the following theorem asserts that, when the domain is simply connected, the covering map *is* regular.

Lemma. Let X and Y be path-connected topological spaces. Let $\pi : X \to Y$ be a covering map. Let $\Gamma := \{\text{deck transformations of } \pi\}$. Then

(1) the Γ-action on X is principal; and
(2) if X is simply connected and locally path-connected, then π is regular.

We defer the proof until later.

Definition. Let Y be a path-connected topological space. A **universal cover** of Y is a simply connected topological space X, together with a covering map $X \to Y$.

By the preceding lemma, if the path-connected, locally path-connected topological space Y has a universal cover (X, π), then there is a discrete group Γ and a principal action of Γ on X such that $\Gamma \backslash X$ is homeomorphic to Y. One simply lets $\Gamma :=$ {deck transformations of π}, with the discrete topology. In fact, even more is true: $\Gamma \backslash X$ is isomorphic to (Y, π) in the category of topological spaces under X.

Recall our earlier

Question: Given a path-connected topological space Y, can we always find a simply connected topological space X, a discrete group Γ and a principal action of Γ on X such that $\Gamma \backslash X$ is homeomorphic to Y?

This is now transformed to:

Question: Does every path-connected topological space have a universal cover?

EXERCISE 6D: Let $\pi_0 : \mathbb{R} \to S^1$ be defined by $\pi_0(t) = (\cos(2\pi t), \sin(2\pi t))$. Find all deck transformations of π_0.

EXERCISE 6E: Let X and Y be path-connected topological spaces. Let $\pi : X \to Y$ be a covering map. Let $\Gamma :=$ {deck transformations of π}. Show

(1) the Γ-action on X is free; and
(2) if π is regular, then for all $y \in Y$, the Γ-action on $\pi^{-1}(y)$ is transitive.

Note: The proof of (1) of Exercise 6E actually appears later in these notes, and you may simply copy what I've written, if you wish.

Definition. Let Y be a topological space. We say that Y is **locally simply connected** if, for any $y \in Y$, for any open neighborhood V of y in Y, there is a simply connected open neighborhood U of y in Y such that $U \subseteq V$.

More concisely: Y is locally simply connected if every point has arbitrarily small simply connected open neighborhoods.

EXERCISE 6F: Show that any locally simply connected topological space is locally path-connected.

Let A and B be groups and let $h : A \to B$ be a homomorphism. Recall that h is said to be **trivial** if $h(A) = \{1_B\}$.

EXERCISE 6G: Let Y be a topological space and let $U \subseteq Y$. Assume that U is path-connected. For all $u \in U$, let $i_u : (U, u) \to (Y, u)$ be the arrow in the category {pointed topological spaces} defined by the inclusion map $U \to Y$. Assume, for some $u_0 \in U$, that the map $(i_{u_0})_* : \pi_1(U, u_0) \to \pi_1(Y, u_0)$ is trivial. Show, for all $u \in U$, that the map $(i_u)_* : \pi_1(U, u) \to \pi_1(Y, u)$ is trivial.

Definition. Let Y be a topological space and let $U \subseteq Y$. Assume that U is path-connected. For all $u \in U$, let $i_u : (U, u) \to (Y, u)$ be the arrow in the category {pointed topological spaces} defined by the inclusion map $U \to Y$. We say that U is **relatively simply connected** in Y if: for all $u \in U$, the map $(i_u)_* : \pi_1(U, u) \to \pi_1(Y, u)$ is trivial. (Here, "trivial" means that its image consists of the identity alone.)

Example. Let $o := (0,0) \in \mathbb{R}^2$. Let $D := \{p \in \mathbb{R}^2 \mid d(o,p) \leq 1\}$ be the closed unit disk in the plane. Then $S^1 \subseteq D$. Note that S^1 is not simply connected, but *is* relatively simply connected in D. On the other hand, S^1 is *not* relatively simply connected in $D \backslash \{o\}$.

Definition. Let Y be a topological space. We say that Y is **semi-locally simply connected** if: if, for any $y \in Y$, for any open neighborhood V of y in Y, there is a path-connected open neighborhood U of y in Y such that $U \subseteq V$ and such that U is relatively simply connected in Y.

More concisely: Y is semi-locally simply connected if every point has arbitrarily small relatively simply connected (path-connected) open neighborhoods.

Theorem. Let Y be a connected, locally path-connected Hausdorff topological space. Then Y has a universal cover iff Y is semi-locally simply connected.

We defer the proof until later. Recall our earlier

Question: Does every path-connected topological space have a universal cover?

Note that a locally path-connected topological space is connected iff it is path-connected. So, for locally path-connected topological spaces, the above question is now transformed to:

Question: Is every connected, locally path-connected topological space semi-locally simply connected?

Exercise 7A below asserts that most of the topological spaces that concern us are, in fact, semi-locally simply connected. On the other hand, part of Exercise 7B below is to show that there do exist topological spaces that are not.

EXERCISE 7A:

(1) Show that if a topological space is locally simply connected, then it is semi-locally simply connected.
(2) Show that any locally path-connected, simply connected topological space is semi-locally simply connected.

EXERCISE 7B:

(1) Show that a simply connected topological space is not necessarily locally simply-connected.
(2) Show that there is a connected, locally path-connected topological space that is not semi-locally simply connected.
(3) Show that a path-connected topological space that is semi-locally simply connected need not be locally simply connected.

(*Note:* I do not require that you prove that your counterexamples work, but they must be stated very carefully. You will probably not find these counterexamples without help. Ask around, consult references. Think about the "Hawaiian earring" and the cone over the Hawaiian earring.)

By (2) of Exercise 7A, your answer to (1) of Exercise 7B, if locally path-connected, will also be an answer to (3) of Exercise 7B.

Note that, any semi-locally simply connected implies locally path-connected, which implies locally connected. Note that, for locally path-connected topological spaces, all the path components are open. Thus, if X is a locally path-connected topological space, then: X is connected iff X is path-connected.

Remark. Let X be a path-connected topological space, let Y be a topological space and let $\pi : X \to Y$ be a covering map. Let $\Gamma := \{\text{deck transformations of } \pi\}$. Then the Γ-action on X is free.

Proof: Fix $x_0 \in X$ and $\gamma \in \Gamma$. Assume that $\gamma x_0 = x_0$. We wish to show that $\gamma = 1_\Gamma$. Let $x \in X$. We wish to show that $\gamma x = x$.

Since $\gamma \in \Gamma$, we have $\pi \circ \gamma = \pi$. Since X is path-connected, let $\widehat{\alpha} \in P_{x_0}^x(X)$. Let $I := [0,1]$. Let $\widehat{\beta} := \gamma \circ \widehat{\alpha} : I \to X$. Let $\alpha := \pi \circ \widehat{\alpha} : I \to Y$. Then

$$\pi \circ \widehat{\beta} = \pi \circ (\gamma \circ \widehat{\alpha}) = (\pi \circ \gamma) \circ \widehat{\alpha} = \pi \circ \widehat{\alpha} = \alpha.$$

Then both $\widehat{\alpha}$ and $\widehat{\beta}$ are π-lifts of α. Moreover, $\widehat{\beta}(0) = \gamma(\widehat{\alpha}(0)) = \gamma x_0 = x_0 = \widehat{\alpha}(0)$.

Since $\pi : X \to Y$ is a covering map, it follows that π has unique $(\{0\}, I)$ lifting. So, as $\widehat{\alpha}$ and $\widehat{\beta}$ are both π-lifts of α and as $\widehat{\alpha}(0) = \widehat{\beta}(0)$, we conclude that $\widehat{\alpha} = \widehat{\beta}$. Then $x = \widehat{\alpha}(1) = \widehat{\beta}(1) = \gamma(\widehat{\alpha}(1)) = \gamma x$. **QED**

EXERCISE 7C: Let D be a discrete topological space. Let V be a topological space. Let $d_0 \in D$. Let $U := V \times \{d_0\} \subseteq V \times D$. Let $p : V \times D \to V$ be the projection defined by $p(v, d) = v$. Let $\gamma : V \times D \to V \times D$ be a deck transformation of p. Show, for all $u \in (\gamma U) \cap U$, that $\gamma u = u$.

EXERCISE 7D: Let X and Y be topological spaces. Let $\pi : X \to Y$ be a trivial covering map. Let Γ be a subgroup of $\{\text{deck transformations of } \pi\}$. Assume that Γ acts freely on X. Show that Γ acts principally on X. (Hint: Use Exercise 7C and Exercise 5F.)

Corollary. Let X be a path-connected topological space. Let Y be a topological space. Let $\pi : X \to Y$ be a covering map. Let $\Gamma := \{\text{deck transformations of } \pi\}$. Then the Γ-action on X is principal.

Proof: Fix $x \in X$. We will show that there is a Γ-invariant open neighborhood X_0 of x in X such that the Γ-action on X_0 is principal.

Let Y_0 be an open neighborhood of $\pi(x)$ in Y such that $\pi|[\pi^{-1}(Y_0)] : \pi^{-1}(Y_0) \to Y_0$ is a trivial covering map.

Let $X_0 := \pi^{-1}(Y_0)$ and let $\pi_0 := \pi|X_0$. Then $\pi_0 : X_0 \to Y_0$ is a trivial covering map. Let $\Gamma_0 := \{\text{deck transformations of } \pi_0\}$. Define $R : \Gamma \to \Gamma_0$ by $R(\gamma) = \gamma|X_0$. Let $\Gamma_0 := R(\Gamma)$. By the preceding remark, the Γ-action on X is free. Then $R : \Gamma \to \Gamma_0$ is a group isomorphism.

Then the Γ_0-action on X_0 is free. Then, by Exercise 7D, the Γ_0-action on X_0 is principal. Then, as $R : \Gamma \to \Gamma_0$ is a group isomorphism, it follows that the Γ-action on X_0 is principal. **QED**

EXERCISE 7E: Let X and Y be topological spaces and let $\pi : X \to Y$ be a trivial covering map. Assume that Y is connected. Let Γ be a subgroup of $\{$deck transformations of $\pi\}$. Assume, for some $y_0 \in Y$, that the Γ-action on $\pi^{-1}(y_0)$ is transitive. Show, for all $y \in Y$, that the Γ-action on $\pi^{-1}(y)$ is transitive.

EXERCISE 7F: Let X and Y be topological spaces and let $\pi : X \to Y$ be a covering map. Assume that Y is locally connected. Let $\Gamma := \{$deck transformations of $\pi\}$. Let T be the set of all $y \in Y$ such that the Γ-action on $\pi^{-1}(y)$ is transitive. Show that T is a clopen subset of Y. (Hint: Openness follows from Exercise 7E. Openness of the complement follows from a variant of Exercise 7E, in which "transitive" is replaced by "nontransitive".)

Definition. Let X and Y be topological spaces and let $f : X \to Y$ be a function. We say that f is **open** or that $f : X \to Y$ is an **open mapping** if: for every open subset U of X, we have that $f(U)$ is open in Y. We say that f is a **local homeomorphism** if, for any point $x \in X$, there is an open neighborhood U of x in X such that $f|U : U \to f(U)$ is a homeomorphism.

EXERCISE 7G: Let Γ be a topological group acting continuously on a topological space X. Show that the canonical map $X \to \Gamma\backslash X$ is an open mapping.

EXERCISE 7H: Show that any covering map is an open mapping.

Lemma. Let X be a topological space. Let Y be a connected, locally connected topological space. Let $\pi : X \to Y$ be a covering map and let $\Gamma := \{$deck transformations of $\pi\}$. Assume, for some $y_0 \in Y$, that the Γ-action on $\pi^{-1}(y_0)$ is transitive. Then, for all $y \in Y$, the Γ-action on $\pi^{-1}(y)$ is transitive.

Proof: Let T denote the collection of all $y \in Y$ such that the Γ-action on $\pi^{-1}(Y)$ is transitive. We wish to show that $T = Y$.

By Exercise 7F, T is clopen in Y. As Y is connected, we know that $\emptyset$ and Y are the only clopen subsets of Y. As $y_0 \in T$, we see that $T \neq \emptyset$. Thus $T = Y$. **QED**

Lemma. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. Let $\Gamma := \{$deck transformations of $\pi\}$. Then π is regular iff, for all $y \in Y$, the Γ-action on $\pi^{-1}(y)$ is transitive.

Proof: "Only if" is (2) of Exercise 6E. Assume, for all $y \in Y$, that the Γ-action on $\pi^{-1}(y)$ is transitive. Let $W := \Gamma\backslash X$. Let $p : X \to W$ be the canonical map. We wish to show that there is a homeomorphism $h : W \to Y$ such that $h \circ p = \pi$.

By transitivity on fibers of π, we conclude that $\{\pi^{-1}(y) \,|\, y \in Y\}$ is the set of Γ-orbits in X. Since $\{p^{-1}(w) \,|\, w \in W\}$ is also the set of Γ-orbits in X, we see that

$$\{\pi^{-1}(y) \,|\, y \in Y\} = \{p^{-1}(w) \,|\, w \in W\},$$

which implies that there is a bijective map $h : W \to Y$ such that $h \circ p = \pi$. We must show that h is continuous and open.

For any subset $Y_0 \subseteq Y$, we have $h^{-1}(Y_0) = p(\pi^{-1}(Y_0))$. So, since $\pi : X \to Y$ is continuous and since (by Exercise 7G) $p : X \to W$ is open, we conclude that $h : W \to Y$ is continuous.

For any subset $W_0 \subseteq W$, we have $h(W_0) = \pi(p^{-1}(W_0))$. So, since $p : X \to W$ is continuous and since (by Exercise 7H) $\pi : X \to Y$ is open, we conclude that $h : W \to Y$ is open. **QED**

Theorem. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a covering map. Let Z be a simply connected, locally path-connected topological space. Let $z_0 \in Z$. Then π has unique $(\{z_0\}, Z)$ lifting.

Proof: Let $f : Z \to Y$ be a continuous map, let $y_0 = f(z_0)$ and let $x_0 \in \pi^{-1}(y_0)$. We wish to show that there is a continuous map $\widehat{f} : Z \to X$ such that $\pi \circ \widehat{f} = f$ and such that $\widehat{f}(z_0) = x_0$.

Let $I := [0,1]$. For any $z \in Z$, choose $\alpha_z \in P_{z_0}^z(Z)$ Then $(f \circ \alpha_z)(0) = f(z_0) = y_0$, so $x_0 \in \pi^{-1}((f \circ \alpha_z)(0))$. Since π is a covering map, it follows that π has unique $(\{0\}, I)$ lifting. Let $\beta_z : I \to X$ be the π-lift of $f \circ \alpha_z$ such that $\beta_z(0) = x_0$. Define $\widehat{f} : Z \to X$ by $\widehat{f}(z) = \beta_z(1)$.

Then, for all $z \in Z$, we have $\pi(\widehat{f}(z)) = \pi(\beta_z(1)) = f(\alpha_z(1)) = f(z)$, so $\pi \circ \widehat{f} = f$.

Claim: If $z \in Z$, if $\alpha \in P_{z_0}^z(Z)$, if β is a π-lift of $f \circ \alpha$ and if $\beta(0) = x_0$, then $\widehat{f}(z) = \beta(1)$.
Proof of Claim: Since Z is simply connected, for all $\rho, \sigma : I \to Z$, if $\rho(0) = \sigma(0)$ and if $\rho(1) = \sigma(1)$, then ρ and σ are endpoint fixed homotopic. (We leave this as an unassigned exercise.) It follows that α is endpoint fixed homotopic to α_z. Then $f \circ \alpha$ is endpoint fixed homotopic to $f \circ \alpha_z$. Then, by the theorem following Exercise 4F, we see that $\beta(1) = \beta_z(1)$. As $\widehat{f}(z) = \beta_z(1)$, we are done. *End of proof of Claim.*

Let $\alpha_* : I \to Z$ be the constant map at z_0, defined by $\alpha_*(t) = z_0$. Let $\beta_* : I \to X$ be the constant map at x_0, defined by $\beta_*(t) = x_0$. Then $\alpha_* \in P_{z_0}^{z_0}(Z)$, β_* is a π-lift of $f \circ \alpha_*$ and $\beta_*(0) = x_0$. By the claim, we conclude that $\widehat{f}(z_0) = \beta_*(1) = x_0$.

It remains to prove that $\widehat{f} : Z \to X$ is continuous. Fix $z \in Z$, let $x := \widehat{f}(z)$ and let X_1 be an open neighborhood of x in X. We wish to show that there is an open neighborhood Z_1 of z in Z such that $\widehat{f}(Z_1) \subseteq X_1$.

We leave it as an unassigned exercise to show that any covering map is a local homeomorphism. Then, replacing X_1 by a smaller open neighborhood of x in X, we may assume that $\pi : X_1 \to \pi(X_1)$ is a homeomorphism. By Exercise 7H, $Y_1 := \pi(X_1)$ is an open neighborhood of y in Y. Let $\pi_1 := \pi|X_1 : X_1 \to Y_1$. Then $\pi_1 : X_1 \to Y_1$ is a homeomorphism.

Since $f : Z \to Y$ is continuous, since Z is locally path-connected and since $f(z) = \pi(\widehat{f}(z)) = \pi(x) \in \pi(X_1) = Y_1$, let Z_1 be a path-connected open neighborhood of z in Z such that $f(Z_1) \subseteq Y_1$. We wish to show that $\widehat{f}(Z_1) \subseteq X_1$. Fix $z_1 \in Z_1$. We wish to show that $\widehat{f}(z_1) \in X_1$.

Choose $\alpha \in P_z^{z_1}(Z_1)$. Let $\beta := \pi_1^{-1} \circ f \circ \alpha : I \to X_1$. Then $\beta \in P_x^{x_1}(X)$ is a π-lift of $f \circ \alpha$. Recall that $\beta_z \in P_{x_0}^x(X)$ is a π-lift of $f \circ \alpha_z$. Then $\beta_z \| \beta$ is a π-lift of $f \circ (\alpha_z \| \alpha)$ and $(\alpha_z \| \alpha)(0) = \alpha_z(0) = z_0$ and $(\alpha_z \| \alpha)(1) = \alpha(1) = z_1$. so, by the claim, we have $\widehat{f}(z_1) = (\beta_z \| \beta)(1) = \beta(1) \in \beta(I)$. Since $\beta : I \to X_1$, we conclude that $\widehat{f}(z_1) \in X_1$, as desired. **QED**

Corollary. Let X be a simply connected, locally path-connected topological space. Let Y be a topological space. Let $\pi : X \to Y$ be a covering map. Then π is regular.

Proof: Let $\Gamma := \{\text{deck transformations of } \pi\}$. Fix $y \in Y$. By the preceding lemma, we wish to show that the Γ-action on $\pi^{-1}(y)$ is transitive. Fix $x, x' \in \pi^{-1}(y)$. We wish to show that there exists $\gamma \in \Gamma$ such that $\gamma x = x'$.

Let $Z := X$, let $z_0 := x$. By the preceding theorem, π has unique $(\{z_0\}, Z)$ lifting. Let $f := \pi : Z \to X$. Let $x_0 = x'$. Then $f(x_0) = f(x') = \pi(x') = y = \pi(x)$. Therefore there is a unique π-lift $\widehat{f} : Z \to X$ of $f : Z \to Y$ such that $\widehat{f}(z_0) = x_0$.

We have $\widehat{f} : X \to X$ and $\widehat{f}(x) = \widehat{f}(z_0) = x_0 = x'$. Similarly, let $\widehat{f}^* : X \to X$ be the π-lift of f satisfying $\widehat{f}^*(x') = x$. Let $I := \widehat{f}^* \circ \widehat{f}$. Then $I : X \to X$ is the unique π-lift of f satisfying $I(x) = x$. Since the identity map $X \to X$ is another, we conclude from uniqueness that $I : X \to X$ is the identity, *i.e.*, that $\widehat{f}^* \circ \widehat{f} : X \to X$ is the identity. Similarly $\widehat{f} \circ \widehat{f}^* : X \to X$ is the identity. Then $\widehat{f} : X \to X$ is a homeomorphism.

We have $\pi \circ \widehat{f} = f = \pi$. Then $\widehat{f}$ is a deck transformation of π, *i.e.*, $\widehat{f} \in \Gamma$. Let $\gamma := \widehat{f}$. Then $\gamma x = \widehat{f}(x) = x'$. **QED**

EXERCISE 7I: Let Y be a connected, locally path-connected topological space. Assume, for some simply connected topological space X, that there exists a covering map $X \to Y$. Show that Y is semi-locally simply connected.

We are now ready to prove:

Theorem. Let Y be a connected, locally path-connected topological space. Then: Y has a universal cover iff Y is semi-locally simply connected.

Proof: "Only if" is given in Exercise 7I. Assume that Y is semi-locally simply connected. We wish to show, for some simply connected topological space X, that there is a covering map $\pi : X \to Y$.

For any topological space T, for any $t \in T$, let $P_t(T) := \bigcup\limits_{t' \in T} P_t^{t'}(T)$.

Fix $y_0 \in Y$. Let $\widehat{X} := P_{y_0}(Y)$. For any $\alpha \in \widehat{X}$, let $[\alpha]$ denote the endpoint fixed homotopy class of α. Let $X := \{[\alpha] \,|\, \alpha \in \widehat{X}\}$. Define $\pi : X \to Y$ by $\pi([\alpha]) = \alpha(1)$. Note that, if $\alpha, \alpha' \in \widehat{X}$ and if α is endpoint fixed homotopic to α', then $\alpha(1) = \alpha'(1)$, so π is a well-defined function. We now wish to show that there is a simply connected topology on X with respect to which $\pi : X \to Y$ is a covering map.

For any $y \in Y$, let $\mathcal{V}_y$ denote the collection of all open neighborhoods of y in Y that are relatively simply connected in Y. For any $\alpha \in \widehat{X}$, for any $V \in \mathcal{V}_{\alpha(1)}$, let

$$N_{[\alpha]}^V := \{[\alpha \| \beta] \,|\, \beta \in P_{\alpha(1)}(V)\} \subseteq X;$$

note that $[\alpha] \in N_{[\alpha]}^V$.

EXERCISE 7J: For any $\alpha \in \widehat{X}$, for any $V \in \mathcal{V}_{\alpha(1)}$, show that $\pi | N_{[\alpha]}^V : N_{[\alpha]}^V \to Y$ is injective.

EXERCISE 7K: Let $\alpha \in \widehat{X}$ and let $V \in \mathcal{V}_{\alpha(1)}$. Let $\beta \in P_{\alpha(1)}(V)$ and let $W \in \mathcal{V}_{\beta(1)}$. Assume that $W \subseteq V$. Show that $N^W_{[\alpha\|\beta]} \subseteq N^V_{[\alpha]}$.

EXERCISE 7L: Show that $\{N^V_{[\alpha]} \,|\, \alpha \in \widehat{X}, V \in \mathcal{V}_{\alpha(1)}\}$ is a basis of a topology on X. (*Hint:* Use Exercise 7K and Exercise 2H.)

We now give X the topology described in Exercise 7L.

Claim 1: $\pi : X \to Y$ is continuous. Proof of Claim 1: Let $\alpha \in \widehat{X}$ and let W be a open neighborhood of $\pi([\alpha])$ in Y. We wish to show that there is a open neighborhood N of $[\alpha]$ in X such that $\pi(N) \subseteq W$.

Since Y is semi-locally simply connected, choose $V \in \mathcal{V}_{\alpha(1)}$ such that $V \subseteq W$. Let $N := N^V_{[\alpha]}$. Then, by definition of $N^V_{[\alpha]}$ and of π, we have $\pi(N) \subseteq V$. Then $\pi(N) \subseteq W$. *End of proof of Claim 1.*

Claim 2: $\pi : X \to Y$ is open. Proof of Claim 2: Recall that $\{N^V_{[\alpha]} \,|\, \alpha \in \widehat{X}, V \in \mathcal{V}_{\alpha(1)}\}$ is a basis for the topology on X. For all $\alpha \in \widehat{X}$, for all $V \in \mathcal{V}_{\alpha(1)}$, we have $\pi(N^V_{[\alpha]}) = V$. Thus, the image of every basic open set is open, and it follows that the image of every open set is open. *End of proof of Claim 2.*

Claim 3: For all $\alpha \in \widehat{X}$, for all $V \in \mathcal{V}_{\alpha(1)}$, the map $\pi|N^V_{[\alpha]} : N^V_{[\alpha]} \to V$ is a homeomorphism. Proof of Claim 3: By Exercise 7J, $\pi|N^V_{[\alpha]}$ is injective. By Claim 1, this map is continuous. By Claim 2, $\pi|N^V_{[\alpha]} : N^V_{[\alpha]} \to V$ is open. So, as $\pi|N^V_{[\alpha]} : N^V_{[\alpha]} \to V$ is surjective, we are done. *End of proof of Claim 3.*

EXERCISE 7M: Let $y \in Y$ and let $\alpha, \alpha' \in P^y_{y_0}(Y) \subseteq \widehat{X}$. Let $V \in \mathcal{V}_y$. Assume that $N^V_{[\alpha]} \cap N^V_{[\alpha']} \neq \emptyset$. Show that $[\alpha] = [\alpha']$.

For any topological space A, a **pairwise disjoint open covering** of A, is a subset $\mathcal{U} \subseteq \{\text{open subsets of } A\}$ such that $\cup\mathcal{U} = A$ and such that, for any $U, U' \in \mathcal{U}$, we have: if $U \cap U' \neq \emptyset$, then $U = U'$. For any topological spaces A and B, for any continuous $p : A \to B$, we have: $p : A \to B$ is a trivial covering map iff there is an pairwise disjoint open covering $\mathcal{U}$ of A such that, for all $U \in \mathcal{U}$, $p|U : U \to B$ is a homemorphism. We leave the proof of this last statement as an unassigned exercise.

Then, by Claim 3 and Exercise 7M, we see, for any $y \in Y$, for any $V \in \mathcal{V}_y$, that $\pi|[\pi^{-1}(V)] : \pi^{-1}(V) \to V$ is a trivial covering map; we use the pairwise disjoint open cover $\{N^V_{[\alpha]} \,|\, \alpha \in P^y_{y_0}\}$ of $\pi^{-1}(V)$. We conclude that $\pi : X \to Y$ is a covering map.

It remains to show that X is simply connected. Let $c : I \to Y$ be the constant map defined by $c(t) = y_0$ Let $x_0 := [c]$. We wish to show that $\pi_1(X, x_0)$ is trivial. Let $\widehat{c} : I \to X$ be the constant map defined by $\widehat{c}(t) = x_0$. Note that $\pi \circ \widehat{c} = c$. We wish to show that any element of $P^{x_0}_{x_0}(X)$ is endpoint fixed homotopic to $\widehat{c}$.

Claim 4: For all $x \in X$, for all $\sigma \in P^x_{x_0}(X)$, we have $[\pi \circ \sigma] = x$. Proof of Claim 4: Let $\alpha := \pi \circ \sigma$. Let $I := [0, 1]$. For all $s \in I$, let $\alpha_s : I \to Y$ be defined by $\alpha_s(t) = \alpha(st)$. Then $\alpha_0 = c$ and $\alpha_1 = \alpha$. Define $\widehat{\alpha} : I \to X$ by $\widehat{\alpha}(s) = [\alpha_s]$. We leave it as an unassigned exercise to show that $\widehat{\alpha}$ is continuous. For all $s \in I$, we have $\pi(\widehat{\alpha}(s)) = \pi([\alpha_s]) = \alpha_s(1) = \alpha(s)$, so $\pi \circ \widehat{\alpha} = \alpha$. So, since $\alpha = \pi \circ \sigma$, we conclude that both $\widehat{\alpha}$ and σ are π-lifts of α. Moreover, we have $\widehat{\alpha}(0) = [\alpha_0] = [c] = x_0 = \sigma(0)$. By the proposition preceding Exercise 4F, we see that π has unique $(\{0\}, I)$-lifting. It follows that $\widehat{\alpha} = \sigma$. Then $[\pi \circ \sigma] = [\alpha] = [\alpha_1] =$

$\widehat{\alpha}(1) = \sigma(1) = x$. *End of proof of Claim 4.*

Let $\sigma \in P_{x_0}^{x_0}(X)$. We wish to show that σ is endpoint fixed homotopic to $\widehat{c}$.

By Claim 4, we see that $[\pi \circ \sigma] = x_0$. Then $[\pi \circ \sigma] = [c] = [\pi \circ \widehat{c}]$. That is, $\pi \circ \sigma$ is endpoint fixed homotopic to $\pi \circ \widehat{c}$. We also have $\sigma(0) = x_0 = \widehat{c}(0)$. It then follows, from the theorem following Exercise 4F, that σ is endpoint fixed homotopic to $\widehat{c}$. **QED**

We now return to the question we asked before: Are all covering maps are regular? Our approach will be to develop, for any connected, semi-locally simply connected Y, a "classification" of covering maps to Y, and to calculate for each one, its group of deck transformations, and to analyze, for each one, whether it is regular. In the process, we will discover many non-regular covering maps.

First, though, we need some basic results. The following is our **main lifting theorem**:

Theorem. Let X, Y and Z be topological spaces. Assume that Z is connected and locally path-connected. Let $\pi : X \to Y$ be a covering map. Let $f : Z \to Y$ be continuous. Let $x_0 \in X$ and $z_0 \in Z$. Assume that $\pi(x_0) = f(z_0)$. Then the following are equivalent:

(1) there is a unique π-lift $\widehat{f}$ of f such that $\widehat{f}(z_0) = x_0$; and

(2) $f_*(\pi_1(Z, z_0)) \subseteq \pi_*(\pi_1(X, X_0))$.

Proof: By Exercise 8A below, (1) implies (2).

EXERCISE 8A: Show: If there is a π-lift $\widehat{f}$ of f such that $\widehat{f}(z_0) = x_0$, then (2) is true.

Now assume that (2) is true. We wish to prove (1).

EXERCISE 8B: Let $\gamma \in P_{z_0}^{z_0}(Z)$. Let $\widehat{\gamma}$ be the π-lift of $f \circ \gamma$ such that $\widehat{\gamma}(0) = x_0$. Show that $\widehat{\gamma}(1) = x_0$.

EXERCISE 8C: Let $z \in Z$. Let $\alpha, \beta \in P_{z_0}^{z}(Z)$. Let $\widehat{\alpha}$ and $\widehat{\beta}$ be the π-lifts of $f \circ \alpha$, $f \circ \beta$, respectively, such that $\widehat{\alpha}(0) = x_0 = \widehat{\beta}(0)$. Show that $\widehat{\alpha}(1) = \widehat{\beta}(1)$.

By the axiom of choice, for all $z \in Z$, let $\alpha_z \in P_{z_0}^{z}(Z)$; then $\alpha_z(0) = z_0$ and $\alpha_z(1) = z$. For all $z \in Z$, let $\widehat{\alpha}_z$ be the unique π-lift of $f \circ \alpha_z$ such that $\widehat{\alpha}_z(0) = x_0$. Then $\pi \circ \widehat{\alpha}_z = f \circ \alpha_z$. Define $\widehat{f} : Z \to X$ by $\widehat{f}(z) = \widehat{\alpha}_z(1)$.

Let $I := [0, 1]$. Let $c : I \to Z$ be the constant loop at z_0, defined by $c(t) = z_0$. Let $\widehat{c} : I \to X$ be the constant loop at x_0, defined by $\widehat{c}(t) = x_0$. Then $\widehat{c}$ is the unique π-lift of $f \circ c$ such that $\widehat{c}(0) = x_0$. By Exercise 8C, we have $\widehat{\alpha}_{z_0}(1) = \widehat{c}(1)$. Then $\widehat{f}(z_0) = \widehat{\alpha}_{z_0}(1) = \widehat{c}(1) = x_0$. For all $z \in Z$, we have $(\pi \circ \widehat{f})(z) = \pi(\widehat{f}(z)) = \pi(\widehat{\alpha}_z(1)) = (\pi \circ \widehat{\alpha}_z)(1) = (f \circ \alpha_z)(1) = f(\alpha_z(1)) = f(z)$.

EXERCISE 8D: Show that $\widehat{f}$ is continuous. That is show the following: Let $z \in Z$. Let V be an open neighborhood of $\widehat{f}(z)$ in X. Show that there is an open neighborhood U of z in Z such that $\widehat{f}(U) \subseteq V$.

It remains to prove uniqueness. Let $\widehat{g}$ be a π-lift of f such that $\widehat{g}(z_0) = x_0$. We wish to show that $\widehat{g} = \widehat{f}$. Fix $z \in Z$. We wish to show that $\widehat{g}(z) = \widehat{f}(z)$.

Because $\pi \circ \widehat{g} = f$, we see that $\pi \circ (\widehat{g} \circ \alpha_z) = f \circ \alpha_z$. Moreover, $(\widehat{g} \circ \alpha_z)(0) =$

$\widehat{g}(\alpha_z(0)) = \widehat{g}(z_0) = x_0$. Then, by definition of $\widehat{\alpha}_z$, we see that $\widehat{g} \circ \alpha_z = \widehat{\alpha}_z$. Then $\widehat{g}(z) = \widehat{g}(\alpha_z(1)) = (\widehat{g} \circ \alpha_z)(1) = \widehat{\alpha}_z(1) = \widehat{f}(z)$, as desired. **QED**

EXERCISE 8E: Let Y be a path-connected, locally path-connected topological space and let (W, p) be a universal cover of Y.

(1) Show that W is locally path-connected.
(2) Let (W', p') be another universal cover of Y. Show that (W, p) is isomorphic to (W', p') in the category of topological spaces over Y. That is, show that there is a homeomorphism $h : W \to W'$ such that $p' \circ h = p$.

Remark. Let Γ be a group and let F be a free, transitive Γ-set. Let $f \in F$. Then the orbit map $\gamma \mapsto \gamma f : \Gamma \to F$ based at f is a bijection. Consequently, there is a uniqe group structure on F such that $\gamma \mapsto \gamma f : \Gamma \to F$ is a group isomorphism. With this group structure on F, note that f becomes the identity element.

Corollary. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a regular covering map. Let $\Gamma := \{\text{deck transformations of } \pi\}$. Let $y_0 \in Y$ and let $F := \pi^{-1}(y_0)$. Then, for any $x_0 \in F$, there is a unique group structure on F such that $\gamma \mapsto \gamma x_0 : \Gamma \to F$ is an isomorphism.

Proof: Since Γ acts freely on X, it follows that Γ acts freely on F. Since π is regular, it follows that Γ acts transitively on F. The result then follows from the preceding remark. **QED**

The preceding corollary is summarized by the statement: "In a regular covering, any fiber with a chosen point has a unique group structure making the orbit map at that point is an isomorphism with the group of deck transformations." Note that, under that group structure, the chosen point becomes the identity of the group. Thus we may think of the fibers of a regular covering map as groups isomorphic to the group of deck transformations. However, once again, obtaining the group structure requires first picking a particular point in the fiber that you'll make the identity.

Recall that we have proved

Fact. Let Y be a path-connected topological space and let (X, π) be a universal cover of Y. Let $x_0 \in X$ and let $y_0 := \pi(x_0)$. For all $\alpha \in P_{y_0}^{y_0}(Y)$, let $\widehat{\alpha}$ be the unique π-lift of α satisfying $\widehat{\alpha}(0) = x_0$. Then the map $[\alpha] \mapsto \widehat{\alpha}(1) : \pi_1(Y, y_0) \to \pi^{-1}(y_0)$ is a well-defined bijection.

Let Y be a connected, locally path-connected topological space. Let (X, π) be a universal cover of Y. Let $x_0 \in X$ and let $y_0 := \pi(x_0)$. Let $\Gamma := \{\text{deck transformations of } \pi\}$. Define $\phi : \pi_1(Y, y_0) \to \pi^{-1}(y_0)$ as in the preceding fact. Let $\psi : \Gamma \to \pi^{-1}(y_0)$ be the orbit map defined by $\psi(\gamma) = \gamma x_0$. Then, by the preceding corollary and fact, we obtain a bijection $\psi^{-1} \circ \phi : \pi_1(Y, y_0) \to \Gamma$, and it is an exercise to see that this map is a group isomorphism. Thus we see that $\pi_1(Y, y_0)$ and Γ are isomorphic (as we have noted before), but the point we're making now is that this isomorphism is not "natural" until we have picked a point in the fiber over y_0.

To make this more precise, let $\mathcal{C}$ be the category whose objects are

(1) a connected, locally path connected topological space Y; together with
(2) a universal cover (X, π) of Y; together with
(3) a point $x_0 \in X$.

(What are the arrows in $\mathcal{C}$?) Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \{\text{groups}\}$ be functors defined by

$$\mathcal{F}(Y, X, \pi, x_0) = \pi_1(Y, \pi(x_0))$$

and $\mathcal{G}(Y, X, \pi, x_0) = \{\text{deck transformations of } \pi\}$. Then the paragraph following the preceding fact asserts that $\mathcal{F}$ and $\mathcal{G}$ are equivalent functors.

We now return to classifying covering map and determining which covering maps are regular.

Setup for covering maps: Let Y be a connected, locally path-connected topological space. Let (X, p) be a universal cover of Y. Let $\Gamma := \{\text{deck transformations of } p\}$. For all $\Delta \leq \Gamma$, let $X_\Delta := \Delta \backslash X$ and let $c_\Delta : X \to X_\Delta$ be the canonical map and let $p_\Delta : X_\Delta \to Y$ be the unique function such that $p = p_\Delta \circ c_\Delta$.

It is an unassigned exercise to show that such a function p_Δ exists and is unqiue. It is a further unassigned exercise to show that it is a covering map.

Theorem. Assume the "Setup for covering maps". Let Z be a connected topological space. Let $q : Z \to Y$ be a covering map. Then there exists $\Delta \leq \Gamma$ such that (Z, q) is isomorphic to (X_Δ, p_Δ) in the category of topological spaces over Y.

We defer the proof momentarily.

The preceding theorem raises the question of whether (X_Δ, p_Δ) and (X_Λ, p_Λ) might be isomorpic in the category of topological spacdes over Y for different $\Delta, \Lambda \leq \Gamma$.

Remark. Let $\Delta, \Lambda \leq \Gamma$. Then: (X_Δ, p_Δ) is isomorphic to (X_Λ, p_Λ) in the category of topological spaces over Y iff there exists $\gamma \in \Gamma$ such that $\gamma \Delta \gamma^{-1} = \Lambda$.

We defer the proof momentarily.

Proof of the theorem:

EXERCISE 9A: Show that there is a covering map $r : X \to Z$ such that $q \circ r = p$. (Hint: First argue that Z is locally path-connected and semi-locally simply connected. Let $(\widetilde{Z}, r_0)$ be a universal cover of Z. Fix $y_0 \in Y$. Choose $x_0 \in p^{-1}(y_0)$, $z_0 \in q^{-1}(y_0)$ and $\widetilde{z}_0 \in r_0^{-1}(z_0)$. Using the lifthing theorem which precedes Exercise 8A to construct a continuous map $a : \widetilde{Z} \to X$ such that $p \circ a = q \circ r_0$ and such that $a(\tilde{z}_0) = x_0$. Using the lifting theorem again, construct a continuous map $r : X \to Z$ such that $q \circ r = p$ and such that $r(x_0) = z_0$. Using the lifting theorem again, construct a continuous map $b : X \to \widetilde{Z}$ such that $r_0 \circ b = r$ and such that $r(x_0) = \widetilde{z}_0$. We have $p \circ a \circ b = q \circ r_0 \circ b = q \circ r = p$. Then, by uniqueness in the lifting theorem, argue that $a \circ b : X \to X$ is the identity. We have $q \circ r \circ a = p \circ a = q \circ r_0$. Then, by uniqueness in the lifting theorem, argue that $r \circ a = r_0$. We have $r_0 \circ b \circ a = r \circ a = r_0$. Then, by uniqueness in the lifting theorem, argue that $b \circ a : \widetilde{Z} \to \widetilde{Z}$ is the identity. Since $a \circ b : X \to X$ and $b \circ a : \widetilde{Z} \to \widetilde{Z}$ are both identity maps, we conclude that $a : \widetilde{Z} \to X$ and $b : X \to \widetilde{Z}$ are both homeomorphisms. Then, as r_0 is a covering map, so is $r_0 \circ b$. However $r_0 \circ b = r$.)

Fix such a map r. Let $\Delta := \{\text{deck transformations of } r\}$.

EXERCISE 9B: Show that $\Delta \leq \Gamma$.

As X is simply connected and Z is locally path-connected, it follows that r is regular. Then (Z, r) is isomorphic to (X_Δ, c_Δ) in the category of topological spaces under X. That is, there is a homeomorphism $h : Z \to X_\Delta$ such that $h \circ r = c_\Delta$. We wish to show that (Z, q) is isomorphic to (X_Δ, p_Δ) in the category of topological spacaces over Y, so it suffices to show that $p_\Delta \circ h = q$. Fix $z \in Z$. We wish to show that $p_\Delta(h(z)) = q(z)$.

Choose $x \in X$ such that $r(x) = z$. Then $h(z) = h(r(x)) = c_\Delta(x)$. So, since $p_\Delta \circ c_\Delta = p = q \circ r$, we have $p_\Delta(h(z)) = p_\Delta(c_\Delta(x)) = q(r(x)) = q(z)$. **QED**

Proof of the remark:

EXERCISE 9C: Show the easy "if" direction of the remark. That is, show that if there is some $\gamma \in \Gamma$, such that $\gamma\Delta\gamma^{-1} = \Lambda$, then there is a homeomorphism $h : X_\Delta \to X_\Lambda$ such that $p_\Lambda \circ h = p_\Delta$.

We now prove "only if". Let $h : X_\Delta \to X_\Lambda$ be a homomorphism such that $p_\Lambda \circ h = p_\Delta$. We wish to show, for some $\gamma \in \Gamma$, that $\gamma\Delta\gamma^{-1} = \Lambda$.

By the theorem preceding Exercise 8A, let $\widehat{h} : X \to X$ be a c_Λ-lift of $h \circ c_\Delta : X \to X_\Lambda$. Then $c_\Lambda \circ \widehat{h} = h \circ c_\Delta$. Then $p_\Lambda \circ (c_\Lambda \circ \widehat{h}) = p_\Lambda \circ (h \circ c_\Delta)$. Then $(p_\Lambda \circ c_\Lambda) \circ \widehat{h} = (p_\Lambda \circ h) \circ c_\Delta$. Then $p \circ \widehat{h} = p_\Delta \circ c_\Delta = p$. Then $\widehat{h} \in \Gamma$. Let $\gamma := \widehat{h}$. It remains to show that $\gamma^{-1}\Delta\gamma = \Lambda$.

EXERCISE 9D: Show that Δ is the group of deck transformations of c_Δ.

EXERCISE 9E: Show, for all $\delta \in \Delta$, that $\gamma\delta\gamma^{-1} \in \Lambda$.

According to Exercise 9C, $\gamma\Delta\gamma^{-1} \subseteq \Lambda$, so it remains only to show that $\Lambda \subseteq \gamma\Delta\gamma^{-1}$.

A similar argument to that of Exercise 9C shows that $\gamma^{-1}\Lambda\gamma \subseteq \Delta$. Then $\Lambda = \gamma\gamma^{-1}\Lambda\gamma\gamma^{-1} \subseteq \gamma\Delta\gamma^{-1}$. **QED**

Recall that, for any group Γ, for any subgroup Δ of Γ, the **normalizer in Γ of Δ** is $N_\Gamma(\Delta) := \{\gamma \in \Gamma \,|\, \gamma\Delta\gamma^{-1} = \Delta\}$; it is the largest subgroup of Γ in which Δ is contained as a normal subgroup. In particular, $N_\Gamma(\Delta) = \Gamma$ iff Δ is a normal subgroup of Γ.

Recall that the kernel of an action of a group G on a set X is $\{g \in G \,|\, \forall x \in X, gx = x\}$; it is the intersection of all the stabilizers of elements of X. Denoting this kernel by K, we have that K is normal in G and that, if $c : G \to G/K$ denotes the canonical homomorphism, then there is a unique action of G/K on X such that, for all $g \in G$, for all $x \in X$, we have $gx = (c(g))x$. That is, any action induces an action of the group modulo this kernel. Moreover, if X is a topological space and if the G-action on X is continuous, then the resulting action of G/K on X is continuous as well.

We say that an action is **faithful** or **effective** if its kernel is trivial. Note that the induced action of G/K on X is faithful. That is, "once you mod out by the kernel of the action, the resulting action has no kernel".

Fact. Assume the "Setup for covering maps". Then, for any subgroup Δ of Γ, the action of $N_\Gamma(\Delta)$ on X factors to an action of $N_\Gamma(\Delta)$ on X_Δ. (That is, there is a unique action of $N_\Gamma(\Delta)$ on X_Δ such that the map p_Δ becomes $(N_\Gamma(\Delta))$-equivariant.) The kernel of this action of $N_\Gamma(\Delta)$ on X_Δ is exactly Δ. Thus $(N_\Gamma(\Delta))/\Delta$ acts continuously and faithfully on X_Δ. The resulting collection of maps

$$\{x \mapsto \tau x : X_\Delta \to X_\Delta \,|\, \tau \in (N_\Gamma(\Delta))/\Delta\}$$

is the group of deck transformations of p_Δ. In particular, the group of deck transformations of p_Δ is isomorphic to $N_\Gamma(\Delta))/\Delta$.

We leave the proof of this fact as an exercise for the interested reader. Here's another unassigned exercise:

Unassigned exercise. Assume the "Setup for covering maps". Let Δ be a subgroup of Γ. Then, for all $y \in Y$, we have $|(p_\Delta)^{-1}(y)| = |\Gamma/\Delta|$.

Now define $S_0 := \{1, 2, 3\}$ and let Γ_0 be the set of all bijections $S_0 \to S_0$. That is, Γ_0 is the set of all permutations of $\{1, 2, 3\}$, *i.e.*, it is the symmetric group on three letters. Let $\Delta_0 := \{\gamma \in \Gamma_0 \,|\, \gamma(1) = 1\}$. We leave it as an unassigned exercise to show that $N_{\Gamma_0}(\Delta_0) = \Delta_0$. Note that $|\Gamma_0| = 3!$ and $|\Delta_0| = 2!$, so $|\Gamma_0/\Delta_0| = (3!)/(2!) = 3$.

In the theorem below, combined with Exercise 9F below, we will show that there is a locally path-connected, contractible, principal topological Γ_0-space $X := E_{\Gamma_0}$. Let $Y := \Gamma_0\backslash X$. Then the group Γ of deck transformations of the canonical map $p : X \to Y$ is, by covering space theory, isomorphic to Γ_0, and therefore contains a subgroup Δ such that $N_\Delta(\Gamma) = \Delta$ and such that $|\Gamma/\Delta| = 3$. Note that $(N_\Delta(\Gamma))/\Delta$ is the trivial group.

Define $X_\Delta := \Delta\backslash X$ and define $p_\Delta : X_\Delta \to Y$ as in the "Setup for covering maps". Recall that the fibers of p_Δ all have cardinalty $|\Gamma/\Delta|$. So every fiber of p_Δ has three elements. By the fact stated above, the group of deck transformations of p_Δ is isomorphic to $(N_\Delta(\Gamma))/\Delta$, which is the trivial group. Thus the group of deck transformations of p_Δ, being trivial, cannot act transitively on the fibers of p_Δ, as they all have three elements. Thus, p_Δ is not a regular covering, which answers a question posed long ago.

This entire discussion was based on:

Theorem. Let $I := [0, 1]$. Let Γ be a discrete group. Then there is a locally path-connected, principal topological Γ-space E_Γ and a continuous map $h_\Gamma : E_\Gamma \times E_\Gamma \times I \to E_\Gamma$ such that:

(1) for all $\gamma \in \Gamma$, for all $x, y \in E_\Gamma$, we have $h_\Gamma(\gamma x, \gamma y, t) = \gamma(h_\Gamma(x, y, t))$; and
(2) for all $x, y \in E_\Gamma$, we have $h_\Gamma(x, y, 0) = x$ and $h_\Gamma(x, y, 1) = y$.

EXERCISE 9F: Show that E_Γ is contractible.

We define $B_\Gamma := \Gamma\backslash E_\Gamma$ and note that, because of the covering space theory that we have developed, $\pi_1(B_\Gamma)$ is isomorphic to Γ. Thus, a consequence of the preceding theorem and exercise is the fact that every discrete group is the fundamental group of some locally path-connected, semi-locally simply connected topological space.

Proof of the preceding theorem: Let $\mathbb{N} := \{0, 1, 2, \ldots\}$. For any $c \in I^{\mathbb{N}}$, let

$$\mathrm{supp}(c) := \{i \in \mathbb{N} \,|\, c_i \neq 0\}.$$

Let $\widehat{E}_\Gamma := \{(c,\gamma) \in I^{\mathbb{N}} \times \Gamma^{\mathbb{N}} \quad | \quad |\mathrm{supp}(c)| < \infty, \quad \sum c_i = 1\}$. We define an equivalence relation $\sim$ on $\widehat{E}_\Gamma$ by: $(c,\gamma) \sim (c',\gamma')$ iff, for all $i \in \mathbb{N}$, we have

(1) $c_i = c_i'$; and

(2) if $c_i \neq 0 \neq c_i'$, then $\gamma_i = \gamma_i'$.

Let $E_\Gamma := \widehat{E}_\Gamma / \sim$. Let $q : \widehat{E}_\gamma \to E_\Gamma$ be the canonical map. For each $(c,\gamma) \in \widehat{E}_\Gamma$, we denote $q(c,\gamma)$ by $c_0\gamma_0 + c_1\gamma_1 + \cdots$. (Warning: This $+$ is not commutative. Moreover, for any $i \in \mathbb{N}$, if $c_i = 0$, then we may and will replace the term $c_i\gamma_i$ by 0, since γ_i is irrelevant; however, we cannot omit this term 0.)

For all $j \in \mathbb{N}$, define $t_j : E_j \to I$ by $t_j(c_0\gamma_0 + c_1\gamma_1 + \cdots) = c_j$, let $E_\Gamma^j := t_j^{-1}((0,1])$ and define $x_j : E_\Gamma^j \to \Gamma$ by $x_j(c_0\gamma_0 + c_1\gamma_1 + \cdots) = \gamma_j$. Give E_Γ the coarsest topology such that all these maps $t_0, t_1, \ldots, x_0, x_1, \ldots$ are continuous. That is, give E_Γ the topology generated by

$$\{t_j^{-1}(U) \quad | \quad j \in \mathbb{N}, \quad U \text{ open in } I\} \qquad \cup \qquad \{x_j^{-1}(V) \quad | \quad j \in \mathbb{N}, \quad V \subseteq \Gamma\}.$$

We leave it as an unassigned exercise to show that E_Γ is locally path connected.

For any $x = \sum c_i\gamma_i \in E_\Gamma$ and any $x' = \sum c_i'\gamma_i' \in E_\Gamma$, if,

$$\text{for all } i \in (\mathrm{supp}(c)) \cap (\mathrm{supp}(c')), \qquad \text{we have } \gamma_i = \gamma_i',$$

then we define the **straight path** from x to x' to be the path $\tau_x^{x'} \in P_x^{x'}(E_\Gamma)$ defined as follows: Define $\gamma'' \in \Gamma^{\mathbb{N}}$ by

(1) for all $i \in \mathrm{supp}(c)$, $\gamma_i'' := \gamma_i$;

(2) for all $i \in \mathrm{supp}(c')$, $\gamma_i'' := \gamma_i'$; and

(3) for all $i \in \mathbb{N} \backslash [(\mathrm{supp}(c)) \cup (\mathrm{supp}(c'))]$, $\gamma_i'' := 1_\Gamma$.

Finally, define $\tau_x^{x'} : I \to E_\Gamma$ by $\tau_x^{x'}(t) = \sum [(1-t)c_i + tc_i']\gamma_i''$.

Let

$$E_\Gamma' := \{c_0\gamma_0 + 0 + c_1\gamma_1 + 0 + c_2\gamma_2 + 0 + \cdots \mid c_0, c_1, c_2, \ldots \in I, \gamma_0, \gamma_1, \gamma_2, \ldots \in \Gamma\}.$$

For all $x = c_0\gamma_0 + 0 + c_1\gamma_1 + 0 + c_2\gamma_2 + 0 + \cdots \in E_\Gamma'$, let $x_0 := x$; let

$$x_1 := c_0\gamma_0 + c_1\gamma_1 + 0 + c_2\gamma_2 + 0 + c_3\gamma_3 + \cdots$$

be obtained from x by dropping the second term (and no others); let

$$x_2 := c_0\gamma_0 + c_1\gamma_1 + c_2\gamma_2 + 0 + c_3\gamma_3 + 0 + \cdots$$

be obtained from x by dropping the second and fourth terms (and no others); let

$$x_3 := c_0\gamma_0 + c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 + 0 + c_4\gamma_4 + \cdots$$

be obtained from x by dropping the second and fourth and sixth terms (and no others); etc. For all $x = c_0\gamma_0 + 0 + c_1\gamma_1 + 0 + c_2\gamma_2 + 0 + \cdots \in E_\Gamma'$, let

$$x_\infty := c_0\gamma_0 + c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 + \cdots$$

be obtained from x by dropping all even terms (and no odd terms). For all $x \in E'_\Gamma$, for all $n \in \mathbb{N}$, let $\alpha_x^n := \tau_{x_n}^{x_{n+1}}$ be the straight path from x_n to x_{n+1}.

For all $n \in \mathbb{N}$, let $s_n := 1 - 2^{-n}$. For all $n \in \mathbb{N}$, let $I_n := [s_n, s_{n+1}]$. For all $n \in \mathbb{N}$, let $l_n : I_n \to I$ be the linear map such that $l_n(s_n) = 0$ and $l_n(s_{n+1}) = 1$; it is defined by $l_n(t) = (1-t)s_n + ts_{n+1}$.

For all $x \in E'_\Gamma$, define $\alpha_x : I \to E_\Gamma$ by the rules:

(1) for all $n \in \mathbb{N}$, for all $t \in I_n$, $\alpha_x(t) = \alpha_x^n(l_n(t))$; and

(2) $\alpha_x(1) = x_\infty$;

we leave it as an unassigned exercise to verify that α_x is continuous. Then, for all $x \in E'_\Gamma$, we have $\alpha_x \in P_x^{x_\infty}(E_\Gamma)$.

For all $x = c_0\gamma_0 + c_1\gamma_1 + \cdots \in E_\Gamma$, define

$$x^* := c_0\gamma_0 + 0 + c_1\gamma_1 + 0 + c_2\gamma_2 + 0 + \cdots \in E'_\Gamma;$$

note that $(x^*)_\infty = x$. For all $x \in E_\Gamma$, let $\beta_x \in P_x^{x^*}(E_\Gamma)$ be defined by $\beta_x(t) = \alpha_{x^*}(1-t)$.

For all $x = c_0\gamma_0 + c_1\gamma_1 + \cdots \in E_\Gamma$, define

$$x^\# := 0 + c_0\gamma_0 + 0 + c_1\gamma_1 + 0 + c_2\gamma_2 + 0 + \cdots \in E_\Gamma.$$

Note, for all $x, y \in E_\Gamma$, that $(\mathrm{supp}(x^*)) \cap (\mathrm{supp}(y^\#)) = \emptyset$.

For all integers $k \in [1,4]$, let $J_k := [(k-1)/4, k/4]$ and let $r_k : J_k \to I$ be the linear map such that $r_k((k-1)/4) = 0$ and $r_k(k/4) = 1$; then $r_k : J_k \to I$ is defined by $r_k(t) = (1-t)((k-1)/4) + t(k/4)$. For all $x, y \in E_\Gamma$ we define a path $\lambda_x^y \in P_x^y(E_\Gamma)$ by

(1) for $t \in J_1$, $\lambda_x^y(t) = \beta_x(r_1(t))$; then $\lambda_x^y(1/4) = x^*$;

(2) for $t \in J_2$, $\lambda_x^y(t) = \tau_{x^*}^{y^\#}(r_2(t))$; then $\lambda_x^y(1/2) = y^\#$;

(3) for $t \in J_3$, $\lambda_x^y(t) = \tau_{y^\#}^{y^*}(r_3(t))$; then $\lambda_x^y(3/4) = y^*$; and

(4) for $t \in J_4$, $\lambda_x^y(t) = \alpha_{y^*}(r_4(t))$; then $\lambda_x^y(1) = (y^*)_\infty = y$.

We now define $h_\Gamma : E_\Gamma \times E_\Gamma \times I \to E_\Gamma$ by $h(x,y,t) = \lambda_x^y(t)$. Note, for all $x, y \in E_\Gamma$, that $h(x,y,0) = \lambda_x^y(0) = x$ and $h(x,y,1) = \lambda_x^y(1) = y$. We leave it as an unassigned exercise to show that it has the required properties of continuity and equivariance, as stated in the theorem. **QED**

For any arrow $f : C \to C'$ in a category $\mathcal{C}$, we will define $\mathrm{dom}(f) = C$ and $\mathrm{tar}(f) = C'$. A **diagram** in $\mathcal{C}$ is a set $\mathcal{D}$ consisting of objects and arrows in $\mathcal{C}$ such that: for any arrow $f \in \mathcal{D}$, we have $\mathrm{dom}(f) \in \mathcal{D}$ and $\mathrm{tar}(f) \in \mathcal{D}$. We denote, by $\mathrm{Obj}(\mathcal{D})$, the set of all objects in $\mathcal{D}$. We denote, by $\mathrm{Arr}(\mathcal{D})$, the set of all arrows in $\mathcal{D}$

Given a diagram $\mathcal{D}$ in a category $\mathcal{C}$, given $D, D' \in \mathrm{Obj}(\mathcal{D})$, a **path** in $\mathcal{D}$ from D to D' in $\mathcal{D}$ is a finite sequence $p = (f_1, \ldots, f_m)$ of arrows in $\mathcal{D}$ such that

(1) $\mathrm{dom}(f_1) = D$, $\mathrm{tar}(f_m) = D'$; and

(2) for any integer $i \in [2, m]$, we have: $\mathrm{dom}(f_i) = \mathrm{tar}(f_{i-1})$.

In this case, we define $\circ p := f_m \circ \cdots \circ f_1 : D \to D'$.

Let $\mathcal{D}$ be a diagram in a category $\mathcal{C}$. Then we say $\mathcal{D}$ is **commutative** if, for all $D, D' \in \mathrm{Obj}(\mathcal{D})$, for any two paths p, q from D to D' in $\mathcal{D}$, we have: $\circ p = \circ q$.

Definition. Let $\mathcal{D}$ be a commutative diagram in a category $\mathcal{C}$. A **"receiver"** of $\mathcal{D}$ in $\mathcal{C}$ consists of

(1) an object R in $\mathcal{C}$; and
(2) a set of arrows $\mathcal{A}$ in $\mathcal{C}$

such that

(A) for all $f \in \mathcal{A}$, we have $\mathrm{dom}(f) \in \mathrm{Obj}(\mathcal{D})$ and $\mathrm{tar}(f) = R$;
(B) for all $D \in \mathrm{Obj}(\mathcal{D})$, there is a unique $f \in \mathcal{A}$ such that $\mathrm{dom}(f) = D$; and
(C) $\mathcal{D} \cup \{R\} \cup \mathcal{A}$ is a commutative diagram in $\mathcal{C}$.

Definition. Let $\mathcal{D}$ be a commutative diagram in a category $\mathcal{C}$ and let $(R, \mathcal{A})$ be a receiver of $\mathcal{D}$ in $\mathcal{C}$. Then $(R, \mathcal{A})$ is a **direct limit** of $\mathcal{D}$ in $\mathcal{C}$ means that: for any receiver $(S, \mathcal{B})$ of $\mathcal{D}$ in $\mathcal{C}$, there is a unique arrow $h : R \to S$ such that $\mathcal{B} = \{h \circ f \,|\, f \in \mathcal{A}\}$.

EXERCISE 10A: For all integers $n \geq 1$, let $\iota_n : S^n \to S^{n+1}$ be the function defined by $\iota_n(x_0, \ldots, x_n) = (x_0, \ldots, x_n, 0)$. Let $\mathcal{D} := \{S^1, S^2, \ldots\} \cup \{\iota_1, \iota_2, \ldots\}$. Construct a direct limit of $\mathcal{D}$ in the category $\{\text{sets}\}$. (Show that it is a direct limit.)

Definition. We say that a category $\mathcal{C}$ **has direct limits** if, for any commutative diagram $\mathcal{D}$ in $\mathcal{C}$, there exists a direct limit in $\mathcal{C}$ of $\mathcal{D}$.

Fact. The following categories have direct limits: $\{\text{sets}\}$, $\{\text{groups}\}$, $\mathcal{TS} = \{\text{topological spaces}\}$, $\{\text{pointed topological spaces}\}$. Somehow, "most" basic categories do seem to have direct limits.

To give some idea about this, let $\mathcal{D}$ be a diagram in the category $\{\text{sets}\}$, and we will describe how to construct a direct limit of $\mathcal{D}$. To simplify matters assume, for all $A, B \in \mathrm{Obj}(\mathcal{D})$, that either $A \cap B = \emptyset$ or $A = B$; otherwise the construction is made slightly more difficult. Let $R_0 := \cup[\mathrm{Obj}(\mathcal{D})]$. For all $A, B \in \mathrm{Obj}(\mathcal{D})$, let S_{AB} be the set of all $(a, b) \in A \times B$ such that, for some path p in $\mathcal{D}$ from A to B, we have $(\circ p)(a) = b$. Let $\sim$ be the smallest equivalence relation on R_0 containing $\cup\{S_{AB} \,|\, A, B \in \mathrm{Obj}(\mathcal{D})\}$. Then $R_0/\sim$ is the direct limit:

EXERCISE 10B: Show that $R := R_0/\sim$ is a direct limit of $\mathcal{D}$ in $\{\text{sets}\}$.

In the other categories mentioned in the preceding fact, the construction of direct limits is at a similar level of difficulty.

Be aware that there is some ambiguity in the meaning of the term "direct limit":

Definition. Let $\mathcal{D}$ be a commutative diagram in a category $\mathcal{C}$. An object R in $\mathcal{C}$ is said to be a **direct limit** of $\mathcal{D}$ in $\mathcal{C}$ if: there exists a set of arrows $\mathcal{A}$ in $\mathcal{C}$ such that $(R, \mathcal{A})$ is a direct limit of $\mathcal{D}$ in $\mathcal{C}$.

EXERCISE 10C: Let $\mathcal{D}$ be a commutative diagram in a category $\mathcal{C}$. Let R and R' be direct limits of $\mathcal{D}$ in $\mathcal{C}$. Show that R and R' are isomorphic in $\mathcal{C}$.

As a result of Exercise 10C, one sometimes talks about *the* direct limit of a diagram, instead of *a* direct limit.

Given any set S, let $\langle S \rangle$ denote the free group on S. Let S_0 be a two element set and let a and b be the distinct elements of S_0. Define $A := \langle\{a\}\rangle$, $B := \langle\{b\}\rangle$ and $G := \langle\{a, b\}\rangle$. Let $f : \{1\} \to A$ and $g : \{1\} \to B$ be the trivial maps. Let $p : A \to G$ and $q : B \to G$

be the inclusion maps. We leave it as an unassigned exercise to show that $(G, \{p, q\})$ is a direct limit of the diagram $\{\{1\}, A, B, f, g\}$ in {groups}.

Incidentally, to save some writing, one often simply writes $\langle a, b\rangle$ instead of the more technically correct $\langle\{a, b\}\rangle$.

The type of direct limit described in the preceding paragraph is sometimes called a "pushout":

Definition. Let $\mathcal{D}$ be a diagram in a category $\mathcal{C}$. Suppose that
(1) there are two arrows and three objects in $\mathcal{D}$;
(2) for all $f, g \in \mathrm{Arr}(\mathcal{D})$, we have $\mathrm{dom}(f) = \mathrm{dom}(g)$; and
(3) for all $f, g \in \mathrm{Arr}(\mathcal{D})$, if $f \neq g$, then $\mathrm{tar}(f) \neq \mathrm{tar}(g)$.
In this case, a direct limit of $\mathcal{D}$ in $\mathcal{C}$ is sometimes called a **pushout** of $\mathcal{D}$ in $\mathcal{C}$.

Let I be a set. For all $i \in I$, let S_i be a set, and let $R_i \subseteq \langle S_i\rangle$ be a subset of the free group generated by S_i. For all $i \in I$, let $G_i := \langle S_i \,|\, R_i\rangle$ be the group with generators S_i and relations R_i. (That is, G_i is the quotient of $\langle S_i\rangle$ by the smallest normal subgroup of $\langle S_i\rangle$ which contains R_i. This "smallest normal subgroup" is exactly the subgroup generated by $\displaystyle\bigcup_{g\in\langle S_i\rangle} gR_ig^{-1}$.)

For a given $i \in I$, the pair (S_i, R_i) is sometimes called a "presentation" of G_i; it completely determines G_i, up to isomorphism.

Let K be a set. For all $k \in K$, let $i_k, j_k \in I$ and let $f_k : G_{i_k} \to G_{j_k}$ be a group homomorphism. Let $\mathcal{D} := \{S_i \,|\, i \in I\} \cup \{f_k \,|\, k \in K\}$. Then $\mathcal{D}$ is a diagram in the category {groups} and we set ourselves to the task of computing the direct limit of $\mathcal{D}$.

To simplify matters, let's assume, for all $i, j \in I$, that: if $i \neq j$, then $S_i \cap S_j = \emptyset$. For all $i \in I$, let $c_i : \langle S_i\rangle \to G_i$ be the canonical map. For all $k \in K$, for all $s \in S_{i_k}$, choose $w_{s,k} \in S_{j_k}$ such that $f_k(c_{i_k}(s)) = c_{j_k}(w_{s,k})$.

For a given $k \in K$, the mapping $s \mapsto w_{s,k} : S_{i_k} \to S_{j_k}$ is sometimes called a "presentation" of f_k; it completely determines f_k.

Let $S := \displaystyle\bigcup_{i\in I} S_i$ and let $R := \displaystyle\bigcup_{i\in I} R_i$. For all $k \in K$, let

$$R'_k := \{s^{-1}w_{s,k} \,|\, s \in S_{i_k}\} \subseteq \langle S_{i_k} \cup S_{j_k}\rangle \subseteq \langle S\rangle.$$

Let $R' := \displaystyle\bigcup_{k\in K} R'_k$. It is an unassigned exercise to demonstrate that $\langle S \,|\, R \cup R'\rangle$ is a direct limit of $\mathcal{D}$.

The point of the preceding is simply that, if a diagram in the category of groups is presented completely in terms of generators and relations, then it is straightforward to write down a presentation of its direct limit. Let's refer to this as the **"group direct limit algorithm"**.

If $\mathcal{D}$ is a set of groups (*i.e.*, a diagram in {groups} with no arrows), then the direct limit of $\mathcal{D}$ is often denoted $*\mathcal{D}$ and is called the **free product** of $\mathcal{D}$. When $\mathcal{D}$ is finite, one often writes down the elements of $\mathcal{D}$ with "$*$"s separating them. Thus, for example, the free product of $\{A, B\}$ is denoted $A * B$, and is often referred to as "the free product of A and B".

A notational comment: Consider the group

$$\langle\{a,b,c,p,q,r,s,y,z\}\,|\,\{a^{-1}bc, pqrs, yzy^{-1}z^{-1}\}\rangle.$$

It would be typical to write this in a different way: $\langle a,b,c,x,y\,|\,a=bc, pqrs=1, yz=zy\}$. On the left and right of the vertical bar ($|$), one removes the braces. On the right, for each relator, one sets the relator to 1 and then tries to manipulate the resulting equation to be more readable. For example, $yzy^{-1}z^{-1}=1$ can be manipulated to $yz=zy$.

Let A, B and C be groups. Let $f:C\to A$ and $g:C\to B$ be group homomorphisms. Then the pushout of $\{A,B,C,f,g\}$ in {groups} is often denoted $A*_C B$. Note that f and g are suppressed from the notation, although they really should not be. One might write $A*_C^{f,g} B$ if there were any confusion. This group is called the **almagamated product** of A and B, with amalgamation over C. It can be described exactly in an algebraic way: It is the quotient of the free product $A*B$ modulo the smallest normal subroup of $A*B$ containing $\{(f(c))(g(c))^{-1}\,|\,c\in C\}$.

Consider the special case where $C=\langle c\,|\,c^2=1\rangle$, where $A=\langle a\,|\,a^4=1\rangle$ and where $B=\langle b\,|\,b^6=1\rangle$. Let $f:C\to A$ be the homomorphism which sends (the image of) c to (the image of) a^2. Let $g:C\to B$ be the homomorphism which sends (the image of) c to (the image of) b^3. Then, according to group direct limit algorithm, one can immediately write down a presentation of $A*_C B$, namely:

$$\langle a,b,c\,|\,c^2=1, a^4=1, b^6=1, c=a^2, c=b^3\rangle.$$

The generator c is not really needed; we leave it to the interested reader to verify that $A*_C B$ is isomorphic to

$$\langle a,b\,|\,a^4=1, b^6=1, a^2=b^3\rangle.$$

Definition. Let $\mathcal{P}$ be the category of pointed topological spaces. Let $(X,x),(Y,y)\in\mathcal{P}$. Let $f,g:(X,x)\to(Y,y)$ be arrows in $\mathcal{P}$. Let $f_0,g_0:X\to Y$ be the continuous maps underlying f,g, respectively. (Then $f_0(x)=g_0(x)=y$.) Let $I:=[0,1]$. A **pointed homotopy** from f to g is a continuous map $h:I\times X\to Y$ such that

(1) $h(0,\cdot)=f_0$, $h(1,\cdot)=g_0$; and
(2) for all $s\in I$, $h(s,x)=y$.

Let X be a topological space. An open cover $\mathcal{U}$ of X is said to be **locally finite** if, for any $x\in X$, there is an open neighborhood V of x in X such that $|\{U\in\mathcal{U}\,|\,U\cap V\neq\emptyset\}|<\infty$. A **refinement** of an open cover $\mathcal{V}$ of X is an open cover $\mathcal{U}$ of X with the property that: for all $U\in\mathcal{U}$, there exists $V\in\mathcal{V}$ such that $U\subseteq V$. We say that X is **paracompact** if every open cover of X has a locally finite refinement.

Let $\mathcal{F}:\mathcal{C}\to\mathcal{D}$ be a functor. Let $\mathcal{S}$ be a set of objects and arrows in $\mathcal{C}$. Then, by $\mathcal{F}(\mathcal{S})$, we mean $\{\mathcal{F}(s)\,|\,s\in\mathcal{S}\}$.

Theorem (van Kampen's Theorem). Let $\mathcal{TS}^*$ be the category of pointed, path-connected, locally path-connected, semi-locally simply connected, Hausdorff, regular, second countable topological spaces whose arrows are pointed homotopy classes of pointed continuous maps. Then $\pi_1 : \mathcal{TS}^* \to \{\text{countable groups}\}$ preserves direct limits. That is, if $\mathcal{D}$ is a commutative diagram in $\mathcal{TS}^*$ and if $(R, \mathcal{A})$ is a direct limit of $\mathcal{D}$ in $\mathcal{TS}^*$, then $(\pi_1(R), \pi_1(\mathcal{A}))$ is a direct limit of $\pi_1(\mathcal{D})$ in $\{\text{countable groups}\}$.

We defer the proof, and proceed to an application of van Kampen's Theorem:

Let S be the genus two orientable surface obtained from a closed solid octagon $\overline{K}$ in $\mathbb{R}^2$ after the $(a, b, \overleftarrow{a}, \overleftarrow{b}, c, d, \overleftarrow{c}, \overleftarrow{d})$-identification of boundary points. We set ourselves to the task of computing $\pi_1(S)$.

Let $K \subseteq S$ denote the interior in $\mathbb{R}^2$ of $\overline{K}$ Let U_0 be an open disk in $\mathbb{R}^2$ such whose closure is contained in K, *i.e.*, $\mathrm{Cl}_{\mathbb{R}^2}(U_0) \subseteq K$. Let D_0 be a closed disk in $\mathbb{R}^2$ such that $D_0 \subseteq U_0$. Let U and D be the images in S of U_0 and D_0. Let $V := S \backslash D$. Let $s \in U \cap V$. For any two subsets $P, Q \subseteq S$, if $s \in P \subseteq Q$, then we let $\iota_P^Q : (P, s) \to (Q, s)$ denote pointed homotopy class of the pointed inclusion map.

Then (S, s) is a pushout in $\mathcal{TS}^*$ of $\{(U \cap V, s), (U, s), (V, s), \iota_{U\cap V}^U, \iota_{U\cap V}^V\}$. Let

$$A := \pi_1(U \cap V, s), \qquad B := \pi_1(U, s), \qquad C := \pi_1(V, s).$$

Then, by van Kampen's Theorem, we see that $\pi_1(S, s) = A *_C B$, where the maps $A \to B$ and $A \to C$ are $\pi_1(\iota_{U\cap V}^U)$ and $\pi_1(\iota_{U\cap V}^V)$. Consequently, if we can calculate A, B, C, f and g, then the problem of calculating $\pi_1(S, s)$ is reduced to the algebraic problem of understanding pushouts in the category $\{\text{countable groups}\}$.

We come back to this in a moment but pause now to note that, quite generally,

if X is a paracompact topological space,
if $\mathcal{U}$ is an open cover of X with nonempty intersection,
if $x \in \cap\mathcal{U}$ and
if, for all $U, V \in \mathcal{U}$, we have that $U \cap V$ is path-connected,

then one can form a diagram $\mathcal{D}$ in $\mathcal{TS}^*$

whose set of objects is $\{(U \cap V, x) \,|\, U, V \in \mathcal{U}\}$,
whose arrows come from inclusions and
whose direct limit is (X, x).

Van Kampen's Theorem then reduces the calculation of $\pi_1(X, x)$ to the calculation of $\pi_1(\mathcal{D})$ combined with the algebraic problem of computing the direct limit of $\pi_1(\mathcal{D})$ in the category $\{\text{countable groups}\}$.

Returning to the specific case of the genus two orientable surface S, we have a presentation of the fundamental grou $\pi_1(U \cap V, s) = \langle \epsilon \rangle$, where ϵ is the endpoint fixed homotopy class of a parameterized circle going once around the annulus $U \cap V$ in the clockwise direction. Let s' be the image in S of a vertex on the boundary of the octagon. (Note that the vertices all have the same image.) Let ∂K be the boundary of the octagon $\overline{K}$. Let α_* be a parametrization running along one of the sides of ∂K labeled a. Let $\alpha' \in P_{s'}^{s'}(S)$ be the image of α_* in S. Using sides b, c and d, we similarly obtain loops $\beta', \gamma', \delta' \in P_{s'}^{s'}(S)$. Then a simple application of van Kampen's theorem shows that $\pi_1(V, s') = \langle \alpha', \beta', \gamma', \delta' \rangle$. That is, $\pi_1(V, s')$ is a free group on four letters. Let ρ be a path in S from s to s'. Let $\alpha := (\rho \| \alpha') \| \overleftarrow{\rho} \in P_s^s(S)$. Similarly, define $\beta, \gamma, \delta \in P_s^s(S)$. Then $\pi_1(V, s) = \langle \alpha, \beta, \gamma, \delta \rangle$. The inclusion $(U \cap V, s) \to (V, s)$ induces the map $g : \pi_1(U \cap V, s) \to \pi_1(V, s)$ defined by $\epsilon \mapsto \alpha\beta\alpha^{-1}\beta^{-1}\gamma\delta\gamma^{-1}\delta^{-1}$. According to standard notation, $[\alpha, \beta] := \alpha\beta\alpha^{-1}\beta^{-1}$ and $[\gamma, \delta] := \gamma\delta\gamma^{-1}\delta^{-1}$, so $\epsilon \mapsto [\alpha, \beta][\gamma, \delta]$.

As U is contractible, $\pi_1(U, s) \cong \langle \eta \,|\, \eta = 1 \rangle$. The map $f : \pi_1(U \cap V, s) \to \pi_1(U, s)$ induced by inclusion is then the trivial map, defined by $\epsilon \mapsto 1$.

Then $\pi_1(S, s)$ is the pushout in the category {countable groups} of the diagram

$$\{\pi_1(U \cap V, s), \pi_1(U, s), \pi_1(V, s), f, g\}.$$

We can therefore use the group direct limit algorithm to see that $\pi_1(S, s)$ is isomorphic to

$$\langle \alpha, \beta, \gamma, \delta, \epsilon, \eta \,|\, \eta = 1, \epsilon = [\alpha, \beta][\gamma, \delta], \epsilon = 1 \rangle.$$

It is then an unassigned group theory exercise to see that ϵ and η are not needed. In fact, we have that $\pi_1(S, s)$ is isomorphic to

$$\langle \alpha, \beta, \gamma, \delta \,|\, [\alpha, \beta][\gamma, \delta] = 1 \rangle.$$

We now turn to preparations for the proof of van Kampen's theorem. The fundamental category theoretic notion we will need is that of an "adjoint pair".

Given two categories $\mathcal{C}$ and $\mathcal{E}$, define $\mathcal{C} \times \mathcal{E}$ to be the category such that an object in $\mathcal{C} \times \mathcal{E}$ is

(α) a pair (C, E), where C is an object in $\mathcal{C}$ and E is an object in $\mathcal{E}$;

and such that an arrow $(C, E) \to (C', E')$ in $\mathcal{C} \times \mathcal{E}$ is

(β) a pair of arrows (f, g), where $f : C' \to C$ is an arrow in $\mathcal{C}$ and $g : E \to E'$ is an arrow in $\mathcal{E}$.

Note, above, that the arrow f goes from C' to C, and not the other way around.

Definition. Let $\mathcal{C}$ and $\mathcal{E}$ be categories and let $\mathcal{F} : \mathcal{C} \to \mathcal{E}$ and $\mathcal{G} : \mathcal{E} \to \mathcal{C}$ be functors. We say that $(\mathcal{F}, \mathcal{G})$ is an **adjoint pair** the following two functors $\mathcal{C} \times \mathcal{E} \to$ {sets} are equivalent:

($*$) $(C, E) \mapsto \mathrm{Hom}_{\mathcal{C}}(C, \mathcal{G}E)$; and

($**$) $(C, E) \mapsto \mathrm{Hom}_{\mathcal{E}}(\mathcal{F}C, E)$.

Let η be an equivalence between the functors ($*$) and ($**$). Then, for any (C, E), $\eta_{(C,E)}$ is a bijection between $\mathrm{Hom}_{\mathcal{C}}(C, \mathcal{G}E)$ and $\mathrm{Hom}_{\mathcal{E}}(\mathcal{F}C, E)$. This gives us four ways of moving arrows between $\mathcal{C}$ and $\mathcal{E}$:

(1) for any arrow $a : C \to C'$ in $\mathcal{C}$, $\mathcal{F}a : \mathcal{F}C \to \mathcal{F}C'$ is an arrow in $\mathcal{E}$;

(2) for any arrow $a : E \to E'$ in $\mathcal{E}$, $\mathcal{G}a : \mathcal{G}E \to \mathcal{G}E'$ is an arrow in $\mathcal{C}$;

(3) for any arrow $C \to \mathcal{G}E$ in $\mathcal{C}$; there is an η-corresponding arrow $\mathcal{F}C \to E$ in $\mathcal{E}$;

(4) for any arrow $\mathcal{F}C \to E$ in $\mathcal{E}$; there is an η-corresponding arrow $C \to \mathcal{G}E$ in $\mathcal{C}$.

Since η is an equivalence between ($*$) and ($**$), we have:

(A) for any objects C, C' in $\mathcal{C}$, for any object E in $\mathcal{E}$, for any arrows $f : C' \to C$ and $g : C \to \mathcal{G}E$ in $\mathcal{C}$, if $g' : \mathcal{F}C \to E$ corresponds to g under η, then $g' \circ (\mathcal{F}f)$ corresponds to $g \circ f$ under η.

Since η is an equivalence between ($*$) and ($**$), we also have:

(B) for any objects E, E' in $\mathcal{E}$, for any object C in $\mathcal{C}$, for any arrows $f : \mathcal{F}C \to E$ and $g : E \to E'$ in $\mathcal{E}$, if $f' : C \to \mathcal{G}E$ corresponds to f under η, then $(\mathcal{G}g) \circ f'$ corresponds to $g \circ f$ under η.

We now come to the main result we'll be using on adjoint pairs:

Theorem. Let $\mathcal{C}$ and $\mathcal{E}$ be categories. Let $\mathcal{F} : \mathcal{C} \to \mathcal{E}$ and $\mathcal{G} : \mathcal{E} \to \mathcal{C}$ be functors. Assume that $(\mathcal{F}, \mathcal{G})$ is an adjoint pair. Then $\mathcal{F}$ preserves direct limits. That is, if $\mathcal{D}$ is a diagram in $\mathcal{C}$ and if (L, f) is a direct limit for $\mathcal{D}$ in $\mathcal{C}$, then $(\mathcal{F}L, \mathcal{F}f)$ is a direct limit for $\mathcal{F}\mathcal{D}$ in $\mathcal{E}$.

Proof: Let (R, g) be a receiver of $\mathcal{F}\mathcal{D}$ in $\mathcal{E}$. We wish to show that there is a unique $h : \mathcal{F}L \to R$ such that $\{h \circ (\mathcal{F}f_0) \,|\, f_0 \in f\} = g$.

Let η be an equivalence from

$$(C, E) \mapsto \mathrm{Hom}_{\mathcal{C}}(C, \mathcal{G}E) \quad : \quad \mathcal{C} \times \mathcal{E} \to \{\text{sets}\}$$

to

$$(C, E) \mapsto \mathrm{Hom}_{\mathcal{E}}(\mathcal{F}C, E) \quad : \quad \mathcal{C} \times \mathcal{E} \to \{\text{sets}\}.$$

For each $D \in \mathrm{Obj}(\mathcal{D})$, let g_D be the unique arrow in $g \cap (\mathrm{Hom}(\mathcal{F}D, R))$. Then set $g' := \{\eta^{-1}_{(D,R)}(g_D) \,|\, D \in \mathrm{Obj}(\mathcal{D})\}$. Then $(\mathcal{G}R, g')$ is a receiver for $\mathcal{D}$.

Since (L, f) a direct limit of $\mathcal{D}$ in $\mathcal{C}$, choose $h' : L \to \mathcal{G}R$ such that $g' = \{h' \circ f_0 \,|\, f_0 \in f\}$. Now set $h := \eta_{(L,R)}(h') \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{F}L, R)$. Then, by the naturality properties of η, we see that $g = \{h \circ (\mathcal{F}f_0) \,|\, f_0 \in f\}$, as desired.

We leave the uniqueness of h as an exercise. **QED**

EXERCISE 11A: Let $\mathcal{C}$ and $\mathcal{E}$ be categories. Let $\mathcal{F} : \mathcal{C} \to \mathcal{E}$ and $\mathcal{G} : \mathcal{E} \to \mathcal{C}$ be functors. Assume that $(\mathcal{F}, \mathcal{G})$ is an adjoint pair. Let $\mathcal{D}$ be a diagram in $\mathcal{C}$. Let (L, f) be a direct limit of $\mathcal{D}$ in $\mathcal{C}$. Let (R, g) be a receiver of $\mathcal{F}\mathcal{D}$ in $\mathcal{G}$. Let $h, \widetilde{h} \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{F}L, R)$. Assume that $\{h \circ (\mathcal{F}f_0) \,|\, f_0 \in f\} = g = \{\widetilde{h} \circ (\mathcal{F}f_0) \,|\, f_0 \in f\}$. Show that $h = \widetilde{h}$.

Recall, for any group Γ, that E_Γ denotes the set of all

$$c_0\gamma_0 + c_1\gamma_1 + \cdots$$

such that $c_0, c_1, \ldots \in [0, 1]$, such that $\gamma_0, \gamma_1, \ldots \in \Gamma$, such that $|\{i \in \mathbb{N} \,|\, c_i \neq 0\}| < \infty$ and such that $c_0 + c_1 + \cdots = 1$, appropriately topologized. For any group Γ, let $1 \cdot 1_\Gamma + 0 + 0 + 0 + \cdots$ be the basepoint of E_Γ. For any group Γ, we have that Γ acts on E_Γ via $\gamma(c_0\gamma_0 + c_1\gamma_1 + \cdots) = c_0\gamma\gamma_0 + c_1\gamma\gamma_1 + \cdots$. Recall, for any group Γ, that $B_\Gamma := \Gamma \backslash E_\Gamma$; let the basepoint of B_Γ be the Γ-orbit of the basepoint $1 \cdot 1_\Gamma + 0 + 0 + 0 + \cdots$ of E_Γ.

Then $\Gamma \mapsto B_\Gamma : \{\text{countable groups}\} \to \mathcal{TS}^*$ is a functor, which, in the following proposition, we will denote by B. (What does B do to arrows in $\{$countable groups$\}$?)

The functor $\pi_1 : \mathcal{TS} \to \{\text{groups}\}$ also induces a functor $\mathcal{TS}^* \to \{\text{countable groups}\}$, which we will also denote by π_1 in the following proposition.

In view of the preceding theorem, van Kampen's Theorem follows from:

Proposition. The pair (π_1, B) is an adjoint pair.

Sketch of proof: Fix $X \in \mathcal{TS}^*$ and fix a group Γ. We wish to construct functions

$$\eta : \mathrm{Hom}(X, B_\Gamma) \to \mathrm{Hom}(\pi_1 X, \Gamma)$$

and

$$\zeta : \mathrm{Hom}(\pi_1 X, \Gamma) \to \mathrm{Hom}(X, B_\Gamma).$$

After we complete this construction below, we then leave it as an unassigned exercise to the reader to show that η and ζ are inverses. We also leave it as an unassigned exercise to prove the naturality requirements of η and ζ, as (X, Γ) varies.

Since we have a chosen basepoint in E_Γ, this determines an isomorphism between $\pi_1 B_\Gamma$ and the group of deck transformations of the canonical map $E_\Gamma \to B_\Gamma$. However, this group is exactly Γ, so we have an isomorphism $\tau : \pi_1 B_\Gamma \to \Gamma$. Similarly, we define an isomorphism $\mu : \pi_1 B_{\pi_1 X} \to \pi_1 X$.

We define

$$\eta : \mathrm{Hom}(X, B_\Gamma) \to \mathrm{Hom}(\pi_1 X, \Gamma)$$

by $\eta(f) = \tau \circ (\pi_1 f)$. It remains to construct ζ.

Claim: There is a unique $\alpha \in \mathrm{Hom}(X, B_{\pi_1 X})$ such that $\mu \circ (\pi_1 \alpha) \in \mathrm{Hom}(\pi_1 X, \pi_1 X)$ is the identity. *Proof of Claim:* Let $\Xi := \pi_1(X)$ and let $I := [0,1]$. Let $(\widetilde{X}, q)$ be a pointed universal cover of X. Let $r : E_\Xi \to B_\Xi$ be the canonical map.

For any pointed space S, let p_S denote the basepoint of S. The underlying topological space of S will also be denoted by S. Let $e := p_{E_\Xi}$ be the basepoint of E_Ξ.

We first prove *uniqueness*. Let $\alpha, \beta \in \mathrm{Hom}(X, B_\Xi)$ and assume that both $\mu \circ (\pi_1 \alpha)$ and $\mu \circ (\pi_1 \beta)$ are equal to the identity arrow $\pi_1 X \to \pi_1 X$. We wish to show that $\alpha = \beta$. Fix $\alpha_0 \in \alpha$ and $\beta_0 \in \beta$. We wish to show that α_0 is pointed homotopic to β_0.

Let $\widetilde{X}$ be a pointed universal cover of X. By lifting theory, there exist Ξ-equivariant pointed maps $\widehat{\alpha}_0, \widehat{\beta}_0 : \widetilde{X} \to E_\Xi$ such that $r \circ \widehat{\alpha}_0 = \alpha_0 \circ q$ and $r \circ \widehat{\beta}_0 = \beta_0 \circ q$

We wish to show that there exists a continuous map $H : \widetilde{X} \times I \to E_\Xi$ such that

(1) for all $x \in \widetilde{X}$, we have $H(x, 0) = \widehat{\alpha}_0(x)$; and $H(x, 1) = \widehat{\beta}_0(x)$;

(2) for all $\xi \in \Xi$, for all $x \in \widetilde{X}$ for all $t \in I$, we have $H(\xi x, t) = \xi(H(x, t))$; and

(3) for all $t \in I$, we have $H(p_{\widetilde{X}}, t) = e$.

EXERCISE 11B: Show that there exists a continuous map $k : E_\Xi \times E_\Xi \times I \to E_\Xi$ such that

(A) for all $x, y \in E_\Xi$, we have $k(x, y, 0) = x$ and $k(x, y, 1) = y$;

(B) for all $\xi \in \Xi$, for all $x, y \in E_\Xi$, for all $t \in I$, we have $k(\xi x, \xi y, t) = \xi(k(x, y, t))$; and

(C) for all $t \in I$, we have $k(e, e, t) = e$.

Fix $k : E_\Xi \times E_\Xi \times I \to E_\Xi$ as in Exercise 11B. Now define $H : \widetilde{X} \times I \to E_\Xi$ by $H(x, t) = k(\widehat{\alpha}_0(x), \widehat{\beta}_0(x), t)$. This ends the proof of *uniqueness*.

We now prove *existence*. It suffices to show that there exists a continuous Ξ-invariant map $\widehat{\alpha}_0 : \widetilde{X} \to E_\Xi$ such that $\alpha_0(p_{\widetilde{X}}) = e$.

Let $\overline{V}_0, \overline{V}_1, \overline{V}_2, \ldots$ be a locally finite open cover of X such that, for all integers $i \geq 0$, we have that $V_i := q^{-1}(\overline{V}_i)$ is a trivial principal topological Ξ-space. Reordering, if ncessary, we may assume that $p_X \in \overline{V}_0$. For each integer $i \geq 1$, by, if necessary, replacing $\overline{V}_i$ by $\overline{V}_i \backslash \{p_X\}$, we may assume that $p_X \notin \overline{V}_i$.

For all integers $i \geq 0$, choose a topological space U_i and an isomorphism of topological Ξ-spaces $\phi_i : V_i \to \Xi \times U_i$. We may assume that $\phi_0(p_{\widetilde{X}}) \in \{1_\Xi\} \times U_i$. For all integers $i \geq 0$, let $\psi_i : \Xi \times U_i \to \Xi$ be projection onto the first coordinate, and let $\lambda_i := \psi_i \circ \phi_i : V_i \to \Xi$; then λ_i is Ξ-equivariant, *i.e.*, for all $v \in V_i$, for all $\xi \in \Xi$, we have $\lambda_i(\xi v) = \xi(\lambda_i(v))$. For all integers $i \geq 0$, let $\kappa_i : \widetilde{X} \to \Xi$ be any function such that $\kappa_i | V_i = \lambda_i$. Note that $\kappa_0(p_{\widetilde{X}}) = 1_\Xi$.

By paracompactness, let $\overline{V}'_0, \overline{V}'_1, \overline{V}'_2, \ldots$ be a *shrinking* of $\overline{V}_0, \overline{V}_1, \overline{V}_2, \ldots$; this means that $\overline{V}'_0, \overline{V}'_1, \overline{V}'_2, \ldots$ is an open cover of X such that: for all integers $i \geq 0$, the closure in X of $\overline{V}'_i$ is contained in $\overline{V}_i$.

For all integers $i \geq 0$, by Urysohn's Lemma, let $\overline{f}'_i : X \to [0,1]$ be a continuous map such that $\overline{f}'_i(\overline{V}'_i) = \{1\}$ and such that $\overline{f}'_i(X \backslash \overline{V}_i) = \{0\}$. Let $\overline{f}' := \sum_{i=0}^{\infty} \overline{f}'_i$. For all integers $i \geq 0$, define $\overline{f}_i := \overline{f}'_i / \overline{f}$.

For all integers $i \geq 0$, let $f_i := \overline{f}_i \circ q : \widetilde{X} \to [0,1]$. Now define $\widehat{\alpha}_0 : \widetilde{X} \to E_\Xi$ by:

$$\widehat{\alpha}_0(x) = [f_0(x)][\kappa_0(x)] + [f_1(x)][\kappa_1(x)] + [f_2(x)][\kappa_2(x)] + \cdots$$

This ends the proof of *existence*. *End of proof of claim.*

Fix $\alpha \in \mathrm{Hom}(X, B_{\pi_1 X})$ as in the claim. We now define

$$\zeta : \mathrm{Hom}(\pi_1 X, \Gamma) \to \mathrm{Hom}(X, B_\Gamma)$$

by $\zeta(f) = B_f \circ \alpha$. **QED**

For all integers $n \geq 0$, let $e_0^n, \ldots, e_n^n$ be the standard basis of $\mathbb{R}^{n+1}$, let $\nu_n := \{e_0^n, \ldots, e_n^n\}$ and let

$$\sigma_n := \{c_0 e_0^n + \cdots + c_n e_n^n \mid c_0, \ldots, c_n \in [0,1], c_0 + \cdots + c_n = 1\}$$

be the closed conved hull of ν_n.

Let S be a finite set and let $n := |S| - 1$. A **labeling** of S is a bijective map $S \to \nu_n$.

Let S be a set and let $R \subseteq S$. Let $m := |R| - 1$ and $n := |S| - 1$. Let $\phi : R \to \nu_m$ and $\psi : S \to \nu_n$ be labelings. Let $\chi := \psi \circ (\phi^{-1})$. Then we define $I_\phi^\psi : \sigma_m \to \sigma_n$ by

$$I_\phi^\psi(c_0 e_0^m + \cdots + c_m e_m^m) = c_0[\chi(e_0^m)] + \cdots + c_m[\chi(e_m^m)].$$

Definition. An **(abstract) simplicial complex** consists of

(1) a nonempty set S; and
(2) $F \subseteq \{\text{finite subsets of } S\}$

such that

(A) for all $s \in S$, we have $\{s\} \in F$; and
(B) for all $f \in F$, for all $f' \subseteq f$, we have $f' \in F$.

For any integer $n \geq 0$, an **n-face** of a simplicial complex (S, F) is an element $f \in F$ such that $|f| = n + 1$. For any integer $n \geq 0$, a **labeled n-face** of a simplicial complex (S, F) consists of

(α) an n-face f of (S, F); and
(β) a labeling of f.

We define the **dimension**, $\dim(S, F)$, of a simplicial complex (S, F) to be the supremum of the set of integers $n \geq 0$ satisfying: there exists an n-face of (S, F). This supremum may equal ∞.

Definition. Let (S, F) be a simplicial complex. For all integers $n \geq 0$, let F_n be the set of all labeled n-faces of (S, F). Let $X := \bigcup_{n=0}^{\infty} F_n \times \sigma_n$. For each integer $n \geq 0$, give F_n the discrete topology, give each σ_n the inherited topology from the standard topology on $\mathbb{R}^{n+1}$ and let τ_n denote the product topology on $F_n \times \sigma_n$. Let X have the topology generated by $\bigcup_{n=0}^{\infty} \tau_n$. Define a relation $\mapsto$ on X by: $((f, \phi), x) \mapsto ((g, \psi), y)$ iff both $f \subseteq g$ and $I_\phi^\psi(x) = y$. Let $\sim$ be the equivalence relationg generated by $\mapsto$. Then the **realization** of (S, F) is $X/\sim$.

Example. Let $S := \{1, 2, 3\}$ and let $F := \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Then (S, F) is a simplical complex. Its realization is a "triangle", so that it is homeomorphic to S^1. Consequently, the fundamental group of its realization in isomorphic to $\mathbb{Z}$. Let $F' := F \cup \{\{1, 2, 3\}\}$. Then the realization of (S, F') is a solid triangle, and is therefore homeomorphic to a disk. It is therefore contractible and its fundamental group is trivial.

Definition. A **tree** is a simply connected topological space which is homeomorphic to the realization of a simplicial complex of dimension ≤ 1. A **graph** is a locally path-connected topological space with a universal cover that is a tree.

Let $x := (1, 0) \in S^1$ be a point on the circle. Let K be a set. Give K the discrete topology. Then a **bouquet of K-many circles** is the topological space

$$B := (S \times K)/(\{x\} \times K)$$

obtained by identifying $\{x\} \times K$ to a point in $S \times K$. Then B is a graph. Van Kampen's theorem shows that the fundamental group of this space is homeomorphic to the free group $\langle K \rangle$.

Proposition. Let G be a graph. Then there exists $T \subseteq G$ such that T is a tree, such that G/T is a bouquet of circles and such that $\pi_1(G)$ is isomorphic to $\pi_1(G/T)$.

We omit proof except to comment that one takes, in G, a so-called "maximal tree" T, and the proposition follows.

Corollary. For any graph G, we have: $\pi_1(G)$ is a free group.

Our next goal is to use topology to prove the algebraic fact that a subgroup of a free group is free.

However, we need first to address the question of whether a composite of covering maps is again a covering map. In fact, I'm unsure if this is true in complete generality, but I can prove it when the spaces in question are connected and semi-locally simply connected. For the moment, however, let's make the following ad-hoc definition:

Definition. Let P and Q be topological spaces and let $\alpha : P \to Q$ be a function. Then we say that α is a **composite covering map** if there is an integer $k \geq 1$ and there are topological spaces $R_0, \ldots, R_k$ and there are covering maps

$$\beta_1 : R_0 \to R_1, \qquad \ldots, \qquad \beta_k : R_{k-1} \to R_k$$

such that $P = R_0$, $Q = R_k$ and $\alpha = \beta_k \circ \cdots \circ \beta_1$.

That is, a map is a compostite covering map if it is a composition of covering maps.

EXERCISE 12A: Let P, Q and X be connected, locally path-connected topological spaces. Let $\alpha : P \to Q$ be a composite covering map. Let $f : X \to Q$ be continuous. Let $q \in Q$, let $p \in \alpha^{-1}(q)$ and let $x \in f^{-1}(q)$. Assume that $f_*(\pi_1(X, x)) \subseteq \alpha_*(\pi_1(P, p))$. Show that there exists a unique continuous map $\widehat{f} : X \to P$ such that $\alpha \circ \widehat{f} = f$ and such that $\widehat{f}(x) = p$.

Definition. Let Y be a path-connected topological space. A **universal composite cover** of Y is a simply connected topological space X, together with a composite covering map $X \to Y$.

EXERCISE 12B: Let Y be a connected, locally path-connected topological space. Let (X, ϕ) and (X', ϕ') be universal composite covers of Y. Show that (X, ϕ) and (X', ϕ') are isomorphic in the category {topological spaces over Y}.

Fact. Let X and Y be connected, locally path-connected topological spaces. Let $\phi : X \to Y$ be a composite covering map. Let $(\widetilde{X}, \psi)$ be a universal cover of X. Then $(\widetilde{X}, \phi \circ \psi)$ is a universal cover of Y.

Proof: Since X has a universal cover, X is semi-locally simply connected. Then Y is semi-locally simply connected. Let $(\widetilde{Y}, \chi)$ be a universal cover of Y. Then, by Exercise 12B, $(\widetilde{X}, \phi \circ \psi)$ is isomorphic to $(\widetilde{Y}, \chi)$ in the category {topological spaces over Y}. So, since $(\widetilde{Y}, \chi)$ is a universal cover of Y, $(\widetilde{X}, \phi \circ \psi)$ must be as well. **QED**

If X and Y are topological spaces, then we say that X is a **covering** of Y if there exists a covering map $X \to Y$. With this definition, the preceding fact asserts that a universal cover of a cover of Y is a universal cover of Y.

Proposition. Let W, X and Y be connected, semi-locally simply connected topological spacs. Let $\phi : W \to X$ and $\psi : X \to Y$ be covering maps. Then $\psi \circ \phi : W \to Y$ is a covering map.

Proof: Let $(\widetilde{W}, \chi)$ be a universal cover of W. Then $(\widetilde{W}, \phi \circ \chi)$ is a universal cover of X and $(\widetilde{W}, \psi \circ \phi \circ \chi)$ is a universal cover of Y. Then χ, $\phi \circ \chi$ and $\psi \circ \phi \circ \chi$ are all regular covering maps.

Let Γ be the group of deck transformations of χ. Let Λ be the group of deck transformations of $\phi \circ \chi$. Let Δ be the group of deck transformations of $\psi \circ \phi \circ \chi$. Note that $\Gamma \subseteq \Lambda \subseteq \Delta$. Then W, X and Y are isomorphic to $\Gamma \backslash \widetilde{W}$, $\Lambda \backslash \widetilde{W}$ and $\Delta \backslash \widetilde{W}$, respectively, in the category {topological spaces under $\widetilde{W}$}.

Let $\pi : \Gamma \backslash \widetilde{W} \to \Delta \backslash \widetilde{W}$ be the canonical map. A short diagram chase then shows that $\psi \circ \phi$ is isomorphic to π in the arrow category of {topological spaces}. Consequently, it suffices to show that $\pi : \Gamma \backslash \widetilde{W} \to \Delta \backslash \widetilde{W}$ is a covering map.

We leave this as an unassigned exercise. **QED**

Lemma. Any covering of a graph is a graph.

Proof: Let Y be a graph and let X be a covering of Y. Then, by the preceding fact, any universal covering of X is a universal covering of Y, and is therefore a tree. We conclude that X is a graph. **QED**

Theorem. Any subgroup of a free group is free.

Proof: Let S be a set. We wish to show that any subgroup of $\langle S \rangle$ is free.

Let Y be a bouquet of S-many circles. Then, by van Kampen's Theorem, $\pi_1(Y)$ is isomporphic to $\langle S \rangle$. Let (X, ϕ) be a universal cover of Y. Let Γ be the group of deck transformations of ϕ. Then Γ is isomorphic to $\pi_1(Y)$. Let Γ_0 be a subgroup of Γ. We wish to show that Γ_0 is free.

Since $\Gamma \backslash X$ is homeomorphic to Y, it follows that $\Gamma \backslash X$ is a graph. Since $\Gamma_0 \backslash X$ is a covering of $\Gamma \backslash X$, we see, from the preceding lemma, that $\Gamma_0 \backslash X$ is a graph. Then $\pi_1(\Gamma_0 \backslash X)$ is free. However, $\pi_1(\Gamma_0 \backslash X)$ is isomorphic to Γ_0, so we are done. **QED**

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