More notes for Math 8301-8302, Manifolds and Topology, Fall 2004-Spring 2005
Homology Theory

Definition. A **(chain) complex** consists of

(1) a bi-infinite sequence $\ldots, C_1, C_0, C_{-1}, \ldots$ of additive Abelian groups; and

(2) for each integer n, an additive group homomorphism $\partial_n : C_n \to C_{n-1}$

such that, for all integers n, we have $\partial_n \circ \partial_{n+1} = 0$. A chain complex $C_\bullet$ is **nonnegative** if: for all integers $n < 0$, we have $C_n = \{0\}$.

There is a category of chain complexes. (What are the arrows?) An arrow from one chain complex to another is called a **chain map**.

Definition. Let $C_\bullet$ be a chain complex. For all integers n, $Z_n(C_\bullet)$ denotes the kernel of $\partial_n : C_n \to C_{n-1}$, and $B_n(C_\bullet)$ denotes the image of $\partial_{n+1} : C_{n+1} \to C_n$. For all integers n, we define $H_n(C_\bullet) = (Z_n(C_\bullet))/(B_n(C_\bullet))$. For all integers n, C_n is called the n**th chain group** of $C_\bullet$, $B_n(C_\bullet)$ is called the n**th boundary group** of $C_\bullet$, $Z_n(C_\bullet)$ is called the n**th cycle group** of $C_\bullet$ and $H_n(C_\bullet)$ is called the n**th homology group** of $C_\bullet$. For any integer n, an n-cycle is an element of $Z_n(C_\bullet)$ and an n-boundary is an element of $B_n(C_\bullet)$. An n-cycle is said to **bound** if it is an element of $B_n(C_\bullet)$.

Thus $H_n(C_\bullet)$ is the group obtained by modding out n-cycles by those n-cycles that bound.

A **graded group** is a bi-infinite sequence of additive Abelian groups. What are the arrows in the category of graded groups?

With this terminology, $H_\bullet$ is a functor from {chain complexes} to {graded groups}.

For any integer $n \geq 0$, let $\Delta^n := \{(x_0, \ldots, x_n) \in [0,1]^n \,|\, x_0 + \cdots + x_n = 1\} \subseteq \mathbb{R}^{n+1}$. For any topological space, for any integer $n \geq 0$, a **parametric** n**-simplex** in X is a continuous function $\Delta^n \to X$. Thus the set of parametric n-simplices in X is $C(\Delta^n, X)$.

Recall, for any set S, that $\langle S \rangle$ denotes the free group on S. Then $\langle \cdot \rangle$ is a functor from {sets} to {groups}. Let $\mathcal{F}$: {groups} $\to$ {sets} denote the forgetful functor.

EXERCISE 12C: Show that $(\langle \cdot \rangle, \mathcal{F})$ is an adjoint pair.

For any set S, let $\mathbb{Z}[S]$ denote the group of formal finite linear combinations of elements of S with coefficients in $\mathbb{Z}$. This group is called the **free additive Abelian group generated by** S. Then $\mathbb{Z}[\cdot]$ is a functor from from {sets} to {additive Abelian groups}. Let $\mathcal{F}'$: {additive Abelian groups} $\to$ {sets} denote the forgetful functor.

EXERCISE 12D: Show that $(\mathbb{Z}[\cdot], \mathcal{F}')$ is an adjoint pair.

For any integer $n \geq 1$, for any integer $i \in [0, n]$, define $\varepsilon_i^n : \Delta^{n-1} \to \Delta^n$ by $\varepsilon_i^n(x_1, \ldots, x_n) = (x_1, \ldots, x_i, 0, x_{i+1}, \ldots, x_n)$. For any integer $n \geq 1$, for any integer $i \in [0, n]$, define $\partial_i^n : C(\Delta^n, X) \to C(\Delta^{n-1}, X)$ by $\partial_i^n(\sigma) = \sigma \circ \varepsilon_i^n$.

For any topological space X, for any integer $n \geq 0$, let $S_n(X) := \mathbb{Z}[C(\Delta^n, X)]$ be the free additive Abelian group generated by parametric n-simplices in X. For any topological space X, for any integer $n < 0$, let $S_n(X) := \{0\}$ be the trivial additive Abelian group. For any topological space X, for any integer $n \geq 1$, let $\partial_n : S_n(X) \to S_{n-1}(X)$ be the unique $\mathbb{Z}$-linear extension of

$$\sigma \quad \mapsto \quad \sum_{i=0}^{n} (-1)^i [\partial_i^n(\sigma)] \quad : \quad C(\Delta^n, X) \quad \to \quad S_{n-1}(X).$$

For any topological space X, for any integer $n < 1$, let $\partial_n : S_n(X) \to S_{n-1}(X)$ be the zero map.

EXERCISE 12E: Show that $(S_\bullet(X), \partial_\bullet)$ is a nonnegative chain complex. That is, for all integers n, show that $\partial_n \circ \partial_{n+1} = 0$.

For any topological space X, for any integer n, we define $Z_n(X) := Z_n(S_\bullet(X))$, $B_n(X) := B_n(S_\bullet(X))$ and $H_n(X) := H_n(S_\bullet(X))$. Then $H_\bullet$ is a functor from the category {topological spaces} to the category {graded groups}.

Definition. Let X be a topological space, let $n \in Z$ and let $\alpha, \beta \in Z_n(X)$. We say that α is **homologous** to β if $\beta - \alpha$ bounds.

Equivalently, the images of α and β in $H_n(X) = (Z_n(X))/(B_n(X))$ are equal.

Recall, for any chain complex $C_\bullet$, for any $n \in \mathbb{Z}$, that we we defined $Z_n(C_\bullet) := \ker(\partial_n : C_n \to C_{n-1})$ and $B_n(C_\bullet) := \mathrm{im}(\partial_{n+1} : C_{n+1} \to C_n)$. Then Z_n and B_n are functors from {chain complexes} to {additive Abelian groups}. Then $Z_\bullet$ and $B_\bullet$ are functors from {chain complexes} to {graded groups}.

Definition. Let $C_\bullet$ be a chain complex and let $n \in \mathbb{Z}$. For any $\alpha \in C_n$, we say that α is a **cycle** if $\alpha \in Z_n(C)$. For any $\alpha \in Z_n(C_\bullet)$, we say that α **bounds** if $\alpha \in B_n(C_\bullet)$. For any $\alpha, \beta \in Z_n(C_\bullet)$, we say that α and β are **homologous** if $\beta - \alpha$ bounds.

We now turn to the question of computability of homology groups of topological spaces. We begin with H_0 : {topological spaces} $\to$ {additive Abelian groups}.

Define a functor $\mathcal{PC}$ from {topological spaces} to {sets} by letting $\mathcal{PC}(X)$ be the set of path-components of X. We leave it to the reader to consider what $\mathcal{PC}$ does to arrows in {topological spaces}.

EXERCISE 12F: Show that H_0 : {topological spaces} $\to$ {additive Abelian groups} is equivalent to $X \mapsto \mathbb{Z}[\mathcal{PC}(X)]$: {topological spaces} $\to$ {additive Abelian groups}.

By the preceding exercise, if a topological space X has five path-components, then $H_0(X)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Computation of H_0 is equivalent to counting connected components.

We now move on to H_1. We begin with the most basic question:

Question: What is $H_1(S^1)$?

Definition. For any group G, let $[G, G]$ denote the subgroup of G generated by the set $\{xyx^{-1}y^{-1} \,|\, x, y \in G\}$.

Let $\mathcal{N}$ denote the collection of all normal subgroups N of G such that G/N is Abelian. Then it's not hard to show that $[G, G]$ is the unique minimal element of $\mathcal{N}$. In particular,

$[G, G]$ is a normal subgroup of G and $G/[G, G]$ is Abelian. The group $G/[G, G]$ is called the **Abelianization** of G.

EXERCISE 13A: Let $\mathcal{F} : \{\text{groups}\} \to \{\text{Abelian groups}\}$ be the functor $\mathcal{F}(G) = G/[G, G]$ which carries a group to its Abelianization. Let $\mathcal{G} : \{\text{Abelian groups}\} \to \{\text{groups}\}$ be inclusion: $\mathcal{G}(A) = A$. Show that $(\mathcal{F}, \mathcal{G})$ is an Adjoint pair.

Let $I := [0, 1]$. Let $\kappa : I \to \Delta^1$ be defined by $\kappa(t) = (1 - t, t)$.

EXERCISE 13B: Let X be a topological space, let $x_0 \in X$ and let $\sigma \in P_{x_0}^{x_0}(X)$ be a loop at x_0 in X. Then $\sigma \circ \kappa^{-1} \in C(\Delta^1, X) \subseteq S_1(X)$. Show that $\sigma \circ \kappa^{-1} \in Z_1(X)$, *i.e.*, that $\partial_1(\sigma \circ \kappa^{-1}) = 0$.

Let X be a topological space and let $x_0 \in X$. Define a map $\Phi : P_{x_0}^{x_0}(X) \to Z_1(X)$ by $\Phi(\sigma) = \sigma \circ \kappa^{-1}$. Let $C : P_{x_0}^{x_0}(X) \to \pi_1(X_0, x_0)$ and $D : Z_1(X) \to H_1(X)$ be the canonical maps. Then we will show later that there is a unique map $\Psi_{(X,x_0)} : \pi_1(X, x_0) \to H_1(X)$ such that $D \circ \Phi = \Psi_{(X,x_0)} \circ C$. This map $\Psi_{(X,x_0)} : \pi_1(X, x_0) \to H_1(X)$ is a homomorphism of groups and is called the **Hurewicz homomorphism**.

We omit the proof of the following interesting result:

Theorem. Let X be a path-connected topological space and let $x_0 \in X$. Then the Hurewicz homomorphism $\Psi_{(X,x_0)} : \pi_1(X, x_0) \to H_1(X)$ is surjective and its kernel is $[\pi_1(X, x_0), \pi_1(X, x_0)]$.

Let $\mathcal{C}$ be the category {pointed path connected topological spaces}. For any group G, let $A_G := G/[G, G]$ be the Abelianization of G and let $c_G : G \to A_G$ be the canonical map. Because of the preceding theorem, for any pointed path-connected topological space $Z = (X, x_0)$, there is a unique isomorphism of Abelian groups $\overline{\Psi}_Z : A_{\pi_1(Z)} \to H_1(X)$ such that $\overline{\Psi}_Z \circ c_{\pi_1(Z)} = \Psi_Z$. Then $\overline{\Psi}$ is an equivalence between the functor

$$Z \mapsto A_{\pi_1(Z)} : \mathcal{C} \to \{\text{Abelian groups}\}$$

and the functor

$$(X, x_0) \mapsto H_1(X) : \mathcal{C} \to \{\text{Abelian groups}\}.$$

In particular, for any path-connected topological space X, for any $x_0 \in X$, we see that $H_1(X)$ is the Abelianization of $\pi_1(X, x_0)$. This means that, for a person that has facility with the algebraic process of computing the Abelianization of groups, one can get H_1 from π_1. So H_1 is "no harder than" π_1. In fact, in practice, it often happens that computation of π_1 is difficult, whereas H_1 is easier.

Also, we can answer our question about computing $H_1(S^1)$. Let $x \in S^1$ be any point. Then $H_1(S^1)$ is isomorphic to the Abelianization of $\pi_1(S^1, x)$. So $H_1(S^1)$ is isomorphic to the Abelianization of $\mathbb{Z}$. However, $\mathbb{Z}$ is Abelian, and is therefore isomorphic to its own Abelianization. Then $H_1(S^1)$ is isomorphic to $\mathbb{Z}$.

Let X be the genus two orientable surface. Let's next compute $H_1(X)$. Recall that the fundamental group of X is isomorphic to $G := \langle a, b, c, d \,|\, [a, b][c, d] = 1\rangle$. Then $H_1(X)$ is isomorphic to the Abelianization of G, which is isomorphic to

$$\langle a, b, c, d \,|\, [a, b] = 1, [a, c] = 1, [a, d] = 1, [b, c] = 1, [b, d] = 1, [c, d] = 1, [a, b][c, d] = 1\rangle.$$

The last of these relations ($[a,b][c,d]=1$) is implied by the others, and so can be omitted. Then $H_1(X)$ is isomorphic to

$$\langle a,b,c,d \mid [a,b]=1, [a,c]=1, [a,d]=1, [b,c]=1, [b,d]=1, [c,d]=1 \rangle.$$

Up to isomorphism, this is the free Abelian group on $\{a,b,c,d\}$. Then $H_1(X)$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$.

We now take the point of view that we "know" H_0 and H_1. To continue to talk about H_n for $n\geq 2$, we develop some notation about "affine" parametric simplices.

For any integer $m\geq 0$, for any $S\subseteq\mathbb{R}^m$, let $\mathrm{cvx}(S)$ be the set of all $c_0s_0+\cdots+c_ks_k$ such that $k\geq 0$ is an integer, such that $s_0,\ldots,s_k\in S$, such that $c_0,\ldots,c_k\in[0,1]$ and such that $c_0+\cdots+c_k=1$. Then $\mathrm{cvx}(S)$ is called the **convex hull** of S.

Recall, for any integer $m\geq 0$, that $e_0^k,\ldots,e_k^k$ is the standard basis of $\mathbb{R}^{k+1}$ and that $\Delta^k=\mathrm{cvx}\{e_0^k,\ldots,e_k^k\}\subseteq\mathbb{R}^{k+1}$.

Let $k,m\geq 0$ be integers and let $x_0,\ldots,x_k\in\mathbb{R}^m$. Let $X:=\mathrm{cvx}\{x_0,\ldots,x_k\}$. Let $[x_0,\ldots,x_k]$ denote the map $c_0e_0^k+\cdots+c_ke_k^k\mapsto c_0x_0+\cdots+c_kx_k:\Delta^k\to X$. Then $[x_0,\ldots,x_k]\in C(\Delta^k,X)\subseteq S_k(X)$.

Let $k,m\geq 0$ be integers, let X be a convex subset of $\mathbb{R}^m$ and let $c=(c_0,\ldots,c_k)\in X^{k+1}$. We define $[c]:=[c_0,\ldots,c_k]\in S_k(X)$. For any integer $i\in[0,k]$, we define $c^i:=(c_0,\ldots,c_{i-1},c_{i+1},\ldots,c_k)\in X^k$.

Let $k,m\geq 0$ be integers, let X be a convex subset of $\mathbb{R}^m$ and let $c\in X^{k+1}$. We leave it as an unassigned exercise to show that $\partial[c]=\sum_{i=0}^{k}(-1)^i[c^i]$.

For example, we have the following: Let $m\geq 0$ be an integer, let X be a convex subset of $\mathbb{R}^m$ and let $x,y,z\in X$. Then $\partial[x,y,z]=[y,z]-[x,z]+[x,y]$.

Let $I:=[0,1]$. Let $p:=(0,0)$, $q:=(1,0)$, $r:=(0,1)$, $s:=(1,1)$. Then p,q,r,s are the four "corners" of the square $I\times I$. Let $\sigma:=[p,q,r]-[q,r,s]\in S_2(I\times I)$. One now verifies readily that $\partial\sigma=-[r,s]+[p,q]+[q,s]-[p,r]$.

Let X be a topological space and let $x_0\in X$. Recall that $\kappa:I\to\Delta^1$ is defined by $\kappa(t)=(1-t,t)$, so $\kappa(0)=(1,0)=e_0^1$ and $\kappa(1)=(0,1)=e_1^1$. Define $\Phi:P_{x_0}^{x_0}(X)\to Z_1(X)$ by $\Phi(\sigma)=\sigma\circ\kappa^{-1}$. Let $C:P_{x_0}^{x_0}(X)\to\pi_1(X_0,x_0)$ and $D:Z_1(X)\to H_1(X)$ be the canonical maps. As promised earlier, we show that there is a unique map

$$\Psi_{(X,x_0)}:\pi_1(X,x_0)\to H_1(X)$$

such that $D\circ\Phi=\Psi_{(X,x_0)}\circ C$.

This statement is equivalent to the following:

Proposition. Let X be a topological space and let $x_0\in X$. Let $\alpha,\beta:I\to X$ be loops at x_0 in X. Let κ be defined as above. Assume that α and β are endpoint fixed homotopic. Then $(\beta\circ\kappa^{-1})-(\alpha\circ\kappa^{-1})\in Z_1(X)$ bounds.

Proof: Let $H:I\times I\to X$ be an endpoint fixed homotopy from α to β. Let p, q, r, s and σ be as defined above. We aim to show that $\partial(H_*(\sigma))=(\beta\circ\kappa^{-1})-(\alpha\circ\kappa^{-1})$.

Since $\partial\sigma=[q,s]-[p,r]-[r,s]+[p,q]$, we see that

$$\partial(H_*(\sigma))=(H_*([r,s]))-(H_*([p,q]))-(H_*([q,s]))+(H_*([p,r])).$$

We have $H_*([r,s]) = (H(1,\cdot)) \circ \kappa^{-1} = \beta \circ \kappa^{-1}$. Similarly, $H_*([p,q]) = \alpha \circ \kappa^{-1}$. It remains to show that $H_*([q,s]) = H_*([p,r])$.

Let $c : I \to X$ be the constant map at x_0, defined by $c(t) = x_0$. Then $H(\cdot,0) = H(\cdot,1) = c$. Then $H_*([q,s]) = (H(\cdot,1)) \circ \kappa^{-1} = c \circ \kappa^{-1}$. Similarly, $H_*([p,r]) = c \circ \kappa^{-1}$. Then $H_*([q,s]) = H_*([p,r])$. **QED**

Proposition. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. Assume that f and g are homotopic. Then $f_* : H_1(X) \to H_1(Y)$ is equal to $g_* : H_1(X) \to H_1(Y)$.

Proof: Let $h : I \times X \to Y$ be a continuous map such that $h(0,\cdot) = f$ and $h(1,\cdot) = g$.

Let $p,q,r,s \in I \times \Delta^1$ be defined by $p := (0, e_0^1)$, $q := (0, e_1^1)$, $r := (1, e_0^1)$ and $s := (1, e_1^1)$. We leave it as an unassigned exercise to show that there exists $\sigma \in S_2(I \times \Delta^1)$ such that $\partial\sigma = [q,s] - [p,r] - [r,s] + [p,q]$.

For all $\omega \in C(\Delta^1, X)$, define $R_\omega : I \times \Delta^1 \to Y$ by $R_\omega(u,v) = h(u, \omega(v))$. Define $\phi : S_1(X) \to S_2(Y)$ by: for all $\omega \in C(\Delta^1, X)$, $\phi(\omega) = (R_\omega)_*(\sigma)$.

Recall that $\Delta^0 = \{1\} \subseteq \mathbb{R}$. Let $a, b \in I \times \Delta^0$ be defined by $a := (0,1)$ and $b := (1,1)$. Let $\rho := [a,b]$. Then $\partial\rho = [b] - [a]$.

For all $\omega \in C(\Delta^0, X)$ define $L_\omega : I \times \Delta^0 \to Y$ by $L_\omega(u,v) = h(u, \omega(v))$. Define $\psi : S_0(X) \to S_1(Y)$ by: for all $\omega \in C(\Delta^1, X)$, $\psi(\omega) = (L_\omega)_*(\rho)$.

EXERCISE 13C: Show, for all $\omega \in C(\Delta^1, X)$, that $\partial_2(\phi(\omega)) = -[g_*(\omega)] + [f_*(\omega)] - [\psi(\partial_1(\omega))]$.

Let $\omega_0 \in H_1(X)$. We wish to show that $f_*(\omega_0) = g_*(\omega_0)$.

Let $c : Z_1(X) \to H_1(X)$ be the canonical map.

Choose $\omega \in Z_1(X)$ such that $c(\omega) = \omega_0$. Since $\omega \in Z_1(X) = \ker(\partial_1 : S_1(X) \to S_0(X))$, we conclude that $\partial_1(\omega) = 0$. Then, by Exercise 13C, we see that $\partial_2(\phi(\omega)) = [g_*(\omega)] - [f_*(\omega)]$.

Then $[g_*(\omega)] - [f_*(\omega)] \in \operatorname{im}(\partial_2 : S_2(X) \to S_1(X)) = B_1(X)$. So, as $B_1(X) = \ker(c)$, we get $c([g_*(\omega)] - [f_*(\omega)]) = 0$. So, as $c(g_*(\omega)) = g_*(\omega_0)$ and as $c(f_*(\omega)) = f_*(\omega_0)$, we conclude that $f_*(\omega_0) = g_*(\omega_0)$, as desired. **QED**

EXERCISE 14A: Let k and m be integers and let $U \subseteq \mathbb{R}^m$ be star-shaped. Assume that $k \neq 0$. Show, for all $\omega \in Z_k(U)$, that ω bounds. That is, show that $H_k(U) = 0$.

We now extend the preceding theorem slightly, and, at the same time, reorganizing the notation a bit.

Theorem. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. Assume that f is homotopic to g. Then $f_* : H_1(X) \to H_1(Y)$ is equal to $g_* : H_1(X) \to H_1(Y)$ and $f_* : H_2(X) \to H_2(Y)$ is equal to $g_* : H_2(X) \to H_2(Y)$.

Proof: Choose a continuous map $h : I \times X \to Y$ such that $h(0,\cdot) = f$ and such that $h(1,\cdot) = g$.

We make the following definitions:

(A) For all $\omega \in C(\Delta^0, X)$, define $L_\omega : I \times \Delta^0 \to Y$ by $L_\omega(u,v) = h(u, \omega(v))$.

(B) For all $\omega \in C(\Delta^1, X)$, define $R_\omega : I \times \Delta^1 \to Y$ by $R_\omega(u,v) = h(u, \omega(v))$.

(C) For all $\omega \in C(\Delta^2, X)$, define $P_\omega : I \times \Delta^2 \to Y$ by $P_\omega(u,v) = h(u, \omega(v))$.
Above, "L" stands for "line segment", "R" stands for "rectangle" and "P" stands for prism. Note that $I \times \Delta^0$ is a line segment, $I \times \Delta^1$ is a rectangle and $I \times \Delta^2$ is a prism.

Fix $\rho \in S_1(I \times \Delta^0)$ and $\sigma \in S_2(I \times \Delta^1)$ as in the preceding proof. Also, as in the preceding proof, define $\psi : S_0(X) \to S_1(Y)$ and $\phi : S_1(X) \to S_2(Y)$ by:

(A') for all $\omega \in C(\Delta^0, X)$, $\psi(\omega) = (L_\omega)_*(\rho)$; and

(B') for all $\omega \in C(\Delta^1, X)$, $\phi(\omega) = (R_\omega)_*(\sigma)$.

For any $\tau \in S_3(I \times \Delta^2)$, define $\chi_\tau : S_2(X) \to S_3(Y)$ by

(C') for all $\omega \in C(\Delta^2, X)$, $\chi_\tau(\omega) = (P_\omega)_*(\tau)$.

According to Exercise 13C, for all $\omega \in C(\Delta^1, X)$, we have

(1) $(g_* - f_*)(\omega) = (\partial_2 \circ \phi + \psi \circ \partial_1)(\omega))$.

Then, by linearity, (1) holds for all $\omega \in S_1(X)$.

EXERCISE 14B: Show that there exists $\tau \in S_3(I \times \Delta^2)$ such that, for all $\omega \in C(\Delta^2, X)$, we have

(2) $(g_* - f_*)(\omega) = (\partial_3 \circ \chi_\tau + \phi \circ \partial_2)(\omega))$.

(Hint: For all integers $i \in [0,2]$, let $\delta_i := \mathrm{id} \times \varepsilon_i^2 : I \times \Delta^1 \to I \times \Delta^2$. Define $l, r : \Delta^2 \to I \times \Delta^2$ by $l(u) = (0,u)$ and $r(u) = (1,u)$. Let $\tau_0 := r - l + (\delta_0)_*(\sigma) - (\delta_1)_*(\sigma) + (\delta_2)_*(\sigma) \in S_2(I \times \Delta^2)$. Show that $\partial_2 \tau_0 = 0$. Then argue from Exercise 14A that τ_0 bounds. Choose $\tau \in S_3(I \times \Delta^2)$ such that $\partial_3 \tau = \tau_0$. Then show that (2) above holds.)

By linearity, since (2) holds for all $\omega \in C(\Delta^2, X)$, it follows that (2) holds for all $\omega \in S_2(X)$.

By (1), for all $\omega \in Z_1(X)$, $(g_* - f_*)(\omega) = (\partial_2 \circ \phi)(\omega)$, so $(g_* - f_*)(\omega)$ bounds. It follows, for all $\omega \in H_1(X)$, that $(g_* - f_*)(\omega) = 0$.

By (2), for all $\omega \in Z_2(X)$, $(g_* - f_*)(\omega) = (\partial_3 \circ \chi_\tau)(\omega))$, so $(g_* - f_*)(\omega)$ bounds. It follows, for all $\omega \in H_2(X)$, that $(g_* - f_*)(\omega) = 0$. **QED**

For any $k \in \mathbb{Z}$, define a functor $C_\bullet \mapsto C_{\bullet+k}$: {chain complexes} $\to$ {chain complexes} by $(C_{\bullet+k})_i = C_{i+k}$.

The preceding two proofs motivate us to formulate the following definition:

Definition. Let $C_\bullet$ and $D_\bullet$ be chain complexes. Let $f, g : C_\bullet \to D_\bullet$ be chain maps. A **chain homotopy** from f to g is a chain map $h : C_\bullet \to D_{\bullet+1}$ such that $g - f = \partial \circ h + h \circ \partial$. If a chain homotopy from f to g exists, then we say that f and g are **chain homotopic**.

The maps $\psi : S_0(X) \to S_1(Y)$, $\phi : S_1(X) \to S_2(Y)$ and $\chi_\tau : S_2(X) \to S_3(Y)$ of the preceding proof are the beginnings of a map $S_\bullet(X) \to S_{\bullet+1}(Y)$. The formulas (1) and (2) of the preceding proof are the beginnings of the assertion that this map is a chain homotopy.

We leave it to the interested reader to continue the sequence $\psi, \phi, \chi_\tau, \ldots$ and the formulas (1), (2), ..., and prove:

Theorem. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. Assume that f and g are homotopic. Then $f_* : S_\bullet(X) \to S_\bullet(Y)$ and $g_* : S_\bullet(X) \to S_\bullet(Y)$ are chain homotopic.

For any $k \in \mathbb{Z}$, define functors B_k, Z_k : {chain complexes} $\to$ {additive Abelian groups} by $B_k(C_\bullet) := \partial_{k+1}(C_{k+1})$ and $Z_k(C_\bullet) := \ker(\partial_k : C_k \to C_{k-1})$.

We have an easy algebraic result:

Theorem. Let $C_\bullet$ and $D_\bullet$ be chain complexes and let $f, g : C_\bullet \to D_\bullet$ be chain maps. Assume that f and g are chain homotopic. Then $f_* : H_\bullet(C_\bullet) \to H_\bullet(D_\bullet)$ and $g_* : H_\bullet(C_\bullet) \to H_\bullet(D_\bullet)$ are equal.

Proof: Let $k \in \mathbb{Z}$. We wish to show that $f_* : H_k(C_\bullet) \to H_k(D_\bullet)$ and $g_* : H_k(C_\bullet) \to H_k(D_\bullet)$ are equal. Let $\omega \in Z_k(C_\bullet)$. We wish to show that $(g(\omega)) - (f(\omega)) \in B_k(D_\bullet)$.

Let $h : H_\bullet(C_\bullet) \to H_{\bullet+1}(D_\bullet)$ be a chain homotopy. Then $g - f = \partial \circ h + h \circ \partial$. Then $[g(\omega)] - [f(\omega)] = [\partial_{k+1}(h(\omega))] + [h(\partial_k(\omega))]$. Since $\omega \in Z_k(C_\bullet)$, we get $\partial_k(\omega) = 0$. Then $[g(\omega)] - [f(\omega)] = \partial_{k+1}(h(\omega)) \in \partial_{k+1}(D_{k+1}) = B_k(D_\bullet)$. **QED**

Combining the two preceding theorems, we get

Corollary. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. Assume that f and g are homotopic. Then $f_* : H_\bullet(X) \to H_\bullet(Y)$ and $g_* : H_\bullet(X) \to H_\bullet(Y)$ are equal.

Corollary. Let X and Y be homotopy equivalent topological spaces. Then $H_\bullet(X) \cong H_\bullet(Y)$.

Corollary. Let X be a contractible topological space, and let $*$ be a topological space with only one point. Then $H_\bullet(X) \cong H_\bullet(*)$.

Note that $H_0(*) \cong \mathbb{Z}$ and that, for all integers $k \neq 0$, $H_k(*)$ is isomorphic to the trivial additive Abelian group $\{0\}$.

Definition. Let $C_\bullet$ be a nonnegative chain complex. An **augmentation** of $C_\bullet$ is a surjective homomorphism $\varepsilon : C_0 \to \mathbb{Z}$ such that $\varepsilon \circ \partial_1 : C_1 \to \mathbb{Z}$ is equal to zero. An **augmented chain complex** consists of a nonnegative chain complex, together with an augmentation.

Example. Let X be a nonempty topological space, and define $\varepsilon : S_0(X) \to \mathbb{Z}$ by $\varepsilon\left(\sum_i c_i x_i\right) = \sum_i c_i$. Then $\widetilde{S}_\bullet(X) := (S_\bullet, \varepsilon)$ is an augmented chain complex.

With the notation of the preceding example, $\widetilde{S}_\bullet$ is a functor from {topological spaces} to {augmented chain complexes}.

Given an augmented chain complex $(C_\bullet, \varepsilon)$, let $C_\bullet^*$ be the chain complex

$$\cdots \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0 \to 0 \to \cdots$$

where all the maps are boundary maps from $C_\bullet$, except for the map $C_0 \to \mathbb{Z}$ which is ε. Note, for all integers $n \neq -1$, that $C_n^* = C_n$ and that $C_{-1}^* = \mathbb{Z}$. Then $(C_\bullet, \varepsilon) \mapsto C_\bullet^*$ is a functor $\mathcal{F}$ from {augmented chain complexes} to {chain complexes}

With all this notation, we define a functor

$$\widetilde{H}_\bullet := H_\bullet \circ \mathcal{F} : \{\text{augmented chain complexes}\} \to \{\text{graded groups}\}.$$

Precomposing this functor with the functor $\widetilde{S}_\bullet$, we obtain a functor

$$\{\text{topological spaces}\} \to \{\text{graded groups}\}.$$

This last functor is also denoted by $\widetilde{H}_\bullet$, and the reader must pick up from context which of the two functors $\widetilde{H}_\bullet$ is under discussion at a given time.

The **reduced homology** groups of a topological space X are the groups $\widetilde{H}_\bullet(X)$.

We leave it as an unassigned exercise to show, for any augmented chain complex $\widetilde{C}_\bullet = (C_\bullet, \varepsilon)$, for any integer $k \neq 0$, that $H_k(C_\bullet) = \widetilde{H}_k(\widetilde{C}_\bullet)$. We leave it as an unassigned exercise to show, for any chain complex $C_\bullet$, that $H_0(C_\bullet) = \mathbb{Z} \oplus [\widetilde{H}_0(\widetilde{C}_\bullet)]$. Thus "augmenting the chain complex reduces the rank of homology by one in degree zero, and has no effect in other degrees".

For any topological space X, for any integer $k \neq 0$, we have that $H_k(X) = \widetilde{H}_k(X)$. Moreover, for any topological space X, we have that $H_0(X) = \mathbb{Z} \oplus [\widetilde{H}_k(X)]$. We leave it as an unassigned exercise to prove that, if $*$ denotes a topological space with exactly one point, then $\widetilde{H}_\bullet(*) = 0$.

Definition. Let $\mathcal{D}$ be a diagram in the category of additive Abelian groups and let A be an object in $\mathcal{D}$. We will say that $\mathcal{D}$ is **exact** at A if:

(1) there is exactly one arrow $f : X \to A$ in $\mathcal{D}$ whose target is A;
(2) there is exactly one arrow $g : A \to Y$ in $\mathcal{D}$ whose domain is A; and
(3) $\operatorname{im}(f) = \ker(g)$, *i.e.*, $\{f(x) \mid x \in X\} = \{a \in A \mid g(a) = 0\}$.

Definition. A diagram

$$0 \to A' \to A \to A'' \to 0$$

in {additive Abelian groups} is called a **short exact sequence** if it is exact at A', at A and at A''.

For any arrow $f : C_\bullet \to D_\bullet$ in {chain complexes} or {graded groups}, we define $\operatorname{im}(f), \ker(f) \in$ {chain complexes} by $(\operatorname{im}(f))_n = \operatorname{im}(f_n : C_n \to D_n)$ and $(\ker(f))_n := \ker(f_n : C_n \to D_n)$. Then ker and im are functors from the arrow category of the category {chain complexes} to the category {chain complexes}.

Definition. Let $\mathcal{D}$ be a diagram in the category of chain complexes or of graded groups. Let $A_\bullet$ be an object in $\mathcal{D}$. We will say that $\mathcal{D}$ is **exact** at $A_\bullet$ if:

(1) there is exactly one arrow $f : X_\bullet \to A_\bullet$ in $\mathcal{D}$ whose target is $A_\bullet$;
(2) there is exactly one arrow $g : A_\bullet \to Y_\bullet$ in $\mathcal{D}$ whose domain is $A_\bullet$; and
(3) $\operatorname{im}(f) = \ker(g)$.

Definition. A diagram

$$0 \to A'_\bullet \to A_\bullet \to A''_\bullet \to 0$$

in {chain complexes} is called a **short exact sequence** if it is exact at $A'_\bullet$, at $A_\bullet$ and at $A''_\bullet$.

Definition. Let $A_\bullet$, $B_\bullet$ be chain complexes. We say that $A_\bullet$ is a **subcomplex** of $B_\bullet$ if: for all integers n, A_n is a subgroup of B_n.

For any chain complex $B_\bullet$, for any subcomplex $A_\bullet$, we define the chain complex $B_\bullet/A_\bullet$ by $(B_\bullet/A_\bullet)_n = B_n/A_n$.

For any arrow $f : A \to B$ in {additive Abelian groups}, we define $\mathrm{coker}(f) := B/(\mathrm{im}(f))$; note that, with this definition,

$$0 \to \ker(f) \to A \to B \to \mathrm{coker}(f) \to 0$$

is exact at $\ker(f)$, at A, at B and at $\mathrm{coker}(f)$.

For any arrow $f : A_\bullet \to B_\bullet$ in {additive Abelian groups}, we define $\mathrm{coker}(f) := B_\bullet/(\mathrm{im}(f))$; note that, with this definition,

$$0 \to \ker(f) \to A_\bullet \to B_\bullet \to \mathrm{coker}(f) \to 0$$

is exact at $\ker(f)$, at $A_\bullet$, at $B_\bullet$ and at $\mathrm{coker}(f)$.

Definition. A diagram in {additive Abelian groups} of the form

$$0 \to A \to B \to C \qquad\qquad D \to E \to F \to 0$$

is called a **broken six** if it is exact at A, B, E and F. A diagram in {additive Abelian groups} of the form

$$0 \to A \to B \to C \to D \to E \to F \to 0$$

is called a **snake six** if it is exact at A, B, C, D, E and F.

To get into the spirit of the terminology, reorganize the preceding diagram so that D lies under A, E lies under B and F lies under C. Then the arrow from C to D has to follow a path that gives it something of a snake shape.

Let $\mathcal{SESAG}$:= {short exact sequences of additive Abelian groups}. Let $\mathcal{BRSX}$:= {broken sixes}. Let $\mathcal{SNSX}$:= {snake sixes}. Let $\mathcal{A}$ be the arrow category of $\mathcal{SESAG}$. Define a functor $\mathcal{G} : \mathcal{A} \to \mathcal{BRSX}$ by: for all $\phi = (f', f, f'') : (A', A, A'') \to (B', B, B'')$ in $\mathcal{SESAG}$, let $\mathcal{G}(\phi)$ be the diagram

$$0 \to \ker(f') \to \ker(f) \to \ker(f'') \qquad\qquad \mathrm{coker}(f') \to \mathrm{coker}(f) \to \mathrm{coker}(f'') \to 0.$$

Let $\mathcal{F} : \mathcal{SNSX} \to \mathcal{BRSX}$ be the forgetful functor.

Snake Lemma. There is a functor $\mathcal{H} : \mathcal{A} \to \mathcal{SNSX}$ such that $\mathcal{F} \circ \mathcal{H} = \mathcal{G}$.

Proof: See the opening scene in the movie "It's My Turn" with Jill Clayburg and Michael Douglas. Alternatively, spend quite some time chasing diagrams. **QED**

Definition. A diagram in {graded groups} of the form

$$A'_\bullet \to A_\bullet \to A''_\bullet$$

is said to be a **broken long exact sequence** if it is exact at $A_\bullet$. It is said to be a **broken complex** if the composite arrow $A'_\bullet \to A''_\bullet$ is zero.

Definition. A **long exact sequence** consists of

(1) a broken long exact sequence $A'_\bullet \to A_\bullet \to A''_\bullet$; and

(2) a **connecting homomorphism** $A''_\bullet \to A'_{\bullet-1}$

such that

$$A''_{\bullet+1} \to A'_\bullet \to A_\bullet \to A''_\bullet \to A'_{\bullet-1}$$

is exact at $A'_\bullet$, at $A_\bullet$ and at $A''_\bullet$.

In the preceding diagram, the arrow $A''_{\bullet+1} \to A'_\bullet$ is obtained by applying the functor $C_\bullet \mapsto C_{\bullet+1}$ to the connecting homomorphism $A''_\bullet \to A'_{\bullet-1}$.

Let $\mathcal{SESCC}$ be the category of short exact sequences of chain complexes. Let $\mathcal{LES}$ be the category of long exact sequences. Let $\mathcal{BRLES}$ be the category of broken long exact sequences. Define a functor $\mathcal{G}_0 : \mathcal{SESCC} \to \mathcal{BRLES}$ by: for any short exact sequence $0 \to A'_\bullet \to A_\bullet \to A''_\bullet$ of chain complexes, we define $\mathcal{G}_0(0 \to A'_\bullet \to A_\bullet \to A''_\bullet)$ to be the broken long exact sequence:

$$H_\bullet(A'_\bullet) \to H_\bullet(A_\bullet) \to H_\bullet(A''_\bullet).$$

Let $\mathcal{F}_0 : \mathcal{LES} \to \mathcal{BRLES}$ be the forgetful functor.

Snake Corollary. There is a functor $\mathcal{H}_0 : \mathcal{SESCC} \to \mathcal{LES}$ such that $\mathcal{F}_0 \circ \mathcal{H}_0 = \mathcal{G}_0$.

Proof: We leave it as an exercise to show that this follows from the Snake Lemma. Alternatively, spend quite some time chasing diagrams. **QED**

Definition. Let (X, A) be a topological pair. We define $H_\bullet(X, A) := H_\bullet[(S_\bullet(X))/(S_\bullet(A))]$.

The functor $H_\bullet$ defined above goes from the category of topological pairs to the category of graded groups. We make the convention that $\mathbb{Z}[\emptyset] = \{0\}$ so that $S_\bullet(\emptyset) = 0$. Then, for any topological space X, we have $H_\bullet(X, \emptyset) \cong H_\bullet(X)$. More precisely, $H_\bullet(\cdot, \emptyset)$ and $H_\bullet$ are equivalent functors.

For any topological pair (X, B), we have a short exact sequence of chain complexes:

$$0 \to S_\bullet(B) \to S_\bullet(X) \to (S_\bullet(X))/(S_\bullet(B)) \to 0,$$

and, applying the Snake Corollary, we get a long exact sequence whose broken form is

$$H_\bullet(B) \to H_\bullet(X) \to H_\bullet(X, B).$$

Slightly more generally, if X is a topological space and if $A \subseteq B \subseteq X$, then we have a short exact sequence of chain complexes:

$$0 \to (S_\bullet(B))/(S_\bullet(A)) \to (S_\bullet(X))/(S_\bullet(A)) \to (S_\bullet(X))/(S_\bullet(B)) \to 0,$$

and, applying the Snake Corollary, we get a long exact sequence whose broken form is

$$H_\bullet(B, A) \to H_\bullet(X, A) \to H_\bullet(X, B).$$

Example. Let $X := \mathbb{R}^2$ and let $A := \mathbb{R}^2\backslash\{(0,0)\}$. Then, as part of the above long exact sequence we get a diagram in the category of Abelian groups

$$H_2(X) \to H_2(X,A) \to H_1(A) \to H_1(X)$$

which is exact at $H_2(X,A)$ and at $H_1(A)$. Since X is contractible, we see that $H_2(X)$ and $H_1(X)$ are both trivial, and so exactness implies that the map $H_2(X,A) \to H_1(A)$ is an isomorphism. Since A is homotopy equivalent to the circle S^1 and since $H_1(S^1) \cong \mathbb{Z}$, we conclude that $H_2(X,A) \cong \mathbb{Z}$.

To clarify the importance of this last example, we need to have some additional terminology.

Definition. Let X be a topological space, let $A \subseteq X$ and let $n \in \mathbb{Z}$. We define $Z_n(X,A) := \{\gamma \in S_n(X) \,|\, \partial_n\gamma \in S_{n-1}(A)\}$. We define $B_n(X,A) := (B_n(X)) + (S_n(A))$.

It is an unassigned exercise to show that the functor $(X,A) \mapsto (Z_\bullet(X,A))/(B_\bullet(X,A))$ is equivalent to the functor $(X,A) \mapsto H_\bullet(X,A)$.

Definition. Let (X,A) be a topological pair and let $n \in \mathbb{Z}$. For any $\alpha \in S_n(X)$, we say that α is a **cycle mod** A if $\alpha \in Z_n(X,A)$. For any $\alpha \in Z_n(X,A)$, we say that α **bounds mod** A if $\alpha \in B_n(X,A)$. For any $\alpha, \beta \in Z_n(X,A)$, we say that α and β are **homologous mod** A if $\beta - \alpha$ bounds mod A.

Let $X := \mathbb{R}^2$ and let $A := \mathbb{R}^2\backslash\{(0,0)\}$. Let $\sigma \in C(\Delta^2, X)$ and assume that $\partial_2(\sigma) \in S_1(A)$. Relative homology allows us to define the "number of times" that σ covers $(0,0)$, as follows: We know that σ is a cycle mod A, *i.e.*, that $\sigma \in Z_2(X,A)$. Following the preceding example, fix an isomorphism $H_2(X,A) \to \mathbb{Z}$. Such an isomorphism is called an **orientation** of $\mathbb{R}^2$ at $(0,0)$. Since $H_2(X,A) \cong (Z_2(X,A))/(B_2(X,A))$, we obtain a composite homomorphism $Z_2(X,A) \to (Z_2(X,A))/(B_2(X,A)) \cong H_2(X,A) \cong \mathbb{Z}$. The image of σ under this map is what we will call the "number of times" that σ covers $(0,0)$. Note that this number might be negative. Intuitively, each time that σ covers $(0,0)$, it either covers it in an orientation-preserving way or in an orientation reversing-way, and, to know which is which require fixing an orientation at $(0,0)$ on X. Each orientation-preserving covering of $(0,0)$ counts for $+1$, while orientation-reversing covering of $(0,0)$ counts for -1.

Excision Theorem. Let X be a topological space. For all $S \subseteq X$, let $\overline{S} := \mathrm{Cl}_X(S)$ and let $S^\circ := \mathrm{Int}_X(S)$. Let $U, A \subseteq X$ and assume that $\overline{U} \subseteq A^\circ$. Let $i : (X\backslash U, A\backslash U) \to (X,A)$ be the inclusion map. Then $i_* : H_\bullet(X\backslash U, A\backslash U) \to H_\bullet(X,A)$ is an isomorphism.

We will sketch part of the proof in a moment, but first let's mention an application. Let $X := S^2$ be the two-sphere. Let $x, y \in X$ be distinct points. (Imagine x as the north pole, y as the south pole.) Let $U := \{y\}$ and let $A := X\backslash\{x\}$. Then, by the Excision Theorem, we ahve $H_2(X,A) \cong H_2(X\backslash U, A\backslash U)$. Stereographic projection gives an isomorphism, in the category {topological pairs}, betwen $(X\backslash U, A\backslash U)$ and $(\mathbb{R}^2, \mathbb{R}^2\backslash\{0\})$. Then, by the preceding example, $H_2(X,A) \cong H_2(X\backslash U, A\backslash U) \cong \mathbb{Z}$. An isomorphism $H_2(X,A) \to \mathbb{Z}$ is called an **orientation** of S^2 at x, and allows us to count the number of times that a given parametric 2-simplex σ in S^2 covers the point x, counting $+1$ for orientation-preserving coverings of x and -1 for orientation-reversing coverings of x. This is all provided that $\partial_2(\sigma) \in S_1(A)$.

Given a topological space X and $\mathcal{U} \subseteq \{\text{subsets of } X\}$, of X, we define, for each $n \in \mathbb{Z}$, $S_n^{\mathcal{U}}(X) := \sum_{U \in \mathcal{U}} S_n(U)$; for any $\alpha \in S_n(X)$, we say that α is **subordinate** to $\mathcal{U}$ if $\alpha \in S_n^{\mathcal{U}}(X)$.

EXERCISE 14C: Recall that e_0^1, e_1^1 is the standard basis of $\mathbb{R}^2$. Let $p := e_0^1$, $q := e_1^1$. Recall that $\Delta^1 := \text{cvx}\{p, q\}$. Let $r := (p+q)/2$. Show that $[p,q] - ([p,r] + [r,q])$ bounds.

Let p, q, r be as in Exercise 14C. Let X be a topological space and let $\alpha \in C(\Delta^1, X)$. We define the **subdivision** of α to be $\text{Sd}(\alpha) := \alpha_*([p,r] + [r,q]) \in S_1(X)$ and note, by Exercise 14C, that $\alpha - (\text{Sd}(\alpha))$ bounds. The subdivison map extends by $\mathbb{Z}$-linearity to a map $\text{Sd} : S_1(X) \to S_1(X)$ and this map has the property that, for any $\alpha \in S_1(X)$, we have that $\alpha - (\text{Sd}(\alpha))$ bounds.

Sketch of proof that, in the Excision Theorem, the map $i_* : H_1(X\backslash U, A\backslash U) \to H_1(X, A)$ is surjective: Let $\alpha \in Z_1(X, A)$. We wish to show that α is homologous mod A to an element of $Z_1(X\backslash U, A\backslash U)$.

Let $\mathcal{V} := \{X\backslash \overline{U}, A^\circ\}$. Then $\mathcal{V}$ is an open cover of X. By Exercise 14C, if we subdivide α, then we obtain a chain whose difference with α bounds. Via a Lebesgue number argument, if we subdivide α sufficiently, we will obtain a chain β such that $\beta - \alpha$ bounds and such that β that is subordinate to $\mathcal{V}$. Then $\beta \in S_1^{\mathcal{V}}(X) = (S_1(X\backslash\overline{U})) + (S_1(A^\circ))$. Choose $\gamma \in S_1(X\backslash\overline{U})$ and $\delta \in S_1(A^\circ)$ such that $\beta = \gamma + \delta$. Then β is homologous to γ mod A. Then α is homologous to γ mod A. It suffices to show that $\gamma \in Z_1(X\backslash U, A\backslash U)$.

Since α is a cycle mod A, it follows that γ is a cycle mod A.

Then $\gamma \in (S_1(X\backslash\overline{U})) \cap (Z_1(X, A)) \subseteq Z_1(X\backslash U, A\backslash U)$. **QED**

Definition. A **long exact ladder** is an arrow in the category {long exact sequences}.

For any long exact sequence L, let $A_\bullet^L \to B_\bullet^L \to C_\bullet^L$ denote its broken form, and let $f^L : A_\bullet^L \to B_\bullet^L$ and $g^L : B_\bullet^L \to C_\bullet^L$ be the arrows in this broken form. For any long exact sequence L, let $h_\bullet^L : C_\bullet^L \to A_{\bullet-1}^L$ be the connecting homomorphism of L.

For any long exact ladder $p : L \to M$, let $\alpha^p : A_\bullet^L \to A_\bullet^M$, $\beta^p : B_\bullet^L \to B_\bullet^M$ and $\gamma^p : C_\bullet^L \to C_\bullet^M$ be the underlying arrows in {graded groups}.

A long exact ladder $p : L \to M$ is said to be **right isomorphic** if $\gamma^p : C_\bullet^L \to C_\bullet^M$ is an isomorphism in the category {graded groups}. Let $\mathcal{C}$ be the category {right isomorphic long exact lad

Let $\mathcal{A}$: {long exact ladders} $\to$ {broken complexes} be defined by: for any long exact ladder $p : L \to M$, $\mathcal{A}(p)$ is the broken complex

$$A_\bullet^L \to B_\bullet^L \oplus A_\bullet^M \to B_\bullet^M,$$

where the first arrow is

$$a \mapsto (f^L(a), \alpha^p(a)) : A_\bullet^L \to B_\bullet^L \oplus A_\bullet^M,$$

and where the second arrow is

$$(b,a) \mapsto (\beta^p(b)) - (f^M(a)) : B_\bullet^L \oplus A_\bullet^M \to B_\bullet^M.$$

Proposition. Let $\mathcal{A}$: {long exact ladders} $\to$ {broken complexes} be as described above. Let $\mathcal{F}$: {long exact sequences} $\to$ {broken long exact sequences} be the forgetful functor. There is a functor $\mathcal{BW} : \mathcal{C} \to$ {long exact sequences} such that $\mathcal{A}|\mathcal{C} = \mathcal{F} \circ \mathcal{BW}$.

The functor $\mathcal{BW}$ described in the last proposition is the **Barratt-Whitehead functor**.

Definition. A **Mayer-Vietoris triple** or **MV triple** consists of
(1) a topological space X; and
(2) two subsets $A, B \subseteq X$
such that, if A° and B° denote the interiors of A and B in X, then $A^\circ \cup B^\circ = X$.

Let $\mathcal{D}$ denote the arrow category of {short exact sequences of chain complexes}. Recall that the Snake Corollary gives a functor

$$\{\text{short exact sequences of chain complexes}\} \to \{\text{long exact sequences}\},$$

and this functor induces a functor $\mathcal{S} : \mathcal{D} \to$ {long exact ladders}.

We now define a functor $\mathcal{M}$: {MV triples} $\to \mathcal{D}$, as follows: Let $(X,A,B) \in \mathcal{M}$. Let $S_\bullet(A\cap B) \to S_\bullet(A)$ be induced by the inclusion $A \cap B \to A$. Let

$$S_\bullet(A) \to (S_\bullet(A))/(S_\bullet(A\cap B))$$

be the canonical map. Then

$$0 \to S_\bullet(A\cap B) \to S_\bullet(A) \to (S_\bullet(A))/(S_\bullet(A\cap B)) \to 0$$

is a short exact sequence of chain complexes, which we denote by E. Similary, inclusion and canonical maps give a short exact sequence

$$0 \to S_\bullet(B) \to S_\bullet(X) \to (S_\bullet(X))/(S_\bullet(B)) \to 0,$$

which we denote by F. The inclusions $A\cap B \to B$ and $A \to X$ induce arrows $i : S_\bullet(A\cap B) \to S_\bullet(B)$ and $j : S_\bullet(A) \to S_\bullet(X)$. Then a short diagram chase shows that there is a unique arrow $k : (S_\bullet(A))/(S_\bullet(A\cap B)) \to (S_\bullet(X))/(S_\bullet(B))$ such that $(i,j,k) : E \to F$ is an object in $\mathcal{D}$. We define $\mathcal{M}(X,A,B) = (i,j,k)$.

Theorem (Mayer-Vietoris). For any MV triple (X,A,B), we have $\mathcal{S}(\mathcal{M}(X,A,B)) \in \mathcal{C}$.

Proof: Let $p := \mathcal{S}(\mathcal{M}(X,A,B))$. Let L be the domain of p and let M be the target of p. We wish to show that $\gamma^p : C_\bullet^L \to C_\bullet^M$ is an isomorphism.

We have $C_\bullet^L = H_\bullet(A, A\cap B)$ and $C_\bullet^M = H_\bullet(X,B)$. Moreover, the map

$$\gamma^p : H_\bullet(A, A\cap B) \to H_\bullet(X,B)$$

is induced by the inclusion $(A, A\cap B) \to (X, B)$. Let $S := X\backslash A$. Then the closure in X of S is contained in the interior in X of B, therefore, by the Excision Theorem, the inclusion $(X\backslash S, B\backslash S) \to (X, B)$ induces an isomorphism in homology. However, $(A, A\cap B) = (X\backslash S, B\backslash S)$, so we are done. **QED**

Corollary (Mayer-Vietoris). Let (X, A, B) be a MV triple. Let

$$i : A\cap B \to A, \qquad j : A\cap B \to B, \qquad v : A \to X, \qquad w : B \to X$$

be inclusion maps. Define $f : H_\bullet(A\cap B) \to (H_\bullet(A)) \oplus (H_\bullet(B))$ by $f(\sigma) = (i_*(\sigma), j_*(\sigma))$. Define $g : (H_\bullet(A)) \oplus (H_\bullet(B)) \to H_\bullet(X)$ by $g(\sigma,\tau) = (v_*(\sigma)) - (w_*(\tau))$. Then there is a long exact sequence whose broken form is

$$H_\bullet(A\cap B) \to (H_\bullet(A)) \oplus (H_\bullet(B)) \to H_\bullet(X),$$

where the first arrow is f and the second is g.

Proof: Let $\mathcal{BW}$ be the Barratt-Whitehead functor. Then the required long exact seqence is $\mathcal{BW}(\mathcal{S}(\mathcal{M}(X, A, B)))$. Let $\mathcal{F}$: {long exact sequences} $\to$ {broken long exact sequences} be the forgetful functor. We must show that $\mathcal{F}(\mathcal{BW}(\mathcal{S}(\mathcal{M}(X, A, B))))$ is the broken long exact sequence described in the statement of the theorem.

By the preceding proposition, $\mathcal{F}(\mathcal{BW}(\mathcal{S}(\mathcal{M}(X, A, B)))) = \mathcal{A}(\mathcal{S}(\mathcal{M}(X, A, B)))$. The result then follows directly from the definitions of $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{M}$. **QED**

Example. Let $X := S^2$. Let $x, y \in X$ be distinct. (Think of the north pole and the south pole.) Let $A := X\backslash\{y\}$ and let $B := X\backslash\{x\}$. Then, by the Mayer-Vietoris corollary we have a long exact sequence part of which reads

$$(H_2(A)) \oplus (H_2(B)) \to H_2(X) \to H_1(A\cap B) \to (H_1(A) \oplus H_1(B)).$$

So, as A and B are contractible, we conclude that $H_2(X)$ is isomoprhic to $H_1(A\cap B)$. However $A\cap B$ is homotopy equivalent to S^1, and we deduce from this that $H_1(A\cap B) \cong \mathbb{Z}$. Then $H_2(S^2) \cong \mathbb{Z}$.

We comment that all the Mayer-Vietoris work that was done with homology $H_\bullet$: {topological spaces} $\to$ {graded groups} could as easily have been done with reduced homology $\widetilde{H}_\bullet$: {topological spaces} $\to$ {graded groups}. We would then have the result:

Corollary (Mayer-Vietoris). Let (X, A, B) be a MV triple, and assume that $A\cap B \neq \emptyset$. Let

$$i : A\cap B \to A, \qquad j : A\cap B \to B, \qquad v : A \to X, \qquad w : B \to X$$

be inclusion maps. Define $f : \widetilde{H}_\bullet(A\cap B) \to (\widetilde{H}_\bullet(A)) \oplus (\widetilde{H}_\bullet(B))$ by $f(\sigma) = (i_*(\sigma), j_*(\sigma))$. Define $g : (\widetilde{H}_\bullet(A)) \oplus (\widetilde{H}_\bullet(B)) \to \widetilde{H}_\bullet(X)$ by $g(\sigma,\tau) = (v_*(\sigma)) - (w_*(\tau))$. Then there is a long exact sequence whose broken form is

$$\widetilde{H}_\bullet(A\cap B) \to (\widetilde{H}_\bullet(A)) \oplus (\widetilde{H}_\bullet(B)) \to \widetilde{H}_\bullet(X),$$

where the first arrow is f and the second is g.

EXERCISE DUE AT THE FINAL: Let $d \geq 0$ be an integer. Let C be a closed subset of $\mathbb{R}^d$. Prove, for any $n \in \mathbb{Z}$, that

$$\widetilde{H}_n(\mathbb{R}^d \backslash C) \cong \widetilde{H}_{n+1}(\mathbb{R}^{d+1} \backslash (C \times \{0\})).$$

(*Hint:* Find open contractible subsets $A, B \subseteq \mathbb{R}^{d+1}$ such that $A \cup B = \mathbb{R}^{d+1} \backslash (C \times \{0\})$ and such that $A \cap B = (\mathbb{R}^d \backslash C) \times \mathbb{R}$.)

Corollary. Let $d \geq 0$ be an integer. Let C and D be closed subsets of $\mathbb{R}^d$ and that $C \neq \mathbb{R}^d \neq D$. Assume that C and D are homeomorphic. Then, for all $n \in \mathbb{Z}$, we have $H_n(\mathbb{R}^d \backslash C) \cong H_n(\mathbb{R}^d \backslash D)$.

That is, homology cannot detect the difference between the complements of two homeomorphic closed proper subsets of Euclidean space. Thus, while it is known that "a knot is determined by its complement", this fact cannot be verified using homology, since any two knot complements have the same homology.

One might think that the only use of the last corollary is in showing how weak homology is, but, amazingly, we will soon see that it eventually yields a very strong consequence: an open mapping theorem for maps between equidimensional topological manifolds.

Proof of the last corollary: It suffices to show that $\widetilde{H}_n(\mathbb{R}^d \backslash C) \cong \widetilde{H}_n(\mathbb{R}^d \backslash D)$. Let $C' := C \times \{0\}^d \subseteq \mathbb{R}^{2d}$ and let $D' := D \times \{0\}^d \subseteq \mathbb{R}^{2d}$. Let $f : C \to D$ be a homeomorphism. Let $\Gamma := \{(c, f(c)) \,|\, c \in C\}$ be the graph of f. We leave it as an unassigned exercise to verify, via The Tietze Extension Theorem that $\mathbb{R}^{2d} \backslash C'$ is homeomorphic to $\mathbb{R}^{2d} \backslash \Gamma$. A similar argument shows that $\mathbb{R}^{2d} \backslash D'$ is homeomorphic to $\mathbb{R}^{2d} \backslash \Gamma$. Then $\widetilde{H}_{d+n}(\mathbb{R}^{2d} \backslash C') \cong \widetilde{H}_{d+n}(\mathbb{R}^{2d} \backslash D')$.

By d applications of the preceding exercise, we have $\widetilde{H}_{d+n}(\mathbb{R}^{2d} \backslash C') \cong \widetilde{H}_n(\mathbb{R}^{2d} \backslash C)$. Similarly, we have $\widetilde{H}_{d+n}(\mathbb{R}^{2d} \backslash D') \cong \widetilde{H}_n(\mathbb{R}^{2d} \backslash D)$. The result follows. **QED**

Recall that a topological space is a **simple closed curve** if it is homeomorphic to the circle S^1.

The Jordan Curve Theorem. Let C be a simple closed curve in $\mathbb{R}^2$. Then $\mathbb{R}^2 \backslash C$ has two connected components.

Proof: By the preceding corollary, we have $H_0(\mathbb{R}^2 \backslash C) \cong H_0(\mathbb{R}^2 \backslash S^1)$, so $H_0(\mathbb{R}^2 \backslash C) \cong \mathbb{Z} \oplus \mathbb{Z}$. Then $\mathbb{R}^2 \backslash C$ has two connected components. **QED**

An important issue that often comes up in topology is when one can conclude that a continuous bijection is a homeomorphism. This is not always true (*e.g.*, the map $t \mapsto (\cos(t), \sin(t)) : [0, 2\pi) \to S^1$), but there are a number of "open mapping" theorems in mathematics that clarify important cases where it *is* true.

For example, if K is a compact topological space, and if $g : K \to L$ is continuous, then, because all closed subsets of K are compact, it follows that g is a closed map. Consequently, with the additional assumption that $g : K \to L$ is bijective (which implies that the image of the complement is the complement of the image), we may conclude that $g : K \to L$ is open. A continuous open bijection is a homeomorphism, and so we conclude that: If K is compact, and if $g : K \to L$ is a continuous bijection, then g is a homeomorphism. This then has the following consequence:

Fact. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous bijection. Then, for any compact subset $K \subseteq X$, we have: K and $f(K)$ are homeomorphic.

We aim for another example, proved via homology, that many continuous bijections are open. This is a surprising argument, because it might seem difficult to verify openness of a set via as weak an invariant as homology. In fact, an open disk in $\mathbb{R}^2$ is homotopy equivalent to a closed disk, and therefore the two have the same homology. So, using only homology, it might seem difficult to distinguish between open sets and closed sets. However, the following clever observation belies this impression.

Remark. Let X be a locally path-connected topological space and let $P, Q \subseteq X$. Assume that P is closed in X and that $P \cap Q = \emptyset$. Assume the following:
(1) $H_0(X \backslash P) \ncong \mathbb{Z}$;
(2) $H_0(Q) \cong \mathbb{Z}$; and
(3) $H_0(X \backslash (P \cup Q)) \cong \mathbb{Z}$.
Then Q is open in X.

Proof: Let $Y := X \backslash P$. Then, as X is locally path-connected, Y is as well. Then the path-components of Y are open in Y, and therefore are open in X.

By (2), Q is path-connected. Let A denote the path-component of Y containing Q. It suffices to show that $A = Q$.

By (1), $H_0(Y) \neq \mathbb{Z}$, so Y is not path-connected, so $A \subsetneq Y$, and, in particular, $Y \not\subseteq A$. Then $Q \cup (Y \backslash Q) = Y \not\subseteq A$, so, since $Q \subseteq A$, we conclude that $Y \backslash Q \not\subseteq A$.

By (3), we have $H_0(Y \backslash Q) \cong \mathbb{Z}$, so $Y \backslash Q$ is path-connected. So, because $Y \backslash Q \not\subseteq A$ and because A is a path-component of Y, we conclude that $(Y \backslash Q) \cap A = \emptyset$, or, equivalently, $A \subseteq Q$. Then, as $Q \subseteq A$, we get $A = Q$. **QED**

Proposition. Let $d \geq 0$ be an integer. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous, injective map. Then $f : \mathbb{R}^d \to \mathbb{R}^d$ is open.

Proof: Let B be an open ball in $\mathbb{R}^d$ and let $Q := f(B)$. We wish to show that Q is open in $\mathbb{R}^d$.

Let ∂B denote the boundary of B in $\mathbb{R}^d$. Let $P := f(\partial B)$. As ∂B and $B \cup \partial B$ are both compact subsets of $\mathbb{R}^d$, the preceding Fact yields both that P is homeomorphic to ∂B and that $P \cup Q$ is homeomorphic to $B \cup \partial B$.

Then, by the preceding corollary, we get both that $H_0(\mathbb{R}^d \backslash P) \cong H_0(\mathbb{R}^d \backslash \partial B) \cong \mathbb{Z} \oplus \mathbb{Z}$ and that $H_0(\mathbb{R}^d \backslash (P \cup Q)) \cong H_0(\mathbb{R}^d \backslash (B \cup \partial B)) \cong \mathbb{Z}$. Finally, since B is path-connected, Q is as well, and so $H_0(Q) \cong \mathbb{Z}$. Then, by the last remark, with $X := \mathbb{R}^d$, we see that Q is open in $\mathbb{R}^d$. **QED**

Again, remarkably, although homology seems too weak to prove openness of anything, and although preceding corollary highlights the weakness of homology (it cannot distinguish between the complements of homeomorphic proper closed sets of Euclidean space), nevertheless, the preceding corollary was a primary tool in proving an open mapping theorem.

We can generalize this last proposition, once we have the notion of a topological manifold:

Definition. Let $d \geq 0$ be an integer. A topological space is said to be a **topological d-manifold** if every point has an open neighborhood that is homeomorphic to $\mathbb{R}^d$.

We leave it as an unassigned exercise to verify that, in a topological d-manifold, every point has arbitrarily small open neighborhoods that are homeomorphic to $\mathbb{R}^d$. That is:

Remark. Let $d \geq 0$ be an integer. Let X be a topological d-manifold. Let $x \in X$ and let V be an open neighborhood of x in X. Then there is an open neighborhood X_0 of x in X such that $X_0 \subseteq V$ and such that X_0 is homeomorphic to $\mathbb{R}^d$.

Note that because of the preceding remark, any open subset of a topological d-manifold is again a topological d-manifold.

Theorem. Let $d \geq 0$ be an integer. Let X and Y be topological d-manifolds. Let $f : X \to Y$ be a continuous, injective map. Then $f : X \to Y$ is open.

Proof: Let U be an open subset of X. We wish to show that $f(U)$ is open in Y. Let $y \in f(U)$. We wish to show that there is an open neighborhood N of y in Y such that $N \subseteq f(U)$.

Let Y_0 be an open neighborhood of y in Y such that Y is homeomorphic to $\mathbb{R}^d$. Fix $x \in U$ such that $f(x) = y$. By the preceding remark, let X_0 be an open neighborhood of x in X such that $X_0 \subseteq U \cap (f^{-1}(Y_0))$ and such that X_0 is homeomorphic to $\mathbb{R}^d$.

Then $f|X_0 : X_0 \to Y_0$ is continuous and injective, so, by the preceding proposition, $f|X_0 : X_0 \to Y_0$ is open. Then $N := f(X_0)$ is an open subset of Y. Moreover, we have $y = f(x) \in f(X_0) = N$. Finally, we have $N = f(X_0) \subseteq f(U)$. **QED**

Corollary. Let $d \geq 0$ be an integer. Let X and Y be topological d-manifolds. Let $f : X \to Y$ be a continuous, bijective map. Then $f : X \to Y$ is a homeomorphism.

Corollary. Let $d \geq 0$ be an integer. Let $U, S \subseteq \mathbb{R}^d$. Assume that U is open in $\mathbb{R}^d$. Assume that U and S are homeomorphic. Then S is open in $\mathbb{R}^d$.

Proof: Let $f : U \to S$ be a homeomorphism. By the preceding theorem, $f : U \to \mathbb{R}^d$ is open, so $f(U)$ is open in $\mathbb{R}^d$. That is, S is open in $\mathbb{R}^d$. **QED**

The preceding corollary is sometimes called **Invariance of Domain**.

A classical question is whether a nonempty open subset of Euclidean space can be homeomorphic to a nonempty open subset of another. The fact that the answer is negative is also sometimes called **Invariance of Domain**, since a connected open subset of Euclidean space is sometimes called a "domain", and since such a fact asserts that dimension is a topological invariant of domains.

In fact, we can now prove something *a priori* stronger:

Corollary. Let $d, k \geq 0$ be integers and assume that $d \neq k$. Let X be a topological

k-manifold and let Y be a topological d-manifold. Then X is not homeomorphic to Y.

Proof: Assume, for definiteness, that $d < k$; the proof is otherwise similar. Let $f : X \to Y$ be a homeomorphism, and we aim for a contradiction.

Choose an open subseteq X_0 of X such that X_0 is homeomorphic to $\mathbb{R}^k$. Choose a subset Z of X_0 such that Z is homeomorphic to $\mathbb{R}^d$ and such that Z is not open in X_0. Then Z is not open in X.

The map $f|Z : Z \to Y$ is a continuous injection, so, by the preceding thorem, $f(Z)$ is open in Y. Then, as $f : X \to Y$ is a homeomorphism, we conclude that Z is open X, a contradiction. **QED**

A topological space is a **topological manifold** if there exists some integer $d \geq 0$ such that it is a topological d-manifold.

The preceding corollary tells us that, if X is a topological manifold, then its **dimension**, $\dim(X)$, is well-defined, via: $\dim(X)$ is the unique integer $d \geq 0$ such that X is a topological d-manifold.

Here is an unassigned, but interesting exercise to ponder: Let X and Y be topological manifolds. Let $f : X \to Y$ be a continuous bijection. Show that $f : X \to Y$ is a homeomorphism.

START OF WINTER SEMESTER:

Definition. Let $m \geq 0$ be an integer and let $P : \mathbb{R}^m \to \mathbb{R}$. We say that P is **homogeneous quadratic** if there exist $(c_{ij}) \in \mathbb{R}^{m \times m}$ such that, for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$, we have

$$P(x_1, \ldots, x_m) = \sum_{i,j=1}^{m} c_{ij} x_i x_j.$$

Note that the matrix (c_{ij}) described above is not unique. However, if we require it to be symmetric then it is unique. Alternatively, if we require it to be upper triangular, then it is unique as well.

Definition. Let $m, n \geq 0$ be integers and let $P : \mathbb{R}^m \to \mathbb{R}^n$. For all integers $k \in [1, n]$, let $q_k : \mathbb{R}^n \to \mathbb{R}$ be the kth coordinate projection, $q_k(y_1, \ldots, y_n) = y_k$. Then we say that P is **homogeneous quadratic** if, for all integers $k \in [1, n]$, we have that $q_k \circ P : \mathbb{R}^m \to \mathbb{R}$ is homogeneous quadratic.

By **vector space** we will always mean *real* vector space, unless otherwise specified. By $\dim(V)$ we will always mean the real dimension of the vector space V, unless otherwise specified.

Definition. Let V and W be vector spaces and let $Q : V \to W$. We say that Q is **homogenous quadratic** if there exist integers $m, n \geq 0$ and isomorphsims $f : \mathbb{R}^m \to V$ and $g : W \to \mathbb{R}^n$ such that $g \circ Q \circ f : \mathbb{R}^m \to \mathbb{R}^n$ is homogeneous quadratic.

In freshman calculus, functions $f : \mathbb{R} \to \mathbb{R}$ are approximated by inhomogeneous linear functions $x \mapsto a + bx$, and then, later, by quadratic Taylor polynomials $x \mapsto a + bx + cx^2$.

In higher dimensions, a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is approximated by the sum of a constant and a homogeneous linear map $L : \mathbb{R}^m \to \mathbb{R}^n$. The analysis of L involves developing an entirely new branch of mathematics called "linear algebra".

One also wants to do quadratic Taylor polynomials, in which $f : \mathbb{R}^m \to \mathbb{R}^n$ is approximated by a sum of a constant, a homogeneous linear function $L : \mathbb{R}^m \to \mathbb{R}^n$, and a homogeneous quadratic function $Q : \mathbb{R}^m \to \mathbb{R}^n$.

Question: Why is there no subject called "quadratic algebra"?

Answer: For any vector spaces V and W, for any homogeneous quadratic $Q : V \to W$, there exists a unique symmetric bilinear $B : V \times V \to W$ such that, for all $v \in V$, we have $Q(v) = B(v, v)$.

EXERCISE 15A: Let $V := \mathbb{R}^2$, $W := \mathbb{R}$ and define Q by $Q(x, y) = x^2 + 2xy + 3y^2$. Find $B : V \times V \to W$.

Thus questions about quadratic algebra can be recast as questions about symmetric bilinear algebra. So quadratic algebra is subsumed in bilinear algebra. (In a similar way, cubic algebra is subsumed in trilinear algebra, *etc.*) But this just begs the:

Question: Why is there no subject called "bilinear algebra"?

Definition. Let V, W and X be vector spaces and let $B : V \times W \to X$ be bilinear. Then we say that B is **universal** if, for any vector space Y, for any bilinear map $C : V \times W \to Y$, there is a unique linear map $L : X \to Y$ such that $C = L \circ B$.

Answer: For any vector spaces V and W, there is a vector space X and a universal bilinear map $B : V \times W \to X$.

We will explain a construction of X and B below, but if we can find such X and B, then any question about bilinear maps from $V \times W$ to another vector space Y can be recast as a question about linear maps from X to Y. Thus the subject of bilinear algebra is subsumed in linear algebra.

EXERCISE 15B: Let V, W, X and X' be vector spaces. Let

$$B : V \times W \to X \qquad \text{and} \qquad B' : V \times W \to X'$$

be universal bilinear maps. Prove that there is a vector space isomorphism $L : X \to X'$ such that $L \circ B = B'$.

Exercise 15B asserts that, given V and W, up to simply renaming the elements of X, there is a unique X and universal map $V \times W \to X$.

Let's now address existence. Given V and W, we will construct a vector space denoted $V \otimes_{\mathbb{R}} W$ and a universal bilinear $V \times W \to V \otimes_{\mathbb{R}} W$.

Let $X_0 := \mathbb{R}[V \times W]$ be the vector space of formal finite $\mathbb{R}$-linear combinations of elements of the set $V \times W$. Let K be the vector subspace of X_0 spanned by the union of the following four sets:

$$\{a(v,w) - (av,w) \,|\, a \in \mathbb{R}, v \in V, w \in W\},$$
$$\{a(v,w) - (v,aw) \,|\, a \in \mathbb{R}, v \in V, w \in W\},$$
$$\{(v+v',w) - (v,w) - (v',w) \,|\, v,v' \in V, w \in W\},$$
$$\{(v,w+w') - (v,w) - (v,w') \,|\, v \in V, w,w' \in W\}.$$

Let $V \otimes_{\mathbb{R}} W := X_0/K$.

Let $i : V \times W \to X_0$ be the inclusion and let $p : X_0 \to V \otimes_{\mathbb{R}} W$ be the canonical map. The standard notation for the image of $(p \circ i)(v,w)$ is $v \otimes w$. Then the map

$$p \circ i = (v,w) \mapsto v \otimes w : V \times W \to V \otimes_{\mathbb{R}} W$$

will be called the **canonical bilinear map**, and we leave it as an unassigned exercise to verify that it is a universal bilinear map.

EXERCISE 15C: Show that the matrix multiplication map $R^{3\times 1} \times \mathbb{R}^{1\times 4} \to \mathbb{R}^{3\times 4}$ is universal.

EXERCISE 15D: Compute $\dim(\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4)$.

All rings will be assumed to have multiplicative unity 1.

In fact, in all of the preceding remarks, we may replace $\mathbb{R}$ by any commutative ring R, as long as we use "R-module" as a replacement for "vector space". (Note that an $\mathbb{R}$-module is the same as a vector space.)

Note that any additive Abelian group A may be thought of as a $\mathbb{Z}$-module, in which, for all $n \in \mathbb{Z}$, for all $a \in A$, na is defined to be the n-fold sum $a + \cdots + a$. Moreover any $\mathbb{Z}$-module has an underlying additive Abelian group. Thus the category of $\mathbb{Z}$-modules is isomorphic to the category of additive Abelian groups.

A $\mathbb{Z}$-linear map $A \to B$ is the same as an additive group homomorphism. A $\mathbb{Z}$-bilinear map $A \times B \to C$ will sometimes be called a **bihomomorphism**.

Then, for any additive Abelian groups A and B, we construct an additive Abelian group $A \otimes_{\mathbb{Z}} B$ and a bihomomorphism $S : A \times B \to A \otimes_{\mathbb{Z}} B$ which is **universal** in the sense that, for any additive Abelian group C, for any bihomomorphism $T : A \times B \to C$ there is a homomorphism $H : A \otimes_{\mathbb{Z}} B \to C$ such that $T = H \circ S$.

We will abbreviate $\otimes_{\mathbb{Z}}$ as $\otimes$. For any integer n we abbreviate $\mathbb{Z}/(n\mathbb{Z})$ as Z/n.

As with Exercise 15B, for any universal bihomomorphism $A \times B \to D$, we have that: D is isomorphic to $A \otimes B$.

EXERCISE 15E: Let $m, n \in \mathbb{Z}$ and assume that $(m,n) \neq (0,0)$. Let g be the gcd of m and n. Let $\phi : \mathbb{Z}/m \to \mathbb{Z}/g$ and $\psi : \mathbb{Z}/n \to \mathbb{Z}/g$ be canonical maps. Show that the map

$$(x,y) \quad \mapsto \quad [\phi(x)][\psi(y)] \qquad : \qquad (\mathbb{Z}/m) \times (\mathbb{Z}/n) \quad \to \quad \mathbb{Z}/g$$

is universal.

We conclude, from Exercise 15E, that $(\mathbb{Z}/m) \otimes (\mathbb{Z}/n)$ is isomorphic to $\mathbb{Z}/g$.

EXERCISE 15F: Show, for any additive Abelian group A, that $(n, a) \mapsto na : \mathbb{Z} \times A \to A$ is universal.

We conclude, from Exercise 15F, that $\mathbb{Z} \otimes A$ is isomorphic to A. In fact, one may show that this isomorphism is "natural" in the sense that the functor

$$\mathbb{Z} \otimes \cdot \qquad : \qquad \{\text{additive Abelian groups}\} \to \{\text{additive Abelian groups}\}$$

is equivalent to the identity functor.

EXERCISE 15G: Show, for any additive Abelian groups A, B and C, that

$$((a, b), c) \mapsto (a \otimes c, b \otimes c) \qquad : \qquad (A \oplus B) \times C \to (A \otimes C) \oplus (B \otimes C)$$

is universal.

We conclude, from Exercise 15F, that $(A \oplus B) \otimes C$ is isomorphic to $(A \otimes C) \oplus (B \otimes C)$. A similar argument shows that $A \otimes (B \oplus C)$ is isomorphic to $(A \otimes B) \oplus (A \otimes C)$.

EXERCISE 15H: Compute $[(\mathbb{Z}/3) \oplus (\mathbb{Z}/27) \oplus (\mathbb{Z}/54)] \otimes [(\mathbb{Z}/2) \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/12)]$ up to isomorphism. Express your answer in the form $(\mathbb{Z}/a) \oplus (\mathbb{Z}/b) \oplus (\mathbb{Z}/c) \oplus \cdots$ such that $a|b|c| \cdots$.

Definition. Let A, B, C and D be additive Abelian groups. Let $f : A \to B$ and $g : C \to D$ be homomorphisms. Then we define $f \otimes g : A \otimes C \to B \otimes D$ by $(f \otimes g)(a \otimes c) = (f(a)) \otimes (g(c))$.

We remark that it is an exercise to show that this is well-defined. (Note that every element of $A \otimes C$ is a finite sum of elements from $\{a \otimes c \,|\, a \in A, c \in C\}$, however a given element of $A \otimes C$ may be expressed as such as sum in two very different ways.)

To see that the map is well-defined, one needs only observe that

$$(a, c) \mapsto (f(a)) \otimes (g(c)) : A \times C \to B \otimes D$$

is a bihomomorphism and therefore determines a homomorphism

$$H : A \otimes C \to B \otimes D$$

such that the composite

$$A \times C \to A \otimes C \to B \otimes D$$

is $(a, c) \mapsto (f(a)) \otimes (g(c)) : A \times C \to B \otimes D$. Thus the homomorphism H has the property that $H(a \otimes c) = (f(a)) \otimes (g(c))$. Then $f \otimes g = H$, so, as H is well-defined by construction, we see that $f \otimes g$ is, as well.

For any additive Abelian groups A and B, we set $A^B := A \otimes B$. Then, for any B, we have a functor $A \mapsto A^B : \{\text{additive Abelian groups}\} \to \{\text{additive Abelian groups}\}$ whose effect on arrows is defined by $f^B := f \otimes \mathrm{id}_B$.

Fact. If $0 \to A' \to A \to A'' \to 0$ is exact, then $(A')^B \to A^B \to (A'')^B \to 0$ is exact.

In short, we say that $A \mapsto A^B$ is "right-exact".

EXERCISE 15I: Let $i : 2\mathbb{Z} \to \mathbb{Z}$ be the inclusion map and let $B := \mathbb{Z}/2$. show that i^B is not injective.

For all additive Abelian groups A, let $\phi_A : \mathbb{Z}[A] \to A$ denote the extension by $\mathbb{Z}$-linearity of the identity map $A \to A$, and let K_A denote the kernel of ϕ_A. Let $\iota_A : K_A \to \mathbb{Z}[A]$ denote the injection map.

definition. For any additive Abelian groups A and B, let $\mathrm{Tor}(A, B)$ denote the kernel of $(\iota_A)^B : (K_A)^B \to (\mathbb{Z}[A])^B$. This is called the **torsion product** of A and B and is, alternatively, denoted $A * B$.

An additive Abelian group F is said to be **free Abelian** if there exists a set S such that F is isomorphic to $\mathbb{Z}[S]$. It is a standard fact, which we will assume without proof, that any subgroup of a free Abelian group is again free Abelian.

A subset S of an additive Abelian group A is said to be a **generating set** if the identity map $s \to s$ extends, by $\mathbb{Z}$-linearity, to a surjection $\mathbb{Z}[S] \to A$. It is said to be a **free generating set** if the identity map $s \to s$ extends, by $\mathbb{Z}$-linearity, to an isomorphism $\mathbb{Z}[S] \to A$. Note that an additive Abelian group is free Abelian iff it admits a free generating set.

Let A and B be additive Abelian groups and let $f : A \to B$ be a surjection. A **splitting** of F is a homomorphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. We say that f **splits** if f admits a splitting. A **split surjection** is a surjection which splits.

EXERCISE 15J: Let A and B be additive Abelian groups and let $f : A \to B$ be a split surjection. Let K be the kernel of f. Show that there is an isomorphism $h : K \oplus B \to A$ such that, for all $k \in K$, for all $b \in B$ we have $f(h(k, b)) = b$.

That is, in the category of "additive Abelian groups with a homomorphism to B" the pair (A, f) is isomorphic to $(B \oplus K, \pi)$, where $\pi : K \oplus B \to B$ is projection onto the second factor.

Remark. Let A be an additive Abelian group and let F be a free Abelian group. Then any surjection $f : A \to F$ is split.

Proof: Let S be a free generating set for F. For all $s \in s$, choose $a_s \in A$ such that $f(a_s) = s$. Extend $s \mapsto a_s : S \to A$ by $\mathbb{Z}$-linearity to a map $g : F \to A$. Then $f \circ g = \mathrm{id}_F$. **QED**

EXERCISE 15K: Let A be an additive Abelian group and let F and F' be free Abelian groups. Let $0 \to F' \to F \to A \to 0$ be a short exact sequence. show that $A * B$ is isomorphic to the kernel of $(F')^B \to F^B$.

EXERCISE 15L: Let $m, n \in \mathbb{Z}\backslash\{0\}$. Compute $(\mathbb{Z}/m) * (\mathbb{Z}/n)$.

EXERCISE 15M: For any additive Abelian group A, show that $A * \mathbb{Z} = 0$.

EXERCISE 15N: For any additive Abelian group A, show that $\mathbb{Z} * A = 0$.

EXERCISE 15O: For any additive Abelian groups A, B and C, show that $A*(B\oplus C) = (A*B)\oplus(A*C)$.

EXERCISE 15P: For any additive Abelian groups A, B and C, show that $(A\oplus B)*C = (A*C)\oplus(B*C)$.

In the preceding exercises, do not use the fact that $A*B$ is isomorphic to $B*A$. We will assume this fact from here on out. Note that Exercise 15M and Exercise 15N both guarantee that $A*B$ is not the same as $A\otimes B$.

EXERCISE 15Q: Compute $[(\mathbb{Z}/3)\oplus(\mathbb{Z}/27)\oplus(\mathbb{Z}/54)]*[(\mathbb{Z}/2)\oplus(\mathbb{Z}/4)\oplus(\mathbb{Z}/12)]$, up to isomorphism. Put your answer in the form $(\mathbb{Z}/a)\oplus(\mathbb{Z}/b)\oplus(\mathbb{Z}/c)\oplus\cdots$, with $a|b|c|\cdots$.

For any integer $n\geq 1$, for any integer $i\in[0,n]$, define $\varepsilon_i^n : \Delta^{n-1}\to\Delta^n$ by $\varepsilon_i^n(x_1,\ldots,x_n) = (x_1,\ldots,x_i,0,x_{i+1},\ldots,x_n)$. For any integer $n\geq 1$, for any integer $i\in[0,n]$, define $\partial_i^n : C(\Delta^n,X)\to C(\Delta^{n-1},X)$ by $\partial_i^n(\sigma)=\sigma\circ\varepsilon_i^n$.

For any topological space X, for any integer $n\geq 0$, let $S_n(X;A) := A[C(\Delta^n,X)]$ be the additive Abelian group of formal finite A-linear combinations of elements of $C(\Delta^n,X)$. For any topological space X, for any integer $n<0$, let $S_n(X;A) := \{0\}$ be the trivial additive Abelian group. For any topological space X, for any integer $n\geq 1$, let $\partial_n : S_n(X;A)\to S_{n-1}(X;A)$ be the unique homomorphism extending

$$a\sigma \quad\mapsto\quad \sum_{i=0}^{n}(-1)^i a[\partial_i^n(\sigma)].$$

For any topological space X, for any integer $n<1$, let $\partial_n : S_n(X;A)\to S_{n-1}(X;A)$ be the zero map.

EXERCISE 16A: Show, for all $n\in\mathbb{Z}$, that $S_n(X;A)$ is isomorphic to $(S_n(X))\otimes A$.

In addition to Exercise 15R, we remark that, for all $n\in Z$, the map $\partial_n : S_n(X;A)\to S_{n-1}(X;A)$ is the result of applying the functor $B\mapsto B^A$: {additive Abelian groups} $\to$ {additive Abelian groups} to the map $\partial_n : S_n(X)\to S_{n-1}(X)$.

Note that $B\mapsto B^A$ defines a functor {chain complexes} $\to$ {chain complexes}. With this notation, we summarize the preceding remars as: $S_\bullet(X;A)\cong(S_\bullet(X))^A$.

We now pose a

Question: If we know, for a topological space X, how to compute $H_\bullet(X)$, can we then compute $H_\bullet(X;A)$.

The philosophy of the Universal Coefficient Theorem is that the answer is "yes", provided that we also know how to compute $\otimes$ and $*$, *i.e.*, how to compute tensor and torsion products.

Definition. A chain complex $C_\bullet$ is **free**, if, for all $n\in\mathbb{Z}$, we have that C_n is free.

Note, for any topological space X, that $S_\bullet(X)$ is free.

Universal Coefficient Theorem. Let $C_\bullet$ be a free chain complex, let A be an additive Abelian group. Then, for all $n\in Z$, $H_n((C_\bullet)^A)$ is isomorphic to $[(H_nC_\bullet)^A]\oplus[(H_{n-1}C_\bullet)*A]$.

Corollary. Let X be a topological space, let A be an additive Abelian group and let $n \in \mathbb{Z}$. Then $H_n(X; A)$ is isomorphic to $[(H_n X)^A] \oplus [(H_{n-1}X) * A]$.

For the next exercise, you may assume, for any integer $n \geq 0$, the following facts:
(1) We have $H_0(\mathbb{R}P^n) = \mathbb{Z}$;
(2) For any even integer $k \in [1, n-1]$, we have $H_k(\mathbb{R}P^n) = 0$;
(3) For any odd integer $k \in [1, n-1]$, we have $H_k(\mathbb{R}P^n) = \mathbb{Z}/2$;
(4) If n is even, then $H_n(\mathbb{R}P^n) = 0$;
(5) If n is odd, then $H_n(\mathbb{R}P^n) = \mathbb{Z}$;
(6) For any integer $k \in (-\infty, -1] \cup [n+1, \infty)$, we have $H_k(\mathbb{R}P^n) = 0$.

EXERCISE 16B: Compute $H_\bullet(\mathbb{R}P^3; \mathbb{Z}/2)$.

Proof of the Universal Coefficient Thoerem: For any $n \in \mathbb{Z}$, let $Z_n := Z_n(C_\bullet)$, let $B_n := B_n(C_\bullet)$, let $H_n := H_n(C_\bullet) = Z_n/B_n$. We wish to show that

$$H_n((C_\bullet)^A) \cong [(H_n)^A] \oplus [H_{n-1} * A].$$

For any $n \in \mathbb{Z}$, as B_{n-1} is a subgroup of the free Abelian group C_{n-1}, we see that B_{n-1} is free Abelian, and so the surjection $\partial_n : C_n \to B_{n-1}$ splits. Then, by Exercise 15J, we obtain, for all $n \in \mathbb{Z}$, an isomorphism $C_n \cong Z_n \oplus B_{n-1}$.

For all $n \in \mathbb{Z}$, let $i_n : B_n \to Z_n$ be the inclusion map. Then, in the category of chain complexes, we conclude that

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$$

is isomorphic to

$$\cdots \quad \to \quad Z_{n+1} \oplus B_n \quad \to \quad Z_n \oplus B_{n-1} \quad \to \quad Z_{n-1} \oplus B_{n-2} \quad \to \quad \cdots,$$

where the boundary maps in the latter are

$$\begin{array}{rcl} \ldots, \quad \alpha_n := (x,y) \mapsto (i_n(y), 0) & : & Z_{n+1} \oplus B_n \to Z_n \oplus B_{n-1} \quad , \\ \alpha_{n-1} := (y,z) \mapsto (i_{n-1}(z), 0) & : & Z_n \oplus B_{n-1} \to Z_{n-1} \oplus B_{n-2} \quad , \ldots. \end{array}$$

Applying the functor $B \mapsto B^A$, we obtain a chain complex

$$\cdots \to (Z_{n+1})^A \oplus (B_n)^A \to (Z_n)^A \oplus (B_{n-1})^A \to (Z_{n-1})^A \oplus (B_{n-2})^A \to \cdots,$$

where the boundary maps are

$$\begin{array}{rcl} \ldots, \quad (\alpha_n)^A = (x,y) \mapsto (i_n^A(y), 0) & : & (Z_{n+1})^A \oplus (B_n)^A \to (Z_n)^A \oplus (B_{n-1})^A \quad , \\ (\alpha_{n-1})^A = (y,z) \mapsto (i_{n-1}^A(z), 0) & : & (Z_n)^A \oplus (B_{n-1})^A \to (Z_{n-1})^A \oplus (B_{n-2})^A \quad , \ldots. \end{array}$$

Now fix $n \in \mathbb{Z}$. We wish to show that H_n applied to this chain complex yields an additive Abelian group isomorphic to $[(H_n)^A] \oplus [H_{n-1} * A]$. That is, we wish to show that

$$\frac{\ker[(\alpha_{n-1})^A]}{\mathrm{im}[(\alpha_n)^A]} \cong [(H_n)^A] \oplus [H_{n-1} * A].$$

Because $(\alpha_{n-1})^A(x,y) = ((i_{n-1})^A(y),0)$ and $(\alpha_n)^A(y,z) = ((i_n)^A(z),0)$, it follows that

$$\ker[(\alpha_{n-1})^A] = [(Z_n)^A] \oplus [\ker(i_{n-1})^A],$$
$$\mathrm{im}[(\alpha_n)^A] = [\mathrm{im}(i_n)^A] \oplus \{0\}.$$

It therefore suffices to show both that the cokernel $(i_n)^A : (B_n)^A \to (Z_n)^A$ is isomorphic to $(H_n)^A$ and that the kernel of $(i_{n-1})^A : (B_{n-1})^A \to (Z_{n-1})^A$ is isomorphic to $H_{n-1} * A$.

We have short exact sequences

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1} \to 0,$$
$$0 \to B_n \to Z_n \to H_n \to 0,$$

where B_{n-1}, Z_{n-1}, B_n and Z_n are all free Abelian. If we apply $B \mapsto B^A$, we get two lines of exact sequences

$$(B_{n-1})^A \to (Z_{n-1})^A \to (H_{n-1})^A \to 0,$$
$$(B_n)^A \to (Z_n)^A \to (H_n)^A \to 0,$$

where the leftmost maps are $(i_{n-1})^A : (B_{n-1})^A \to (Z_{n-1})^A$ and $(i_n)^A : (B_n)^A \to (Z_n)^A$.

From the first line, by definition of torsion product, the kernel of $(i_{n-1})^A : (B_{n-1})^A \to (Z_{n-1})^A$ is isomorphic to $H_{n-1} * A$. Exactness on the second line implies that the cokernel of $(i_n)^A : (B_n)^A \to (Z_n)^A$ is isomorphic to $(H_n)^A$.

This is what was required. **QED**

There is some naturality in Universal Coefficients, which yields a slightly stronger result:

Naturality in the Universal Coefficients Theorem. Let A be an additive Abelian group and let $n \in \mathbb{Z}$. Let $\mathcal{B} := \{\text{chain complexes}\}$ and let $\mathcal{C} := \{\text{additive Abelian groups}\}$. Let $\mathcal{F}, \mathcal{G} : \mathcal{B} \to \mathcal{C}$ be defined by $\mathcal{F}(C_\bullet) = (H_n C_\bullet) \otimes A$ and $\mathcal{G}(C_\bullet) = H_n(C_\bullet \otimes A)$. For all $C_\bullet \in \mathcal{B}$, let $T_{C_\bullet} := \mathrm{Tor}(H_{n-1}(C_\bullet), A)$. There exists a natural transformation $\tau : \mathcal{F} \to \mathcal{G}$ such that: for all $C_\bullet \in \mathcal{B}$, there is a map $\iota : T_{C_\bullet} \to \mathcal{G}(C_\bullet)$ such that

$$(x,y) \quad \mapsto \quad (\tau_{C_\bullet}(x)) + (\iota(y)) \quad : \quad (\mathcal{F}(C_\bullet)) \oplus T_{C_\bullet} \quad \to \quad \mathcal{G}(C_\bullet)$$

is an isomorphism of additive Abelian groups.

In particular, for all $C_\bullet \in \mathcal{C}$, we have that $\tau_{C_\bullet} : \mathcal{F}(C_\bullet) \to \mathcal{G}(C_\bullet)$ is injective. Bear in mind that ι is not natural in $C_\bullet$. We omit a proof of the preceding naturality theorem.

We now begin a new topic: The study of finite CW-complexes and celluar cohomology via spectral sequences.

Let $k \geq 0$ and $d \geq 0$ be integers. Let B^d denote the closed unit ball in $\mathbb{R}^d$; its boundary in B^d is S^{d-1}. Let $kB^d := \{1, \ldots, k\} \times B^d$ and $kS^{d-1} := \{1, \ldots, k\} \times S^{d-1}$. Given a topological space X and a continuous map $f : kS^{d-1} \to X$, we define $A(X,f) := (X \coprod kB^d)/f$, and we let $i_{X,f} : X \to A(X,f)$ be the composite of the inclusion and the canonical map, as shown here:

$$X \quad \subseteq \quad X \coprod kB^d \quad \to \quad \left(X \coprod kB^d\right) \Big/ f \quad = \quad A(X,f).$$

Definition. Let $k \geq 0$, $d \geq 0$ be integers, let Y be a topological space, let $X \subseteq Y$ and let $f : kS^{d-1} \to X$ be continuous. We say that Y is **obtained** from X via f if there is a homeomorphism $\psi : Y \to A(X, f)$ such that $\psi|X = i_{X,f}$.

Note that, in this case, the map $\phi : kB^d \to Y$ obtained by composing

$$kB^d \qquad \subseteq \qquad X \coprod kB^d \qquad \to \qquad A(X, f) \qquad \to \qquad Y$$

has the property that $\phi|(kS^{d-1}) = f$. We will say that such a map $\phi : kB^d \to Y$ is **associated** to (Y, X, f). Note that different choices of ψ will yield different possibilities for ϕ, all associated to (Y, X, f).

Definition. Let $k \geq 0$, $d \geq 0$ be integers, let Y be a topological space and let $X \subseteq Y$. We say that Y is **obtained** from X via k d-cells if there is a continouous map $f : kS^{d-1} \to X$ such that Y is obtained from X via f.

Definition. Let $d \geq 0$ be integers, let Y be a topological space and let $X \subseteq Y$. We say that Y is **obtained** from X via d-cells if there is some integer $k \geq 0$ such that Y is obtained from X via k d-cells.

Definition. Let X be a topological space and let $A \subseteq X$. We say that $(X^0, \ldots, X^d)$ is a (d-dimensional) **skeletal filtration** from A to X if

$$A \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X^d = X,$$

if X^0 is obtained from A via 0-cells and if, for any integer $i \in [1, d]$, X^i is obtained from X^{i-1} via d-cells.

Definition. Let X be a topological space. A **skeletal filtration** of X is a skeletal filtration from $\emptyset$ to X.

Definition. Let X be a topological space and let $A \subseteq X$. We say that (X, A) is a **compact CW-pair** if X is a compact topological space, if A is a closed subset of X and if there exists a skeletal filtration from A to X.

Definition. We say that X is a **finite CW-complex** if $(X, \emptyset)$ is a compact CW-pair, *i.e.*, if X admits a skeletal filtration.

We note that many interesting topological spaces either are finite CW-complexes or are homotopy equivalent to finite CW-complexes. For example, the fundamenal result of Morse theory is that any compact smooth manifold is homotopy equivalent to a finite CW-complex. (The term "smooth manifold" will be defined precisely later in this course.) For another example, the realization of any abstract finite simplicial complex is a finite CW-complex.

Let $\mathcal{C} :=$ {finite CW-complexes with skeletal filtration}, $\mathcal{TS} :=$ {topological spaces}. Note that there is a forgetful functor $\mathcal{F} : \mathcal{C} \to \mathcal{TS}$, in which one forgets the skeletal filtration.

A chain complex $C_\bullet$ is **finitely generated** if both $\{n \in \mathbb{Z} \,|\, C_n \neq 0\}$ is finite and, for all $n \in \mathbb{Z}$, C_n is a finitely generated Abelian group. Let $\mathcal{CC} := \{\text{chain complexes}\}$ and $\mathcal{FGCC} := \{\text{finitely generated chain complexes}\}$. Finally, let $\mathcal{GG} := \{\text{graded groups}\}$.

We take the point of view that the homology functor $H_\bullet : \mathcal{CC} \to \mathcal{GG}$ is a computational nightmare, but that its restriction $H_\bullet : \mathcal{FGCC} \to \mathcal{GG}$ is much more feasible since it amounts to computing kernels, images and cokernels of finitely many matrices of finite size.

Our goal in this part of the notes is to develop a functor $\mathrm{Cell}_\bullet : \mathcal{C} \to \mathcal{FGCC}$ such that, for all $Z \in \mathcal{C}$, we have $H_\bullet(\mathrm{Cell}_\bullet(Z)) \cong H_\bullet(\mathcal{F}(Z))$. (More precisely, we will choose $\mathrm{Cell}_\bullet$ such that $H_\bullet \circ \mathrm{Cell}_\bullet : \mathcal{C} \to \mathcal{GG}$ is equivalent to $H_\bullet \circ \mathcal{F} : \mathcal{C} \to \mathcal{GG}$.) Then, given a finite CW-complex X, to compute $H_\bullet(X)$, we first choose a skeletal filtration of X, *i.e.*, we first choose $Z \in \mathcal{C}$ such that $\mathcal{F}(Z) = X$, and then we compute $H_\bullet(X)$ by instead doing the more feasible computation of $H_\bullet(\mathrm{Cell}_\bullet(Z))$.

Fact. Let Y be a topological space and let $X \subseteq Y$. Let $k, d \in \mathbb{Z}$. Assume $k \geq 0$ and $d \geq 1$. Let $f : kS^{d-1} \to X$ be continuous. Assume that Y is obtaind from X via f and let $\phi : kB^d \to Y$ be associated to (Y, X, f). Then $(\phi, f)_* : H_\bullet(kB^d, kS^{d-1}) \to H_\bullet(Y, X)$ is an isomorphism.

We omit the proof and refer the reader intead to Vick's book on "Homology" or to Rotman's book on algebraic topology. (Look up "relative homeomorphism" and read the nearby results.)

Remark. Let $q, k, d \in \mathbb{Z}$. Assume that $k \geq 0$ and $d \geq 1$. If $d \neq q$, then $H_q(kB^d, kS^d) \cong \{0\}$. If $d = q$, then $H_q(kB^d, kS^d) \cong \mathbb{Z}^k \cong \mathbb{Z}^{k \times 1}$.

We leave the proof of this as an unassigned exercise for the interested reader. (Hint: Consider the long exact sequence associated to (kB^d, kS^d).)

We now fix a topological space X and a collection of subsets $X^0, \ldots, X^d$ of X such that $X^0 \subseteq \cdots X^d = X$. Let $\mathcal{X} := (X^0, \ldots, X^d)$ and let $Z := (X, \mathcal{X})$. For all integers $i < 0$, we define $X^i := \emptyset$. For all integers $i > d$, we define $X^i := X$.

If $\mathcal{X}$ is a skeletal filtration, then, for all $q \in \mathbb{Z}$, we define

$$\mathrm{Cell}_q \quad := \quad \mathrm{Cell}_q(Z) \quad := \quad H_q(X^q, X^{q-1}).$$

For all $n \in \mathbb{Z}$, the long exact sequence of (X^n, X^{n-1}) has the broken form

$$H_\bullet(X^{n-1}) \quad \to \quad H_\bullet(X^n) \quad \to \quad H_\bullet(X^n, X^{n-1}).$$

With $n = q$, between the q degree and the $q-1$ degree, this long exact sequence has a connecting homomorphism $\mathrm{Cell}_q \to H_{q-1}(X^{q-1})$. With $n = q-1$, in the $q-1$ degree, this broken long exact sequence has a map $H_{q-1}(X^{q-1}) \to \mathrm{Cell}_{q-1}$. For all $q \in \mathbb{Z}$, let $\partial_q : \mathrm{Cell}_q \to \mathrm{Cell}_{q-1}$ denote the composition of these two maps:

$$\mathrm{Cell}_q \quad \to \quad H_{q-1}(X^{q-1}) \quad \to \quad \mathrm{Cell}_{q-1}.$$

EXERCISE 16C: Show, for all $q \in \mathbb{Z}$, that $\partial_{q-1} \circ \partial_q = 0$.

Example. Let s, t, u be three distinct points in $\mathbb{R}^2$. Let A be the closed line segment from s to t. Let B be the closed line segment from t to u. Let C be the closed line segment from u to s. Let $T := A \cup B \cup C$. Note that $H_1(T) \cong \mathbb{Z}$, since T is homeomorphic to S^1. Consider now the case where $d = 1$, where $X^0 = \{s, t, u\}$ and where $X = X^1 = T$.

Note that X^0 is obtained from $\emptyset$ by attaching 3 0-cells, and that X^1 is obtained from X^0 by attaching 3 1-cells. Choose attaching maps to make the last sentence true. Then the preceding Fact and Remark show, for all $q \in \mathbb{Z}\backslash\{0, 1\}$, that $\mathrm{Cell}_q = 0$. They also show that there are isomorphisms $\mathrm{Cell}_0 \cong \mathbb{Z}^{3\times 1}$ and $\mathrm{Cell}_1 \cong \mathbb{Z}^{3\times 1}$. Choose such isomorphisms and use them to identify $\partial_1 : \mathrm{Cell}_1 \to \mathrm{Cell}_0$ with a 3×3 matrix $B : \mathbb{Z}^{3\times 1} \to \mathbb{Z}^{3\times 1}$.

EXERCISE 16D: Explictly lay out all of the choices mentioned in the preceding example, and then compute $B \in \mathbb{Z}^{3\times 3}$.

EXERCISE 16E: In the preceding example, show that $H_1(\mathrm{Cell}_\bullet) \cong H_1(X)$.

We aim to show, in complete generality, that if $\mathcal{X}$ is a skeletal filtration of X, then we have $H_\bullet(\mathrm{Cell}_\bullet) \cong H_\bullet(X)$.

Example. Fix an integer $n \geq 0$. Let $h : S^n \to \mathbb{R}P^n$ and $h_1 : S^{n+1} \to \mathbb{R}P^{n+1}$ be the canonnical maps. Let $i : S^n \to S^{n+1}$ be defined by $i(p) = (p, 0)$ and let $j : \mathbb{R}P^n \to \mathbb{R}P^{n+1}$ be defined by $j(h(p)) = h_1(i(p))$; note that j is well-defined and continuous. Let $u : B^{n+1} \to S^{n+1}$ be defined by $u(x) = (x, \sqrt{1 - \|x\|^2})$.

Then $j \coprod (h_1 \circ u) : \mathbb{R}P^n \coprod B^{n+1} \to \mathbb{R}P^{n+1}$ is surjective and has the same fibers as the canonical map

$$\mathbb{R}P^n \coprod B^{n+1} \quad \to \quad \left(\mathbb{R}P^n \coprod B^{n+1}\right) \Big/ h \quad = \quad A(\mathbb{R}P^n, h).$$

Then $j \coprod (h_1 \circ u)$ factors to a continuous bijection

$$\rho \quad : \quad A(\mathbb{R}P^n, h) \quad \to \quad \mathbb{R}P^{n+1};$$

by compactness, this map is closed, therefore open, therefore a homeomorphism, showing that $\mathbb{R}P^{n+1}$ is obtained from $j(\mathbb{R}P^n)$ via one $(n+1)$-cell.

The above remarks, for varying n, give injective continuous maps (denoted above by j) of the form:

$$\mathbb{R}P^0 \quad \to \quad \mathbb{R}P^1 \quad \to \quad \mathbb{R}P^2 \quad \to \quad \cdots.$$

Now consider the case where $d = 2$, where $X = X^2 = \mathbb{R}P^2$, where X^1 be the image of $\mathbb{R}P^1$ in $\mathbb{R}P^2$ and where X^0 be the image of $\mathbb{R}P^0$ in $\mathbb{R}P^2$. The above remarks then show that, for each $k \in \{1, 2\}$, X^k is obtained from X^{k-1} by attaching one k-cell. Thus (X^0, X^1, X^2) is a skeletal filtration of $\mathbb{R}P^2$, showing that $\mathbb{R}P^2$ is a finite CW-complex.

EXERCISE 16F: Compute the cellular chain complex of the skeletal filtration of $\mathbb{R}P^2$ described above. Compute its homology (in all degrees).

Recall that we are considering the case of a **filtered topological space**, *i.e.*, a topological space X together with a finite increasing sequence of subsets $X^0 \subseteq X^1 \subseteq$

$X^2 \subseteq \cdots \subseteq X^d = X$. We have defined $\mathcal{X} := (X^0, \ldots, X^d)$. Recall that we have defined, for all $p < 0$, $X^p := \emptyset$. We have also defined, for all $p > d$, $X^p := X$.

Let's pose the question in complete generality: Can we compute $H_\bullet(X)$ if we know, for all p, $H_\bullet(X^p, X^{p-1})$? This is somehow the topology analogue to the question of whether one can reconstruct a group from the quotients of a composition series.

For topology, the answer, developed below, is, "sometimes yes", and the basic tool for handling this question is what are called spectral sequences. The "sometimes" is, of course, disturbing, but we note that in the case of a *skeletal* filtration (in which each skeleton is obtained by the preceding one from addition of cells), the answer is better: The answer is: "Yes, provided you can calculate the homology of the cellular chain complex associated to the filtration". We have already assigned exercises to demonstrate the feasibility of that kind of computation.

Definition. A **bigraded group** is a map $\mathbb{Z} \times \mathbb{Z} \to \{\text{additive Abelian groups}\}$.

We will now adopt the standard notation that the category of graded groups is denoted $\{\text{additive Abelian groups}\}^{\mathbb{Z}}$, and the category of bigraded groups is denoted

$$\{\text{additive Abelian groups}\}^{\mathbb{Z}\times\mathbb{Z}}.$$

For all $p, q \in \mathbb{Z}$, define $D^p_q := H_q(X^p)$ $E^p_q := H_q(X^p, X^{p-1})$. Then $D^\bullet_\bullet, E^\bullet_\bullet \in \{\text{additive Abelian groups}\}^{\mathbb{Z}\times\mathbb{Z}}$, *i.e.*, $D^\bullet_\bullet$ and $E^\bullet_\bullet$ are both examples of bigraded groups. I advise "picturing" a bigraded group $G^\bullet_\bullet$ by placing, at the integer lattice point $(s,t) \in \mathbb{Z}\times\mathbb{Z}$, *neither* the group G^s_t, *nor* the group G^t_s, but rather the group G^{s+t}_s. Then G^1_1 appears over the point $(1,0)$, and G^1_0 appears over the point $(0,1)$. In what follows, discussion of "origin", "up", "high", "low", "down", "left", "right", "horizontal", "vertical", "northeast", "southwest", "northwest", "southeast" and other points-of-compass will assume this pictorial organization.

This might be odd at first and does require some care to implement. If you read the literature, you will notice that the groups we define as E^p_q are in other places defined as $E^1_{p,q-p}$, and, following that notation, horizontal and vertical take on more traditional meanings. Our choice here means that we have the simple formulas: $D^p_q := H_q(X^p)$ and $E^p_q := H_q(X^p, X^{p-1})$, so we have one advantage in formulas, but one disadvantage in picturing our bigraded groups. Let me mention *a propos* of this that the groups we define below as $(rE)^p_q$ are in other expositions defined as $E^{r+1}_{p,q-p}$.

We take the point of view that our goal is to compute $H_\bullet := H_\bullet(X)$, which means computing, for each q, for some $p \geq d$, the group D^p_q. That is, we want to compute the "high" Ds. Note that we already know the "low" Ds to be 0, because, for all $p, q \in \mathbb{Z}$, if $p < 0$, then $D^p_q = H_q(X^p) = H_q(\emptyset) = 0$.

We recall that, in the case where $\mathcal{X}$ is skeletal, we have, for all $p \neq q$, $E^p_q = 0$, and we have, for all q, $E^q_q = \text{Cell}_q$. We are therefore taking the point of view that $E^\bullet_\bullet$ is known, and that, if $\mathcal{X}$ is cellular, then it is 0 except along the horizontal line through the origin. Along that horizontal line, we see the terms of the cellular chain complex.

Note that, for all $p, q \in \mathbb{Z}$, we have a the long exact sequence associated to the pair (X^p, X^{p-1}) a part of which appears as:

$$\cdots \quad \to \quad D^{p-1}_q \quad \to \quad D^p_q \quad \to \quad E^p_q$$

which is followed by a connecting homomorphism (denoted γ) to

$$D^{p-1}_{q-1} \quad \to \quad D^p_{q-1} \quad \to \quad E^p_{q-1}$$

which is followed by a connecting homomorphism (denoted γ) to

$$D^{p-1}_{q-2} \quad \to \quad D^p_{q-2} \quad \to \quad E^p_{q-2} \quad \to \quad \cdots.$$

Thus we obtain bigraded group homomorphisms $\alpha : D^\bullet_\bullet \to D^{\bullet+1}_\bullet$, $\beta : D^\bullet_\bullet \to E^\bullet_\bullet$ and $\gamma : E^\bullet_\bullet \to D^{\bullet-1}_{\bullet-1}$, with the property that $\operatorname{im}\alpha = \ker\beta$, $\operatorname{im}\beta = \ker\gamma$ and $\operatorname{im}\gamma = \ker\alpha$. We should clarify the definitions of im and ker appearing here: For example, we define $\operatorname{im}\alpha$ to be the bigraded group defined by $(\operatorname{im}\alpha)^p_q = \operatorname{im}(\alpha : D^{p-1}_q \to D^p_q)$. On the other hand, $\ker\alpha$ is defined by $(\ker\alpha)^p_q = \ker(\alpha : D^p_q \to D^{p+1}_q)$.

We now pause from this discussion to define, quite generally:

Definition. A $\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}$**-exact couple** consists of

(1) two bigraded groups $D^\bullet_\bullet$, $E^\bullet_\bullet$; and

(2) three maps $\alpha : D^\bullet_\bullet \to D^{\bullet+a}_{\bullet+x}$, $\beta : D^\bullet_\bullet \to E^{\bullet+b}_{\bullet+y}$ and $\gamma : E^\bullet_\bullet \to D^{\bullet+c}_{\bullet+z}$

such that $\operatorname{im}\alpha = \ker\beta$, such that $\operatorname{im}\beta = \ker\gamma$ and such that $\operatorname{im}\gamma = \ker\alpha$.

Note that we have $\gamma\beta = 0$ but $\beta\gamma : E^\bullet_\bullet \to E^{\bullet+b+c}_{\bullet+y+z}$ may be nonzero. However $(\beta\gamma)^2 = \beta(\gamma\beta)\gamma = \beta \cdot 0 \cdot \gamma = 0$. Thus $(E^\bullet_\bullet, \beta\gamma)$ is simply a family of chain complexes.

With the preceding definition, we see that a filtration $\mathcal{X}$ of a topological space X yields a $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$-exact couple, which basically encodes a family of long exact sequences, one for each pair of consecutive terms in the filtration of X. In this case, $\beta\gamma$ has bidegree $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and so $(E^\bullet_\bullet, \beta\gamma)$ is pictured a collection of horizontal chain complexes with all arrows westbound (*i.e.*, horizontal and to the left).

Recall that, in the case where $\mathcal{X}$ is skeletal, $E^\bullet_\bullet$ is zero except on the horizontal line through the origin. In this case, $\beta\gamma$ appears as a family of westbound arrows, and, along the horizontal line through the origin, we see the cellular chain complex $(\mathrm{Cell}_\bullet, \partial)$. (Observe, for all $p \in \mathbb{Z}$, that $\partial_p = (\beta\gamma)^p_p$.) We define $H^C_\bullet := H_\bullet(\mathrm{Cell}_\bullet)$ and we take the point of view that that is known. That is, we know certain subquotients of the various groups E^p_q.

Recall that we wish to compute the high values of $D^\bullet_\bullet$, which are the same as $H_\bullet$. Thus, phrased broadly, our task is to use known information about $(E^\bullet_\bullet, \beta\gamma)$ to find out information about high values of $D^\bullet_\bullet$. (We may also use that the low $D^\bullet_\bullet$ are all zero.)

We further adopt the point of view that any time we can declare that some group appears as a subquotient of some D^p_q, then we have made progress, particularly when $p \geq d$, in which case $H_q = D^p_q$.

In fact, for each q, if we can get enough subquotients of H_q, then perhaps they will fit together to give the quotient groups of a composition series for H_q. Finally, if we're very lucky, all of those groups but one will be nonzero, and we can then declare that H_q is isomorphic to that group.

Amazingly, in the case where $\mathcal{X}$ is skeletal, this is exactly what happens. Moreover, we should point out that in many other applications of spectral sequences, this is exactly what happens.

We now pause from the specific discussion about filtered topological spaces to make a general

Definition. Let $C := (D^\bullet_\bullet, E^\bullet_\bullet, \alpha, \beta, \gamma)$ be a $\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}$-exact couple. Then the **derived couple** of C is the $\begin{pmatrix} a & b-a & c \\ x & y-x & z \end{pmatrix}$-exact couple $\widehat{C} = (\widehat{D}^\bullet_\bullet, \widehat{E}^\bullet_\bullet, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$, defined as follows: Let $\widehat{D}^\bullet_\bullet := \operatorname{im} \alpha$ and let $\widehat{E}^\bullet_\bullet := (\ker \beta\gamma)/(\operatorname{im} \beta\gamma)$. Define $\widehat{\alpha} : \widehat{D}^\bullet_\bullet \to \widehat{D}^{\bullet+a}_{\bullet+x}$ by $\widehat{\alpha} := \alpha|(\widehat{D}^\bullet_\bullet)$. Define $\widehat{\beta} : \widehat{D}^\bullet_\bullet \to \widehat{E}^{\bullet-a+b}_{\bullet-x+y}$ by $\widehat{\beta}(\alpha(d)) = [\beta(d)]$, where $[\cdot] : \ker \beta\gamma \to \widehat{E}^\bullet_\bullet$ is the quotient map. Define $\widehat{\gamma} : \widehat{E}^\bullet_\bullet \to \widehat{D}^{\bullet+c}_{\bullet+z}$ by $\widehat{\gamma}([e]) = \gamma(e)$.

EXERCISE 16G: Show that $\widehat{\beta}$ and $\widehat{\gamma}$ are well-defined in the preceding definition.

EXERCISE 16H: In the preceding definition, show that $\operatorname{im} \widehat{\alpha} = \ker \widehat{\beta}$, that $\operatorname{im} \widehat{\beta} = \ker \widehat{\gamma}$ and that $\operatorname{im} \widehat{\gamma} = \ker \widehat{\alpha}$.

We now return to our discussion in which we have an exact couple obtained from a filtration $\mathcal{X}$ of a topological space X. Note, in this case that the derived couple described above is a $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$-exact couple. In particular $\widehat{\beta}\widehat{\gamma}$ has bidegree $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ and is pictured as a family of arrows that move in a southwesterly direction. Note also that, in $(\widehat{E}^\bullet_\bullet, \widehat{\beta}\widehat{\gamma})$, the terms on the horizontal line through the origin are exactly $H^C_\bullet$, which we are taking as known quantities. As before, $(\widehat{E}^\bullet_\bullet, \widehat{\beta}\widehat{\gamma})$ is a family of chain complexes, but these chain complexes have the nice property that, for each one, all but one term is zero. Thus these are chain complexes with the property that passing to homology does nothing (up to isomorphism)! So, later on, when we discuss $\widehat{\widehat{E}}^\bullet_\bullet$, we will have $\widehat{\widehat{E}}^\bullet_\bullet \cong \widehat{E}^\bullet_\bullet$, and is therefore known, given that we have computed the homology of the cellular chain complex of $\mathcal{X}$.

Moreover, as we continue and take further and further derived exact couples, we can determine that all these exact couples have the property that their E-term is known and agrees, up to isomorphism, with $\widehat{E}^\bullet_\bullet$.

Again, let us recall that, speaking very broadly, we wish to figure out, from all this known information, the subquotients of D^p_q, for $p \geq d$. That is, we wish to determine as many subquotients as possible of the "high" values of $D^\bullet_\bullet$.

We now define $0C := (D^\bullet_\bullet, E^\bullet_\bullet, \alpha, \beta, \gamma)$, and, for all integers $r \geq 0$, we recursively define $(r+1)C = \widehat{rC}$. For all integers $r \geq 0$ define $(rD)^\bullet_\bullet$, $(rE)^\bullet_\bullet$, $r\alpha$, $r\beta$ and $r\gamma$ to be the components of rC, so that we have $rC = ((rD)^\bullet_\bullet, (rE)^\bullet_\bullet, r\alpha, r\beta, r\gamma)$.

By the preceding remarks, if the original exact couple C is obtained from a skeletal filtration of a topological space, then we have that $(2E)^\bullet_\bullet$ is isomorphic to $\widehat{E}^\bullet_\bullet$. Moreover, the bidegree of $(2\beta)(2\gamma)$ is $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ and is pictured as a family of arrows that move in a south-by-southwesterly direction.

So, in $((2E)^\bullet_\bullet, (2\beta)(2\gamma))$, the terms on the horizontal line through the origin are, up to isomorphism, $H^C_\bullet$, which we are taking as known quantities. As before, $((2E)^\bullet_\bullet, (2\beta)(2\gamma))$ is a family of chain complexes, and these chain complexes again have the nice property that, for each one, all but one term is zero. Thus these are chain complexes with the property that passing to homology does nothing (up to isomorphism)! Then $(3E)^\bullet_\bullet$ is isomorphic (in the category of bigraded groups) to $(2E)^\bullet_\bullet$, which, in turn is isomorphic to $\widehat{E}^\bullet_\bullet$.

Still assuming that C is obtained from a skeltal filtration, continuing in this manner, we see, for all integers $r \geq 1$, that $(rE)^\bullet_\bullet$ is isomorphic (in the category of bigraded groups) to $\widehat{E}^\bullet_\bullet$. That is, as we pass to higher and higher derived couples, nothing happens to the E-term after $r = 1$. So, for all integers $r \geq 1$, for all $p, q \in \mathbb{Z}$, if $p \neq q$, then $(rE)^p_q = 0$. Also, all integers $r \geq 1$, for all $q \in \mathbb{Z}$, we have $(rE)^q_q = H^C_q$. Recall, for all $p, q \in \mathbb{Z}$, if $p \geq d$, then $D^p_q = H_q$. We therefore wish to show, for all $q \in \mathbb{Z}$, that there are some integer $r \geq 1$ and some integer $p \geq d$ such that $(rE)^q_q \cong D^p_q$.

More precisely, we will show for all $q \in \mathbb{Z}$, that there are some integer $r \geq 1$ and some integer $p \geq d$ and some composition series of D^p_q such that every quotient of the composition series is 0 except for one, which is isomorphic to $(rE)^q_q$.

It therefore behooves us to find conditions on exact couples under which, a given $(rE)^p_q$ can be seen as a subquotient of some $D^{p'}_{q'}$.

We now return to the discussion of a general exact couple. First, note, for all $p, q, r \in \mathbb{Z}$, that if $r \geq 1$, then $(rD)^p_q$ is a subgroup of $((r-1)D)^p_q$ and that $(rE)^p_q$ is a subquotient of $((r-1)E)^p_q$. It follows that: for all $p, q, r \in \mathbb{Z}$, that if $r \geq 0$, then $(rD)^p_q$ is a subgroup of D^p_q and that $(rE)^p_q$ is a subquotient of E^p_q. Given $p, q \in \mathbb{Z}$, we now attempt to identify these subgroups (as r varies) of D^p_q and subqotients (as r varies) of E^p_q.

Recall that $\widehat{D}^\bullet_\bullet = \operatorname{im}\alpha = \alpha(D^\bullet_\bullet)$ and that $\widehat{\alpha} := \alpha|\widehat{D}^\bullet_\bullet$. Then $(2D)^\bullet_\bullet = \widehat{\alpha}(\widehat{D}^\bullet_\bullet) = (\alpha|\widehat{D}^\bullet_\bullet)(\widehat{D}^\bullet_\bullet) = \alpha(\widehat{D}^\bullet_\bullet) = \alpha(\alpha(D^\bullet_\bullet)) = \operatorname{im}(\alpha^2)$. We leave it as an unassigned exercise to continue this argument to show, for all integers $r \geq 0$, that $(rD)^\bullet_\bullet = \operatorname{im}(\alpha^r)$. That is, for all $p, q, r \in \mathbb{Z}$, if $r \geq 0$, then $(rD)^p_q = \alpha^r(D^{p-ra}_{q-rb})$. This shows $(rD)^p_q$ as a subgroup of D^p_q.

We now turn to the E-terms. First, note that

$$\widehat{E}^\bullet_\bullet = \frac{\ker(\beta\gamma)}{\operatorname{im}(\beta\gamma)} = \frac{\gamma^{-1}(\ker\beta)}{\beta(\operatorname{im}\gamma)} = \frac{\gamma^{-1}(\operatorname{im}\alpha)}{\beta(\ker\alpha)}.$$

Thus we see $\widehat{E}^\bullet_\bullet$ as a subquotient of $E^\bullet_\bullet$.

A similar argument (just put a hat on each step of the preceding paragraph) shows

$$(2E)^\bullet_\bullet = \frac{\widehat{\gamma}^{-1}(\operatorname{im}\widehat{\alpha})}{\widehat{\beta}(\ker\widehat{\alpha})}.$$

Thus we see $(2E)^\bullet_\bullet$ as a subquotient of $\widehat{E}^\bullet_\bullet$. Under the canonical surjection $\ker(\beta\gamma) \to (\widehat{E})^\bullet_\bullet$, pull back the numerator and denominator (namely, pull back $\widehat{\gamma}^{-1}(\operatorname{im}\widehat{\alpha})$ and $\widehat{\beta}(\ker\widehat{\alpha})$) in the above displayed equation. The quotient of the resulting pullbacks is then isomorphic to $(2E)^\bullet_\bullet$ and shows $(2E)^\bullet_\bullet$ as a subquotient of $E^\bullet_\bullet$. This gives enough hints to make the following

EXERCISE 17A: Show that

$$(2E)^{\bullet}_{\bullet} \cong \frac{\gamma^{-1}(\operatorname{im}\alpha^2)}{\beta(\ker\alpha^2)}.$$

Generalizing Exercise 17A, it is an unassigned exercise to see, for all integers $r \geq 1$,

$$(rE)^{\bullet}_{\bullet} \cong \frac{\gamma^{-1}(\operatorname{im}\alpha^r)}{\beta(\ker\alpha^r)}.$$

This shows $(rE)^{\bullet}_{\bullet}$ as a subquotient of $E^{\bullet}_{\bullet}$.

To be precise about indices, we need to do some calculations, given the bidegrees of α, β and γ. For all $p, q, r \in \mathbb{Z}$, if $r \geq 1$, then

$$(rE)^p_q \cong \left(\frac{\gamma^{-1}(\operatorname{im}\alpha^r)}{\beta(\ker\alpha^r)}\right)^p_q = \frac{\gamma^{-1}(\operatorname{im}(\alpha^r : D^{p+c-ra}_{q+z-rx} \to D^{p+c}_{q+z}))}{\beta(\ker(\alpha^r : D^{p-b}_{q-y} \to D^{p-b+ra}_{q-y+rx}))}.$$

Now recall, in the skeletal filtered case, that, for all $p, q \in \mathbb{Z}$, if $p < 0$, then $D^p_q = 0$.

For the general case, fix $p, q, r \in \mathbb{Z}$ and assume that $r \geq 1$ and that $D^{p+c-ra}_{q+z-rx} = 0$. In this case, we have $\operatorname{im}(\alpha^r : D^{p+c-ra}_{q+z-rx} \to D^{p+c}_{q+z}) = 0$ and so the numerator in the preceding displayed equation is just $(\ker\gamma)^p_q$, which is equal to $(\operatorname{im}\beta)^p_q$. So, in this case, we see that

$$(rE)^p_q \cong \left(\frac{\operatorname{im}\beta}{\beta(\ker\alpha^r)}\right)^p_q.$$

EXERCISE 17B: Let A, B and C be groups, and let $f : A \to B$ and $g : A \to C$ be homomorphisms. Show that

$$\frac{\operatorname{im} f}{f(\ker g)} \cong \frac{\operatorname{im} g}{g(\ker f)}.$$

Again, assume that $r \geq 1$ and that $D^{p+c-ra}_{q+z-rx} = 0$. We apply Exercise 17B to the preceding expression for $(rE)^p_q$, with $f : A \to B$ replaced by $\beta : D^{p-b}_{q-y} \to E^p_q$ and with $g : A \to C$ replaced by $\alpha^r : D^{p-b}_{q-y} \to D^{p-b+ra}_{q-y+rx}$. Taking care with our indices, we conclude

$$(rE)^p_q \cong \left(\frac{\operatorname{im}\alpha^r}{\alpha^r(\ker\beta)}\right)^{p-b+ra}_{q-y+rx}.$$

Since $\ker\beta = \operatorname{im}\alpha$, we then have

$$(rE)^p_q \cong \left(\frac{\operatorname{im}\alpha^r}{\operatorname{im}\alpha^{r+1}}\right)^{p-b+ra}_{q-y+rx}.$$

Or, equivalently, we have

$$(rE)^p_q \cong \frac{\alpha^r(D^{p-b}_{q-y})}{\alpha^{r+1}(D^{p-b-a}_{q-y-x})}.$$

Since all powers of α map D-terms to D-terms, we finally see $(rE)^p_q$ as a subquotient of a term in $D^\bullet_\bullet$. Specifially, we see $(rE)^p_q$ as a subquotient of D^{p-b+ra}_{q-y+rx}.

In the case where our original exact couple C is obtained from a skeletal filtration, we have $(a,x) = (1,0)$ and $(b,y) = (0,0)$. So, as $r \to \infty$ this D-term, D^{p-b+ra}_{q-y+rx}, moves "higher and higher". Since the high D-terms give the homology of X, this is exactly as we would wish.

We now complete the details, and we assume from here on out that the exact couple C is obtained from a skeletal filtration. Given $p \in \mathbb{Z}$, letting $r_0 := \max\{1, p\}$, we have, for all integers $r \geq r_0$, that $X^{p-r} = \emptyset$, so, for all $q \in \mathbb{Z}$, we get $D^{p+c-ra}_{q+z-rx} = D^{p-r}_q = H_q(X^{p-r}) = H_q(\emptyset) = 0$, which, by earlier remarks, then implies that

$$(rE)^p_q \cong \frac{\alpha^r(D^p_q)}{\alpha^{r+1}(D^{p-1}_q)} \cong \frac{\operatorname{im}(H_q\, X^p \to H_q\, X^{p+r})}{\operatorname{im}(H_q\, X^{p-1} \to H_q\, X^{p+r})}.$$

Now fix $q \in \mathbb{Z}$. We wish to show that $H^C_q = H_q$.

Recall that $H_q = H_q\, X$. For all $p \in \mathbb{Z}$, let $F^p := \operatorname{im}(H_q\, X^p \to H_q)$. Note that, for all integers $p < 0$, we have $F^p = 0$. Note that, for all integers $p \geq d$, we have $F^p = H_q$. Then, as p varies, $\{F^p\}$ is a composition series of H_q. We wish to show that, as p varies, the quotients $\{F^p/F^{p-1}\}$ are all 0 except for one, which is isomorphic to H^C_q.

For all $p \in \mathbb{Z}$, there exists $r_0 \in \mathbb{Z}$ such that, for all $r \geq r_0$, we have $(rE)^p_q \cong F^p/F^{p-1}$. For all integers $r \geq 1$, we recall that $(rE)^\bullet_\bullet$ is isomorphic to $\widehat{E}^\bullet_\bullet$ and that this bigraded group is zero, except on the line through the origin, where one sees the cellular homology of X. Specifically, for all $p \in \mathbb{Z}$, for all integers $r \geq 1$, if $p \neq q$, then $(rE)^p_q \cong \widehat{E}^p_q \cong 0$. Also, for all $r \geq 1$, we have $(rE)^q_q \cong \widehat{E}^q_q \cong H^C_q$.

Putting all this together, we see, for all $p \in \mathbb{Z}$, there exists $r_0 \in \mathbb{Z}$ such that, for all $r \geq r_0$, we have

$$F^p/F^{p-1} \cong (rE)^p_q \cong \begin{cases} 0 & \text{if } p \neq q \\ H^C_q & \text{if } p = q \end{cases}.$$

So we get what was required: As p varies, all but one of $\{F^p/F^{p-1}\}$ is 0 and that one is isomorphic to H^C_q.

This completes the proof. QED

Recall that a chain complex is a graded group with a differential of degree -1. (A **differential** is a map whose square is zero.)

Definition. A **cochain complex** is a graded additive Abelian group with a diffential of degree $+1$. That is, it is a bi-infinite sequence $\ldots, C^{-1}, C^0, C^1, \ldots$ of additive Abelian groups, together with a map $d : C^\bullet \to C^{\bullet+1}$ such that $d^2 : C^\bullet \to C^{\bullet+2}$ is equal to zero. This map d is called the **coboundary map**.

Traditionally, one uses subscripts to index chain complexes and superscripts to index cochain complexes. We will use the indexing convention that, if $(C^\bullet, d)$ is a cochain complex, then, for all $k \in \mathbb{Z}$, we have a map $d^k : C^k \to C^{k+1}$.

Let A be a fixed additive Abelian group. Then, for all additive Abelian groups B, $\mathrm{Hom}(B, A)$ is an additive Abelian group under the addition $(f+g)(b) = (f(b)) + (g(b))$. Then $\mathcal{H} := \mathrm{Hom}(\cdot, A) : \{\text{additive Abelian groups}\} \to \{\text{additive Abelian groups}\}$ is a contravariant functor. That is, if $\phi : B \to C$ is an arrow in $\{\text{additive Abelian groups}\}$ then we obtain a map $\phi^* = \mathcal{H}(\phi) : \mathcal{H}(C) \to \mathcal{H}(B)$ defined by $\phi^*(f) = f \circ \phi$.

For any chain complex $(C_\bullet, \partial)$, we define a graded group $\mathcal{H}(C_\bullet)$ by $(\mathcal{H}(C_\bullet))_n = \mathcal{H}(C_n)$, and we then have a cochain complex $(\mathcal{H}(C_\bullet), \mathcal{H}(\partial))$. Thus we get a (contravariant) functor

$$(C_\bullet, \partial) \mapsto (\mathcal{H}(C_\bullet), \mathcal{H}(\partial)) : \{\text{chain complexes}\} \to \{\text{cochain complexes}\}.$$

This functor is also denoted $\mathrm{Hom}(\cdot, A)$.

Fact. Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence of additive Abelian groups. Let A be an additive Abelian group and let $\mathcal{H} := \mathrm{Hom}(\cdot, A)$. Then $0 \to \mathcal{H}(B') \to \mathcal{H}(B) \to \mathcal{H}(B'')$ is exact.

That is, $\mathrm{Hom}(\cdot, A)$ is a "left exact" functor. We leave it as an unassigned exercise to show that it is not fully exact, *i.e.*, that $\mathcal{H}(B) \to \mathcal{H}(B'')$ is not necessarily surjective.

Recall, that for any additive Abelian group C, we define K_C to be the kernel of the map $\phi_C : \mathbb{Z}[C] \to C$ which extends, by $\mathbb{Z}$-linearity, the identity map $C \to C$. Recall that $\iota_C : K_C \to \mathbb{Z}[C]$ denotes the injection map.

Recall that the **cokernel** of a map $\sigma : S \to T$ is the group $T/(\sigma(S))$, so the cokernel of σ measures how close σ is to being surjective.

Definition. Let A and C be additive Abelian groups. Let $\mathcal{H} := \mathrm{Hom}(\cdot, A)$. We define $\mathrm{Ext}(C, A)$ to be the cokernel of $\mathcal{H}(\iota_C) : \mathcal{H}(\mathbb{Z}[C]) \to \mathcal{H}(K_C)$. That is,

$$\mathrm{Ext}(C, A) := \frac{\mathcal{H}(K_C)}{\mathrm{im}[\mathcal{H}(\iota_C) : \mathcal{H}(\mathbb{Z}[C]) \to \mathcal{H}(K_C)]}.$$

Remark. Let A and C be additive Abelian groups and let F' and F be free Abelian groups. Let $0 \to F' \to F \to C \to 0$ be an exact sequence. Then $\mathrm{Ext}(C, A)$ is isomorphic to the cokernel of the map $\mathrm{Hom}(F, A) \to \mathrm{Hom}(F', A)$ induced by $F' \to F$.

EXERCISE 17C: Let $m, n \in \mathbb{Z}\backslash\{0\}$. Compute $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z}/n)$.

EXERCISE 17D: Show, for any additive Abelian group A, that $\mathrm{Ext}(\mathbb{Z}, A) = 0$.

EXERCISE 17E: Let $m \in \mathbb{Z}\backslash\{0\}$. Compute $\mathrm{Ext}(\mathbb{Z}/m, \mathbb{Z})$.

Note, based on your answers to Exercise 17D and to Exercise 17E, that $\mathrm{Ext}(A, B)$ is different from $\mathrm{Ext}(B, A)$.

Fact. Let A, B and C be additive Abelian groups. Then

$$\mathrm{Ext}(A \oplus B, C) \cong \mathrm{Ext}(A, C) \oplus \mathrm{Ext}(B, C)$$
$$\mathrm{Ext}(A, B \oplus C) \cong \mathrm{Ext}(A, B) \oplus \mathrm{Ext}(A, C).$$

Let A be an additive Abelian group.

We note that $\mathrm{Ext}(\cdot, A)$ is sometimes called the "derived functor" of $\mathrm{Hom}(\cdot, A)$. One can also prove that $\mathrm{Ext}(A, \cdot)$ is (equivalent to) the derived functor of the covariant functor $\mathrm{Hom}(A, \cdot)$.

Similarly, $\mathrm{Tor}(\cdot, A)$ is the derived functor of $\cdot \otimes A$ and one may prove that $\mathrm{Tor}(A, \cdot)$ is (equivalent to) the derived functor of $A \otimes \cdot$. So, since $A \otimes \cdot$ and $\cdot \otimes A$ are equvalent, it follows that $\mathrm{Tor}(A, \cdot)$ and $\mathrm{Tor}(\cdot, A)$ are equivalent, which implies, for all additive Abelian groups B, that $\mathrm{Tor}(A, B)$ is isomorphic to $\mathrm{Tor}(B, A)$.

On the other hand, we leave it as an unassigned exercise to show that there exist choices of A and B for which $\mathrm{Hom}(A, B)$ is not isomorphic to $\mathrm{Hom}(B, A)$. Thus $\mathrm{Hom}(A, \cdot)$ is not equivalent to $\mathrm{Hom}(\cdot, A)$. Similarly, $\mathrm{Ext}(A, \cdot)$ is not equivalent to $\mathrm{Ext}(\cdot, A)$.

EXERCISE 17F: Compute $\mathrm{Ext}([(\mathbb{Z}/3) \oplus (\mathbb{Z}/27) \oplus (\mathbb{Z}/54)], [(\mathbb{Z}/2) \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/12)])$, up to isomorphism. Put your answer in the form $(\mathbb{Z}/a) \oplus (\mathbb{Z}/b) \oplus (\mathbb{Z}/c) \oplus \cdots$, with $a|b|c|\cdots$.

Note that your answers to Exercise 15H, Exercise 15Q and Exercise 17F are all the same. Moreover, it would be the same if you were to compute $\mathrm{Hom}([(\mathbb{Z}/3) \oplus (\mathbb{Z}/27) \oplus (\mathbb{Z}/54)], [(\mathbb{Z}/2) \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/12)])$.

In fact, we have four "bifunctors" ($\otimes$, Tor, Hom and Ext) on additive Abelian groups, and they all agree when restricted to finite additive Abelian groups. We leave it as an unassigned exercise to verify that they are all different when considering infinite additive Abelian groups.

Given a cochain complex $(C^\bullet, d)$, we define $Z^\bullet(C^\bullet) := \ker d$, we define $B^\bullet(C^\bullet) := \mathrm{im}\, d$, and we define its **cohomology** to be $H^\bullet(C^\bullet) := (\ker d)/(\mathrm{im}\, d)$. Specifically, for all $n \in \mathbb{Z}$, we have

$$\begin{aligned} Z^n(C^\bullet) &= \ker(d : C^n \to C^{n+1}) \\ B^n(C^\bullet) &= \mathrm{im}(d : C^{n-1} \to C^n) \\ H^n(C^\bullet) &= \frac{Z^n(C^\bullet)}{B^n(C^\bullet)}. \end{aligned}$$

We also have a **Universal Coefficients Theorem** for cohomology:

Theorem. Let $(C_\bullet, d)$ be a free chain complex. Then $H^n(\mathrm{Hom}(C_\bullet, A))$ is isomorphic to

$$\mathrm{Hom}(H^n C_\bullet, A) \qquad \oplus \qquad \mathrm{Ext}(H_{n-1} C_\bullet, A).$$

If X is a topological space, then we define the **cohomology of** X as $H^\bullet(X) := H^\bullet(\mathrm{Hom}(S_\bullet X, \mathbb{Z}))$. If, in addition, A is an additive Abelian group, then we define the **cohomology of** X **with coefficients in** A as $H^\bullet(X; A) := H^\bullet(\mathrm{Hom}(S_\bullet X, A))$. Then the preceding theorem implies that $H^\bullet(X; A)$ is isomorphic to

$$\mathrm{Hom}(H_n(X), A)) \qquad \oplus \qquad \mathrm{Ext}(H_{n-1}(X), A).$$

EXERCISE 17G: Compute $H^\bullet(\mathbb{R}P^3)$.

EXERCISE 17H: Compute $H^\bullet(\mathbb{R}P^3; \mathbb{Z}/2)$.

One might ask what value there is in cohomology and cohomology with coefficients (as well as homology with coefficients), given that these are all computable directly from homology.

First, sometimes these other groups are more easily computable than is homology, and they do provide invariants. Moreover, sometimes we gain information through Universal Coefficients about the homology groups, by first discovering facts about homology or cohomology with coefficients.

Second, there are times when working with $\mathbb{Z}/2$ coefficients is preferable to working with $\mathbb{Z}$ coefficients. In fact, a chain with $\mathbb{Z}/2$ coefficients is just the same as a set of parametric simplices, since the only (non-zero) possibility for a coefficient on a parametric simplex is $1 \in \mathbb{Z}/2$. So, when one "pictures" a chain with $\mathbb{Z}/2$ coefficients, one is simply picturing a collection of parametric simplices. This is particularly helpful in dealing with non-orientable manifolds (defined later) where one can't know whether, when a parametric simplex covers a point in the space, the covering should count for $+1$ or -1. With $\mathbb{Z}/2$ coefficients, one doesn't even ask the question; one only asks whether the point is covered or not.

Third, there is a ring structure (to be defined below) in cohomology. Don Kahn wrote me a note showing that $\mathbb{R}P^3$ has the same homology groups and cohomology groups as $\mathbb{R}P^2 \vee S^3$, but that their cohomology rings are different, so that cohomology with the ring structure can sometimes distinguishes spaces that are indistinguishable using just cohomology groups. (Note: If X and Y are connected topological manifolds, then $X \vee Y$ is the topological space obtained by choosing points $x \in X$ and $y \in Y$ and letting $X \vee Y$ the space obtained by identifying x with y in $X \coprod Y$. More precisely, it is the quotient of the disjoint union $X \coprod Y$ by the smallest equivalence relation on $X \coprod Y$ such that x is equivalent to y. One can show that, up to homeomorphism, the result is independent of the choices of x and y. Note, by Mayer-Vietoris, that, for all integers $k \geq 1$, we have $H_k(X \vee Y) = (H_k(X)) \oplus (H_k(Y))$.)

We now turn to the problem of computing $H_\bullet(X \times Y)$, given knowledge of $H_\bullet(X)$ and $H_\bullet(Y)$. This is part of a broader problem of finding the homology of the total space of a fibration given the homology of its fiber and base. That broader problem involves something called the Leray-Serre Spectral Sequence, and we will not present it here.

For the simplest fibrations, where one simply takes the product of two spaces, computation breaks down into two parts, one of which (called the Eilenberg-Zilber Theorem) is topological and one of which (called the Kunneth Theorem) is algebraic. To state these results, we need the following definitions:

Definition. Let $C_\bullet$ and $D_\bullet$ be two graded groups. We then define the graded group $C_\bullet \otimes D_\bullet$ by

$$(C_\bullet \otimes D_\bullet)_n \qquad := \qquad \bigoplus_{p \in \mathbb{Z}} C_p \otimes D_{n-p}.$$

Definition. Let $C_\bullet$ and $D_\bullet$ be two chain complexes. Then $C_\bullet \otimes D_\bullet$ is a chain complex via the diffential defined by, for all $p, q \in \mathbb{Z}$, for all $\sigma \in C_p$, for all $\tau \in D_q$, we have:

$$\partial(\sigma \otimes \tau) \quad = \quad [(\partial\sigma) \otimes \tau] \quad + \quad [(-1)^p][\sigma \otimes (\partial\tau)].$$

Eilenberg-Zilber Theorem. Let X and Y be topological spaces. Then $H_\bullet(X \times Y)$ is naturally isomorphic to $H_\bullet((S_\bullet X) \otimes (S_\bullet Y))$.

By "naturally isomorphic", we mean that the functor

$$(X, Y) \mapsto H_\bullet(X \times Y) : \{\text{ordered pairs of topological spaces}\} \to \{\text{graded groups}\}$$

is equivalent to the functor

$$(X, Y) \mapsto H_\bullet((S_\bullet X) \otimes (S_\bullet Y)) : \{\text{ordered pairs of topological spaces}\} \to \{\text{graded groups}\}.$$

We will comment on the proof later.

Definition. Let $A_\bullet$ and $B_\bullet$ be graded groups. Then we define $\mathrm{Tor}(A_\bullet, B_\bullet)$ to be the graded group defined by

$$(\mathrm{Tor}(A_\bullet, B_\bullet))_k \quad := \quad \bigoplus_{p \in \mathbb{Z}} \mathrm{Tor}(A_p, B_{k-p}).$$

Kunneth Theorem. Let $C_\bullet$ and $D_\bullet$ be chain complexes and let $n \in \mathbb{Z}$. Then $H_n(C_\bullet \otimes D_\bullet)$ is isomorphic to

$$[(H_\bullet C_\bullet) \otimes (H_\bullet D_\bullet)]_n \quad \oplus \quad [\mathrm{Tor}(H_\bullet C_\bullet, H_\bullet D_\bullet)]_{n-1}.$$

You may note some similarity with the Universal Coefficients Theorems appearing above, and we comment that, in fact, the first of those two theorems can be used to prove the Kunneth Theorem, see either Vick's book "Homology" or Rotman's book on algebraic topology. Beyond this, we will not comment on the proof of the Kunneth Theorem.

Putting the two preceding theorems together, for any two topological spaces X and Y, for any $n \in \mathbb{Z}$, we have that $H_n(X \times Y)$ is isomorphic to

$$\left[\bigoplus_{p \in \mathbb{Z}} H_p(X) \otimes H_{n-p}(Y)\right] \quad \bigoplus \quad \left[\bigoplus_{p \in \mathbb{Z}} \mathrm{Tor}(H_p(X), H_{n-1-p}(Y))\right].$$

EXERCISE 17I: Compute $H_\bullet(\mathbb{R}P^3 \times \mathbb{R}P^2)$.

Some comments on the proof of the Eilenberg-Zilber Theorem: We will not prove this theorem fully. Instead we will state another theorem called the Acyclic Models Theorem, and we will show that it implies two results:

(1) The fact that if X and Y are topological spaces, if $f : X \to Y$ and $g : X \to Y$ are continuous maps and if f is homotopic to g, then $f_* : H_\bullet(X) \to H_\bullet(Y)$ and $g_* : H_\bullet(X) \to H_\bullet(Y)$ are equal; and

(2) The Eilenberg-Zilber Theorem.

Recall that we have already proved (1). I will take the point of view that Acyclic Models is simply a "straightforward" generalization of (1), and that a person with a sufficient understanding of the proof of (1) and of the proof that Acyclic Models implies (1) will be able to figure out the proof of Acyclic Models. Since we will show how Acyclic Models implies Eilenberg-Zilber, this will give a quasi-proof of Eilenberg-Zilber.

Before stating Acycic Models, we need some definitions.

Definition. Let A be an additive group and let $S \subseteq A$. We say that S is a **basis** of A if the $\mathbb{Z}$-linear extension $\mathbb{Z}[S] \to A$ of the inclusion $S \to A$ is an isomorphism.

Note that A is free Abelian iff A admits a basis.

Definition. Let $\mathcal{C}$ be a category. Then a **model system** on $\mathcal{C}$ is a subset $\mathcal{M}$ of the class $\{\text{objects in } \mathcal{C}\}$, together with a function $M \mapsto d_M : \mathcal{M} \to \{0, 1, 2, \ldots\}$.

Elements of $\mathcal{M}$ are called the **models** of $(\mathcal{M}, d.)$. For each $M \in \mathcal{M}$, d_M is called the **dimension** of M. We will adopt the notation:

For all integers $n \geq 0$, $\mathcal{M}_n := \{M \in \mathcal{M} \,|\, d_M = n\}$.

Example. For example, consider $\mathcal{C} = \{\text{topological spaces}\}$ with $\mathcal{M} = \{\Delta^0, \Delta^1, \ldots\}$, the set of simplices, and with $d.$ defined by $d_{\Delta^n} = n$. Then $(\mathcal{M}, d.)$ is a model system on $\mathcal{C}$. In this case, for all integers $n \geq 0$, we have $\mathcal{M}_n = \{\Delta^n\}$.

For any category $\mathcal{C}$, for any functor $T_\bullet : \mathcal{C} \to \{\text{chain complexes}\}$, for any $n \in \mathbb{Z}$, we will define $T_n : \mathcal{C} \to \{\text{additive Abelian groups}\}$ by: $T_n(C) = (T_\bullet(C))_n$.

Recall that a chain complex $\mathcal{C}_\bullet$ is said to be **nonnegative** if, for all integers $n < 0$, we have $C_n = 0$.

Definition. Let $\mathcal{C}$ be a category. Let $(\mathcal{M}, d.)$ be a model system on $\mathcal{C}$. Let

$$T_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be a functor. Let $\mathcal{C}_0 := \bigcup_{M \in \mathcal{M}} T_{d_M}(M)$. A $T_\bullet$ **model set** on $(\mathcal{M}, d.)$ is a function

$$M \mapsto e_M : \mathcal{M} \to \mathcal{C}_0$$

such that, for all $M \in \mathcal{M}$, we have $e_M \in T_{d_M}(M)$.

Example. For example, consider $\mathcal{C} = \{\text{topological spaces}\}$ with $\mathcal{M} = \{\Delta^0, \Delta^1, \ldots\}$, the set of simplices, and with $d.$ defined by $d_{\Delta^n} = n$. Consider $T_\bullet = S_\bullet$, the singular chain functor, which we recall is defined by $S_n(X) = \mathbb{Z}[C(\Delta^n, X)]$. For all integers $n \geq 0$, let

$$e_{\Delta^n} := \mathrm{id}_{\Delta^n} \in C(\Delta^n, \Delta^n) \subseteq \mathbb{Z}[C(\Delta^n, \Delta^n)] = S_n(\Delta^n).$$

Then $e.$ is a $T_\bullet$ model set on $(\mathcal{M}, d.)$.

Definition. Let $\mathcal{C}$ be a category. Let $(\mathcal{M}, d.)$ be a model system on $\mathcal{C}$. For all integers $n \geq 0$, let $\mathcal{M}_n := \{M \in \mathcal{M} \mid d_M = n\}$. Let

$$T_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be a functor. Let $e.$ be a $T_\bullet$ model set on $(\mathcal{M}, d.)$. We say that $e.$ is a **basis** for $T_\bullet$ on $(\mathcal{M}, d.)$ if, for all $X \in \mathcal{C}$, for all integers $n \geq 0$, we have that

$$\{(Tf)(e_M) \mid M \in \mathcal{M}_n, f \in \operatorname{Hom}(M, X)\}$$

is a basis for $T_n(X)$.

Definition. Let $\mathcal{C}$ be a category. Let $(\mathcal{M}, d.)$ be a model system on $\mathcal{C}$. Let

$$T_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be a functor. We say that $T_\bullet$ is **free** on $(\mathcal{M}, d.)$ if there exists a basis for $T_\bullet$ on $(\mathcal{M}, d.)$.

Note that, in the preceding example, $e.$ is a basis for $T_\bullet$ on $(\mathcal{M}, d.)$, so we see that $T_\bullet$ is free on $(\mathcal{M}, d.)$.

Definition. Let $\mathcal{C}$ be a category. Let $\mathcal{M}$ be a subset of the class $\{\text{objects in } \mathcal{C}\}$. Let

$$U_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be a functor. We say that $U_\bullet$ is **$\mathcal{M}$-acyclic** if, for all $M \in \mathcal{M}$, for all integers $n > 0$, we have $H_n(U_\bullet M) = 0$.

Example. Consider $\mathcal{C} = \{\text{topological spaces}\}$ with $\mathcal{M} = \{\Delta^0, \Delta^1, \ldots\}$, the set of simplices. Let $d.$ be defined by $d_{\Delta^n} = n$. Let $I := [0, 1]$. Let $U_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$ be defined by $U_\bullet(X) = S_\bullet(X \times I)$. Then $U_\bullet$ is $\mathcal{M}$-acyclic.

Definition. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories. Let $\mathcal{A} : \mathcal{C} \to \mathcal{D}$, $\mathcal{B} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{F} : \mathcal{D} \to \mathcal{E}$ be functors. Let $\tau : \mathcal{A} \to \mathcal{B}$ be a natural transformation. Then we denote by $\mathcal{F}\tau$ the natural transformation from $\mathcal{F}\mathcal{A} : \mathcal{C} \to \mathcal{E}$ to $\mathcal{F}\mathcal{B} : \mathcal{C} \to \mathcal{E}$ defined by $(\mathcal{F}\tau)_C = \mathcal{F}(\tau_C)$.

We may therefore write $\mathcal{F}\tau_C$ without fear of ambiguity.

Definition. Let $\mathcal{C}$ be a category. Let

$$T_\bullet, U_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be functors. Let $\tau, \mu : T_\bullet \to U_\bullet$ be natural transformations. A natural transformation $h : T_\bullet \to U_{\bullet+1}$ is said to be a **natural chain homotopy** from τ to μ, if, for all $X \in \mathcal{C}$, we have that $h_X : T_\bullet X \to U_{\bullet+1} X$ is a chain homotopy from $\tau_X : T_\bullet X \to U_\bullet X$ to $\mu_X : T_\bullet X \to U_\bullet X$. We say that τ and μ are **naturally chain homotopic** if there exists a natural chain homotopy from τ to μ.

Note that if τ and μ are naturally chain homotopic, then $H_\bullet \tau = H_\bullet \mu$.

Definition. Let $\mathcal{C}$ be a category. Let

$$T_\bullet, U_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be functors. Let $\iota_T : T_\bullet \to T_\bullet$ be the identity natural transformation. Let $\iota_U : U_\bullet \to U_\bullet$ be the identity natural transformation. We say that $T_\bullet$ and $U_\bullet$ are **naturally chain homotopy equivalent** if there are natural transformations $\tau : T_\bullet \to U_\bullet$ and $\tau' : U_\bullet \to T_\bullet$ such that

(1) $\tau'\tau$ is naturally chain homotopic to ι_T; and
(2) $\tau\tau'$ is naturally chain homotopic to ι_U.

If $T_\bullet$ and $U_\bullet$ are naturally chain homotopy equivalent, then $H_\bullet T_\bullet$ is equivalent to $H_\bullet U_\bullet$. This, in turn implies, for all $X \in \mathcal{C}$, that $(H_\bullet T_\bullet)(X) \cong (H_\bullet U_\bullet)(X)$.

Acyclic Models Theorem. Let $\mathcal{C}$ be a category. Let $(\mathcal{M}, d.)$ be a model system on $\mathcal{C}$. Let

$$T_\bullet, U_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be functors. Assume that $T_\bullet$ is free on $(\mathcal{M}, d.)$ and that $U_\bullet$ is acyclic on $\mathcal{M}$. Then:

(1) For any natural transformation $\phi : H_0 T_\bullet \to H_0 U_\bullet$, there exists a natural transformation $\tau : T_\bullet \to U_\bullet$ such that $H_0\tau = \phi$; and
(2) Let $\tau, \mu : T_\bullet \to U_\bullet$ be natural transformations and assume that $H_0\tau = H_0\mu$. Then τ and μ are naturally chain homotopic.

Corollary. Let $\mathcal{C}$ be a category. Let $(\mathcal{M}, d.)$ be a model system on $\mathcal{C}$. Let

$$T_\bullet, U_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be functors. Assume that $T_\bullet$ is free on $(\mathcal{M}, d.)$ and that $U_\bullet$ is acyclic on $\mathcal{M}$. Let $\tau, \mu : T_\bullet \to U_\bullet$ be natural transformations and assume that $H_0\tau = H_0\mu$. Then $H_\bullet\tau = H_\bullet\mu$.

Consequently, for all $X \in \mathcal{C}$, we have $H_\bullet(\tau_X) = H_\bullet(\mu_X)$.

Corollary. Let $\mathcal{C}$ be a category. Let $(\mathcal{M}, d.)$ be a model system on $\mathcal{C}$. Let

$$T_\bullet, U_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$$

be functors. Assume that $T_\bullet$ and $U_\bullet$ are free on $(\mathcal{M}, d.)$. Assume that $T_\bullet$ and $U_\bullet$ are acyclic on $\mathcal{M}$. Assume that $H_0 T_\bullet$ is equivalent to $H_0 U_\bullet$. Then $T_\bullet$ and $U_\bullet$ are naturally chain homotopy equivalent.

Consequently $H_\bullet T_\bullet$ is equivalent to $H_\bullet U_\bullet$. Consequently, for all $X \in \mathcal{C}$, we have $(H_\bullet T_\bullet)(X) \cong (H_\bullet U_\bullet)(X)$.

Proof: Let $\iota_T : T_\bullet \to T_\bullet$ and $\iota_U : U_\bullet \to U_\bullet$ denote the identity natural transformations. Let $\iota_T^H : H_0 T_\bullet \to H_0 T_\bullet$ and $\iota_U^H : H_0 U_\bullet \to H_0 U_\bullet$ denote the identity natural transformations.

Choose $\phi : H_0 T_\bullet \to H_0 U_\bullet$ and $\phi' : H_0 U_\bullet \to H_0 T_\bullet$ such that $\phi'\phi = \iota_T^H$ and such that $\phi\phi' = \iota_U^H$. By the existence part of the Acyclic Models Theorem, choose $\tau : T_\bullet \to U_\bullet$ and $\tau' : U_\bullet \to T_\bullet$ such that $H_0\tau = \phi$ and such that $H_0\mu = \phi'$.

We have $H_0(\tau'\tau) = \phi'\phi = \iota_T^H = H_0\iota_T$ and $H_0(\tau\tau') = \phi\phi' = \iota_U^H = H_0\iota_U$. So, by the uniqueness part of the Acyclic Models Theorem, we see that $\tau'\tau$ is naturally chain homotopic to ι_T and that $\tau\tau'$ is naturally chain homotopic to ι_U. **QED**

Application. For example, consider $\mathcal{C} = \{\text{topological spaces}\}$ with $\mathcal{M} = \{\Delta^0, \Delta^1, \ldots\}$, the set of simplices, and with $d.$ defined by $d_{\Delta^n} = n$. Let $I := [0,1]$.

Let $T_\bullet := S_\bullet$ be the singular chain functor, which is defined by $S_n(X) = \mathbb{Z}[C(\Delta^n, X)]$. Let $U_\bullet : \mathcal{C} \to \{\text{nonnegative chain complexes}\}$ be defined by $U_\bullet(X) = S_\bullet(X \times I)$.

Define $\tau : T_\bullet \to U_\bullet$ by $\tau_X = (\cdot, 0)_* : S_\bullet X \to S_\bullet(X \times I)$. Define $\mu : T_\bullet \to U_\bullet$ by $\mu_X = (\cdot, 1)_* : S_\bullet X \to S_\bullet(X \times I)$. By Exercise 18A below, $H_0\tau = H_0\mu$, so, by the first corollary to the Acyclic Models Theorem, we see that $H_\bullet\tau = H_\bullet\mu$.

EXERCISE 18A: Show, in the notation of the previous application, that $H_0\tau = H_0\mu$.

Corollary. Let X and Y be topological spaces and let $f, g : X \to Y$ be homotopic maps. Then $f_*, g_* : H_\bullet X \to H_\bullet Y$ are equal.

Proof: By the previous application, we see that $H_\bullet\tau_X = H_\bullet\mu_X$. Since

$$H_\bullet\tau_X = (\cdot, 0)_* : H_\bullet X \to H_\bullet(X \times I)$$

and

$$H_\bullet\mu_X = (\cdot, 1)_* : H_\bullet X \to H_\bullet(X \times I)$$

we conclude that $(\cdot, 0)_* : H_\bullet X \to H_\bullet(X \times I)$ and $(\cdot, 1)_* : H_\bullet X \to H_\bullet(X \times I)$ are equal.

Let $h : X \times I \to Y$ be a homotopy from f to g. Then $h \circ (\cdot, 0) = f$ and $h \circ (\cdot, 1) = g$. Then

$$h_* \circ (\cdot, 0)_* = f_* : H_\bullet X \to H_\bullet Y$$

and

$$h_* \circ (\cdot, 1)_* = g_* : H_\bullet X \to H_\bullet Y.$$

Then $f_* = h_* \circ (\cdot, 0)_* = h_* \circ (\cdot, 1)_* = g_*$. **QED**

This justifies our point of view that the Acyclic Models Theorem is "just" a generalization of the basic fact that two homotopic maps induce the same map on homology. Since we know this basic fact, we consider the Acyclic Models Theorem to be proved. For a full proof, see Vick's book "Homology" or Rotman's book on algebraic topology.

Our next application of Acyclic Models is the Eilenberg-Zilber Theorem, which we recall states:

Eilenberg-Zilber Theorem. Let X and Y be topological spaces. Then $H_\bullet(X \times Y)$ is naturally isomorphic to $H_\bullet((S_\bullet X) \otimes (S_\bullet Y))$.

Proof: Let $\mathcal{C}$ be the category of ordered pairs of topological spaces. Let

$$\mathcal{M} := \{(\Delta^k, \Delta^l) \,|\, k, l \in \{0, 1, 2, \ldots\}\}.$$

Define $d. : \mathcal{M} \to \{0, 1, 2 \ldots\}$ by $d_{(\Delta^k, \Delta^l)} = k + l$.

Recall that $S_\bullet$: {topological spaces} $\to$ {nonnegative chain complexes} denotes the singular chain complex functor. Define $T_\bullet, U_\bullet : \mathcal{C} \to$ {nonnegative chain complexes} by $T_\bullet(X,Y) = S_\bullet(X \times Y)$ and $U_\bullet(X,Y) = (S_\bullet(X)) \otimes (S_\bullet(Y))$. It suffices to show that $T_\bullet$ and $U_\bullet$ are naturally chain homotopy equivalent.

EXERCISE 18B: Show that $H_0T_\bullet, H_0U_\bullet : \mathcal{C} \to$ {additive Abelian groups} are equivalent.

EXERCISE 18C: Show that $T_\bullet$ and $U_\bullet$ are free on $(\mathcal{M}, d.)$.

EXERCISE 18D: Show that $T_\bullet$ and $U_\bullet$ are acyclic on $\mathcal{M}$.

Then, by the second corollary following the statement of the Acyclic Models Theorem, we conclude that $T_\bullet$ and $U_\bullet$ are naturally chain homotopy equivalent. **QED**

We now begin a new topic: The ring structure in cohomology.

Recall that a **ring** is an additive Abelian group R together with a $\mathbb{Z}$-bilinear "multiplication map" $(a,b) \mapsto ab : R \times R \to R$ such that

(1) $\forall a,b,c \in R$, $a(b+c) = ab+ac$, $(a+b)c = ac+bc$ and $a(bc) = (ab)c$; and

(2) there exists $1 \in R$ such that, for all $a \in R$, we have $1a = a1 = a$.

The ring R is **commutative** if, for all $a, b \in R$, we have $ab = ba$.

Definition. A **pre-graded ring** is a graded group $R_\bullet$ together with a "multiplication map" $\mu : R_\bullet \otimes R_\bullet \to R_\bullet$. By convention, for all $p, q \in \mathbb{Z}$, for all $a \in R_p$, $b \in R_q$ we denote $\mu(a \otimes b)$ by ab. A pre-graded ring $R_\bullet$ is said to be a **graded ring** if

(1) for all $p,q,r \in \mathbb{Z}$, for all $a \in R_p$, $b \in R_q$, $c \in R_r$, we have $a(b+c) = ab+ac$, $(a+b)c = ac+bc$ and $a(bc) = (ab)c$; and

(2) there exists $1 \in R_0$ such that, for all $p \in \mathbb{Z}$, for all $a \in R_p$, we have $1a = a1 = a$.

The graded ring $R_\bullet$ is said to be **graded commutative** if, for all $p, q \in \mathbb{Z}$, for all $a \in R_p$, for all $b \in R_q$, we have $ab = (-1)^{pq}ba$.

A graded commutative graded ring will simply be called a **graded commutative ring**. Note that there is a forgetful functor

$$\mathcal{F} : \{\text{graded commutative rings}\} \to \{\text{graded groups}\},$$

in which one simply forgets the multiplication map.

Let R be a commutative ring whose underlying additive Abelian group is denoted A. (That is, A is equal to: R, after forgetting about multiplication.) Our next goal is to construct a functor

$$H^\bullet(\cdot; R) : \{\text{topological spaces}\} \to \{\text{graded commutative rings}\}$$

such that $\mathcal{F} \circ (H^\bullet(\cdot; R)) = H^\bullet(\cdot; A) : \{\text{topological spaces}\} \to \{\text{graded groups}\}$.

We will also indicate the existence of topological spaces X and Y such that, with $R = \mathbb{Z}$, we have

$$H^\bullet(X; R) \cong H^\bullet(Y; R), \qquad \text{but} \qquad H^\bullet(X; A) \not\cong H^\bullet(Y; A).$$

This shows that, for the case $R = \mathbb{Z}$, the functor $H^\bullet(\cdot; R)$ is better at distinguishing topological spaces than is the functor $H^\bullet(\cdot; A)$. This is true for other rings R, as well.

Finally, we note that we could set up our definitions so that a slightly more sophisticated version of $H(\cdot; R)$ takes values in the category of "graded commutative R-algebras". We do not wish to define this term, since the definitions are already somewhat complicated. However, we do comment that there is a forgetful functor

$$\{\text{graded commutative } R\text{-algebras}\} \to \{\text{graded commutative rings}\}$$

and our $H(\cdot; R)$ is simply obtained by composing this more sophisticated $H(\cdot; R)$ with this forgetful functor. Note that it might be that this more sophisticated $H(\cdot; R)$ is better at distinguishing topological spaces that our $H(\cdot; R)$ is.

Note that, given a topological space X, the meaning of $H^\bullet(X) := H^\bullet(X; \mathbb{Z})$ is now ambiguous, depending on whether we think of $\mathbb{Z}$ as a ring (in which case $H^\bullet(X)$ is a graded commutative ring), or as a group (in which case $H^\bullet(X)$ is a graded group). Generally, the reader must simply figure this out from context. The same comment applies with $\mathbb{Z}$ replaced by $\mathbb{Z}/2$ or with many other commutative rings, since the underlying additive Abelian group is usually written with the exact same notation as the ring itself.

With those general comments behind us, we fix a topological space X and a commutative ring R, and proceed to define the commutative graded ring $H(X; R)$. The (contravariant) functoriality of $H(\bullet; R)$ will be left as an unassigned exercise for the reader.

Fix $p, q \in \mathbb{Z}$. Define $f : \Delta^p \to \Delta^{p+q}$ by $f(t_0, \ldots, t_p) = (t_0, \ldots, t_p, 0, \ldots, 0)$ and define $b_p : \Delta^q \to \Delta^{p+q}$ by $b(t_0, \ldots, t_q) = (0, \ldots, 0, t_0, \ldots, t_q)$. For all $\sigma \in C(\Delta^{p+q}, X)$, we define $\sigma_f := \sigma \circ f \in C(\Delta^p, X)$ and $\sigma_b := \sigma \circ b \in C(\Delta^q, X)$. For all $\alpha \in S^p(X; R)$, for all $\beta \in S^q(X; R)$, define $\alpha \cup \beta \in S^{p+q}(X; R)$ to be the $\mathbb{Z}$-linear extension to $S_{p+q}(X)$ of the map $C(\Delta^{p+q}, X) \quad \to \quad R$ given by

$$\sigma \mapsto (\alpha(\sigma_f))(\beta(\sigma_b)).$$

Note that, in this formula, we use the multiplication in R. That is, R must be a ring, or this would not make sense. Note also that

$$(\alpha, \beta) \quad \mapsto \quad \alpha \cup \beta \quad : \quad (S^p(X; R)) \times (S^q(X; R)) \quad \to \quad S^{p+q}(X; R)$$

is R-bilinear.

We leave it as an unassigned exercise to show, for all $p, q \in \mathbb{Z}$, for all $\alpha \in S^p(X; R)$, $\beta \in S^q(X; R)$, we have $d(\alpha \cup \beta) = [(d\alpha) \cup \beta] + [(-1)^p][\alpha \cup (d\beta)]$. Using this, it is an easy (unassigned) exercise to show, for all $p, q \in \mathbb{Z}$, for all $\alpha \in Z^p(X; R)$, $\beta \in Z^q(X; R)$, that $\alpha \cup \beta \in Z^{p+q}(X; R)$.

Fix $p, q \in \mathbb{Z}$ and let $\alpha' \in H^p(X; R)$ and let $\beta' \in H^q(X; R)$. We define $\alpha' \cup \beta' \in H^{p+q}(X; R)$, as follows: Let

$$\begin{aligned} a &: Z^p(X; R) \to H^p(X; R), \\ b &: Z^q(X; R) \to H^q(X; R), \\ c &: Z^{p+q}(X; R) \to H^{p+q}(X; R) \end{aligned}$$

be the canonical maps. Choose $\alpha \in Z^p(X;R)$, $\beta \in Z^q(X;R)$ such that $a(\alpha) = \alpha'$ and $b(\beta) = \beta'$. We define $\alpha' \cup \beta' := c(\alpha \cup \beta)$. We leave it as an unassigned exercise show that this map is well-defined. (*Hint:* Use the above-mentioned R-bilinearity together with $d(\alpha \cup \beta) = [(d\alpha) \cup \beta] + [(-1)^p][\alpha \cup (d\beta)]$.)

We leave it as an unassigned exercise to show, for any topological space X, for any commutative ring R, that $(H^\bullet(X;R), \cup)$ is a commutative graded ring.

Example. $H^\bullet(\mathbb{R}P^n;\mathbb{Z}/2)$ is isomorphic to the commutative graded ring $\dfrac{(\mathbb{Z}/2)[x]}{(x^{n+1})}$

A map $f : S^3 \to S^3$ will be said to be **odd** if, for all $x \in S^3$, we have $f(-x) = -(f(x))$.

Lemma. Let $\psi : S^3 \to S^3$ be odd. Then ψ is not homotopic to a constant map.

Proof: Let $p : S^3 \to \mathbb{R}P^3$ be the canonical map. Because $\psi : S^3 \to S^3$ is odd, it follows that there is a continuous map $\chi : \mathbb{R}P^3 \to \mathbb{R}P^3$ such that $\chi \circ p = p \circ \psi$.

EXERCISE 19B: Let $s \in \mathbb{R}P^3$ and let $t := \chi(s)$. Show that

$$\chi_* \quad : \quad \pi_1(\mathbb{R}P^3, s) \quad \to \quad \pi_1(\mathbb{R}P^3, t)$$

is nontrivial.

EXERCISE 19C: Show that $p_* : H_3(S^3) \to H_3(\mathbb{R}P^3)$ is nontrivial. (Note: You may assume, without proof, the fact that the antipodal map $A : S^n \to S^n$ induces a map $A_* : H_n(S^n) \to H_n(S^n)$ such that, for all $z \in H_n(S^n)$, we have $A_*(z) = (-1)^{n+1}z$.)

Since $H_3(S^3) \cong \mathbb{Z} \cong H_3(\mathbb{R}P^3)$, it follows, from Exercise 19C, that the homomorphism $p_* : H_3(S^3) \to H_3(\mathbb{R}P^3)$ is injective.

Recall that $\pi_1(\mathbb{R}P^3, s) \cong \mathbb{Z}/2 \cong \pi_1(\mathbb{R}P^3, t)$. Then, by Exercise 19B and the Hurewicz Theorem, we see that $\chi_* : H_1(\mathbb{R}P^3) \to H_1(\mathbb{R}P^3)$ is nontivial. Then, as $H_1(\mathbb{R}P^3) \cong \mathbb{Z}/2$, we conclude that $\chi_* : H_1(\mathbb{R}P^3) \to H_1(\mathbb{R}P^3)$ is the identity map.

We have $\mathrm{Ext}(H^0(\mathbb{R}P^3), \mathbb{Z}/2) = \{0\}$. Then, by the Universal Coefficients Theorem, $H^1(\mathbb{R}P^3;\mathbb{Z}/2) \cong \mathrm{Hom}(H_1(\mathbb{R}P^3), \mathbb{Z}/2) \cong \mathbb{Z}/2$. Also, by naturality in the Universal Coefficients Theorem, the map $\chi^* : H^1(\mathbb{R}P^3;\mathbb{Z}/2) \to H^1(\mathbb{R}P^3;\mathbb{Z}/2)$ is obtained by applying the functor $\mathrm{Hom}(\cdot, \mathbb{Z}/2)$ to the identity map

$$\chi_* : H_1(\mathbb{R}P^3) \to H_1(\mathbb{R}P^3).$$

Then $\chi^* : H^1(\mathbb{R}P^3;\mathbb{Z}/2) \to H^1(\mathbb{R}P^3;\mathbb{Z}/2)$ is the identity map.

Let u be the nontrivial element of $H^1(\mathbb{R}P^3;\mathbb{Z}/2)$. Then $\chi^*(u) = u$. Let

$$v := u \cup u \cup u \in H^3(\mathbb{R}P^3;\mathbb{Z}/2).$$

Then, by the graded ring structure on $H^\bullet(\mathbb{R}P^3;\mathbb{Z}/2)$, described in the preceding Example, we see that $v \neq 0$. Moreover, as $\chi^* : H^\bullet(\mathbb{R}P^3;\mathbb{Z}/2) \to H^\bullet(\mathbb{R}P^3;\mathbb{Z}/2)$ is a homomorphism of graded commutative rings, and as $\chi^*(u) = u$, we see that $\chi^*(v) = v$. Then

$$\chi^* : H^3(\mathbb{R}P^3;\mathbb{Z}/2) \to H^3(\mathbb{R}P^3;\mathbb{Z}/2)$$

is nontrivial.

We have $H^2(\mathbb{R}P^3) = \{0\}$, so $\mathrm{Ext}(H^2(\mathbb{R}P^3), \mathbb{Z}/2) = \{0\}$. Thus, another application of naturality in the Universal Coefficients shows that

$$\chi^* : H^3(\mathbb{R}P^3; \mathbb{Z}/2) \to H^3(\mathbb{R}P^3; \mathbb{Z}/2)$$

is obtained by applying $\mathrm{Hom}(\cdot, \mathbb{Z}/2)$ to

$$\chi_* : H_3(\mathbb{R}P^3) \to H_3(\mathbb{R}P^3).$$

Then $\chi_* : H_3(\mathbb{R}P^3) \to H_3(\mathbb{R}P^3)$ is nontrivial. So, as $H_3(\mathbb{R}P^3) \cong \mathbb{Z}$, we conclude that $\chi_* : H_3(\mathbb{R}P^3) \to H_3(\mathbb{R}P^3)$ is injective.

Since $p_* : H_3(S^3) \to H_3(\mathbb{R}P^3)$ and $\chi_* : H_3(\mathbb{R}P^3) \to H_3(\mathbb{R}P^3)$ are both injective, it follows that $(\chi \circ p)_* : H_3(S^3) \to H_3(\mathbb{R}P^3)$ is injective. Then, as $\chi \circ p = p \circ \psi$, we conclude that $p_* \circ \psi_* : H_3(S^3) \to H_3(\mathbb{R}P^3)$ is injective, so $\psi_* : H_3(S^3) \to H_3(S^3)$ is injective, and therefore nonzero. Then $\psi : S^3 \to S^3$ cannot be homotopic to a constant map. **QED**

We now use the above example and lemma to prove the

Ham Sandwich Theorem. Let λ denote Lebesgue measure in $\mathbb{R}^3$. Let A, B and C be measurable subsets of $\mathbb{R}^3$, all of which have finite Lebesgue measure. Then there exists a half-space $H \subseteq \mathbb{R}^3$ such that

$$\lambda(A \cap H) = \lambda(A \backslash H), \qquad \lambda(B \cap H) = \lambda(B \backslash H) \qquad \text{and} \qquad \lambda(C \cap H) = \lambda(C \backslash H).$$

NOTE: This result implies that, given a ham sandwich with a piece of wheat bread, a piece of ham and a piece of rye bread, one can, with a single stroke of the knife, bisect each of the three parts of the sandwich.

Proof: Define $\rho : \{\text{subsets of } \mathbb{R}^3\} \to \mathbb{R}^3$ by

$$\rho(H) = ((\lambda(A \cap H)) - (\lambda(A \backslash H)), (\lambda(B \cap H)) - (\lambda(B \backslash H)), (\lambda(C \cap H)) - (\lambda(C \backslash H))).$$

We wish to show, for some half-space H of $\mathbb{R}^3$, that $\rho(H) = 0$.

Define $i : \mathbb{R}^3 \to \mathbb{R}^4$ by $i(w) = (w, 1)$. Let $\cdot$ denote the dot product in $\mathbb{R}^4$. Define $\chi : S^3 \to \{\text{subsets of } \mathbb{R}^3\}$ by $\chi(v) = \{w \in \mathbb{R}^3 \mid v \cdot (i(w)) \geq 0\}$. Let $\phi := \rho \circ \chi : S^3 \to \mathbb{R}^3$. Then ϕ is continuous. We wish to show, for some $v \in S^3$, that $\phi(v) = 0$.

EXERCISE 19A: Show, for all $v \in S^3$, that $\phi(-v) = -(\phi(v))$.

By Exercise 19A, $\phi : S^3 \to \mathbb{R}^3$ is odd. It is an unassigned exercise to show that $\phi : S^3 \to \mathbb{R}^3$ is continuous. The remainder of the proof shows that the image of any odd continuous function $S^3 \to \mathbb{R}^3$ must contain the origin $0 := (0, 0, 0)$. Assume, for all $v \in S^3$, that $\phi(v) \neq 0$. We aim for a contradiction.

Define $\psi : S^3 \to S^3$ by

$$\psi(v) = \left(\frac{\phi(v)}{\|\phi(v)\|} \quad , \quad 0 \right).$$

Then ψ is continuous. Also, for all $v \in S^3$, we have $\psi(-v) = -(\psi(v))$, *i.e.*, ψ is odd. Moreover,

$$\psi(S^3) \quad \subseteq \quad S^2 \times \{0\} \quad \subsetneq \quad S^3,$$

so $\psi : S^3 \to S^3$ is not surjective. It follows that $\psi : S^3 \to S^3$ is homotopic to a constant map.

However the preceding lemma asserts that any odd continuous function $S^3 \to S^3$ cannot be homotopic to a constant map. **QED**

Our next "application" of algebraic topology is the

Brouwer Fixed-Point Theorem. Let $n \geq 1$ be an integer and let D denote the closed unit ball in $\mathbb{R}^n$. Let $f : D \to D$ be continuous. Then there exists $x \in D$ such that $f(x) = x$.

Proof: The case $n = 1$ is an exercise in calculus, so we assume that $n \geq 2$. Suppose the result is false. We aim for a contradiction.

For all $x \in X$, let $L_x := \{(1-t)(f(x)) + tx \,|\, t \in \mathbb{R}\}$ be the line through $f(x)$ and x. Then $R_x := \{(1-t)(f(x)) + tx \,|\, t > 0\}$ is that connected component of $L_x \backslash \{f(x)\}$ containing x. We leave it as an unassigned exercise in geometry to show, for all $a, b \in D$, that, if L denotes the line through a and b, then the connected component R of $L\backslash\{a\}$ which contains b has the property that $R \cap S^{n-1}$ contains exactly one point. For all $x \in X$, let $g(x)$ be the unique element of $R_x \cap S^{n-1}$. Then $g : D \to S^{n-1}$ is continuous and, for all $x \in S^{n-1}$, we have $g(x) = x$.

Let $i : S^{n-1} \to D$ be the inclusion map. Let $\mathrm{id} : S^{n-1} \to S^{n-1}$ be the identity map. Then $g \circ i = \mathrm{id}$. Then the composition of $i_* : H_{n-1}(S^{n-1}) \to H_{n-1}(D)$ with $g_* : H_{n-1}(D) \to H_{n-1}(S^{n-1})$ is the identity map $\mathrm{id}_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$. On the other hand, D is contractible, so $H_{n-1}(D) = \{0\}$, so this composite must be trivial. However, $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, so the map $\mathrm{id}_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ cannot be both the identity map and trival at the same time, so we have a contradiction. **QED**

When R is a ring, we have seen that $H^\bullet(X; R)$ has the structure of a graded commutative ring. We now explain how, if R is a ring and M is an R-module, then $H^\bullet(X; M)$ has the structure of a graded $(H^\bullet(X; R))$-module. To do so, we must, for all $p, q \in \mathbb{Z}$, define a multiplication map

$$H^p(X; R) \times H^q(X; M) \to H^{p+q}(X; M).$$

Let X be a topological space, let R be a ring and let M be an R-module. Let $p, q \in \mathbb{Z}$. Define $f : \Delta^p \to \Delta^{p+q}$ by $f(t_0, \ldots, t_p) = (t_0, \ldots, t_p, 0, \ldots, 0)$ and define $b_p : \Delta^q \to \Delta^{p+q}$ by $b(t_0, \ldots, t_q) = (0, \ldots, 0, t_0, \ldots, t_q)$. For all $\sigma \in C(\Delta^{p+q}, X)$, we define $\sigma_f := \sigma \circ f \in C(\Delta^p, X)$ and $\sigma_b := \sigma \circ b \in C(\Delta^q, X)$.

Given $\alpha \in S^p(X; R)$ and $\beta \in S^q(X; M)$, define $\alpha \cup \beta : S_{p+q}(X) \to R$ by: for all $\sigma \in C(\Delta^{p+q}, X)$, $(\alpha \cup \beta)(\sigma) = (\alpha(\sigma_f))(\beta(\sigma_b)) \in RM \subseteq M$. It is an unassigned exercise to show that this map

$$(\alpha, \beta) \mapsto \alpha \cup \beta : S^p(X; R) \times S^q(X; M) \to S^{p+q}(X; M)$$

restricts to a map

$$(\alpha, \beta) \mapsto \alpha \cup \beta : Z^p(X; R) \times Z^q(X; M) \to Z^{p+q}(X; M)$$

and then factors to a map

$$(\alpha, \beta) \mapsto \alpha \cup \beta : H^p(X; R) \times H^q(X; M) \to H^{p+q}(X; M).$$

Here, $\alpha \cup \beta$ is called the **cup product** of α and β. In the above, two important cases are:
(1) R any ring and $M = R$; and
(2) $R = \mathbb{Z}$ and M any additive Abelian group.
This concludes our discussion of "cup products" Next, we define the "cap" product.

Again, let X be a topological space, let R be a ring and let M be an R-module. Let $p, q \in \mathbb{Z}$. We will define a bilinear map

$$(a, \beta) \mapsto a \cap \beta : H_{p+q}(X; R) \times H^q(X; M) \to H_p(X; M).$$

As before, define $f : \Delta^p \to \Delta^{p+q}$ by $f(t_0, \ldots, t_p) = (t_0, \ldots, t_p, 0, \ldots, 0)$ and define $b_p : \Delta^q \to \Delta^{p+q}$ by $b(t_0, \ldots, t_q) = (0, \ldots, 0, t_0, \ldots, t_q)$. As before, for all $\sigma \in C(\Delta^{p+q}, X)$, we define $\sigma_f := \sigma \circ f \in C(\Delta^p, X)$ and $\sigma_b := \sigma \circ b \in C(\Delta^q, X)$.

For all $\beta \in S^q(X; M)$, define $\Phi_\beta : S_{p+q}(X; R) \to S_p(X; M)$ to be the R-linear extension of

$$\sigma \mapsto (\beta(\sigma_b))\sigma_f : C(\Delta^{p+q}, X) \to S_p(X; M).$$

Then, for all $a \in S_{p+q}(X; R)$, for all $\beta \in S^q(X; M)$, define $a \cap \beta = \Phi_\beta(a)$. It is an unassigned exercise to show that this map

$$(a, \beta) \mapsto a \cap \beta : S_{p+q}(X; R) \times S^q(X; M) \to S_p(X; M)$$

restricts to a map

$$(a, \beta) \mapsto a \cap \beta : Z_{p+q}(X; R) \times Z^q(X; M) \to Z_p(X; M)$$

and then factors to a map

$$(a, \beta) \mapsto a \cap \beta : H^p(X; R) \times H^q(X; M) \to H_p(X; M).$$

Here, $a \cap \beta$ is called the **cap product** of a and β.

This concludes our discussion of cap products. We now define the Kronecker pairing.

Let X be a topological space and let $q \in \mathbb{Z}$. Recall that $S^q(X) = \mathrm{Hom}(S_q(X), \mathbb{Z})$. Define $\Phi_0 : S^q(X) \times S_q(X) \to \mathbb{Z}$ by $\Phi_0(f, s) = f(s)$. Let $\Phi_1 : Z^q(X) \times Z_q(X) \to \mathbb{Z}$ be the restriction of Φ_0 to $Z^q(X) \times Z_q(X)$. Let $\alpha : Z^q(X) \to H^q(X)$ and $\beta : Z_q(X) \to H_q(X)$ be the canonical maps. Define $\Phi : H^q(X) \times H_q(X) \to \mathbb{Z}$ by $\Phi(\alpha(a), \beta(b)) = \Phi_1(a, b)$; we leave it as an unassigned exercise to show that this is well-defined. This map $\Phi : H^q(X) \times H_q(X) \to \mathbb{Z}$ is called the **Kronecker pairing** determined by X.

We leave it as an unassigned exercise to verify: If $p, q, r \in \mathbb{Z}$, if $\alpha \in H^p(X)$, if $\beta \in H^q(X)$ if $c \in H_r(X)$ and if $p+q=r$, then $\Phi(\alpha \cup \beta, c) = \Phi(\alpha, c \cap \beta)$. That is, "cupping with β is adjoint to capping with β, with respect to the Kronecker pairing."

We now start on orientation.

Definition. Let M be a connected d-dimensional topological manifold and let S be a subset of M. We say that M is **direct orientable** along S if $H_d(M, M\backslash S)$ is isomorphic to the additive group $\mathbb{Z}$. A **direct orientation** of M along S is an isomorphism $H_d(M, M\backslash S) \to \mathbb{Z}$.

EXERCISE 20A: Let $d \geq 1$ be an integer. Let M be a connected d-dimensional topological manifold and let $m \in M$. Show that M is direct orientable along $\{m\}$.

EXERCISE 20B: Let $d \geq 1$ be an integer. Let M be a connected d-dimensional topological manifold and let $m \in M$. Show that there is a neighborhood V of m in M such that M is direct orientable along V.

Note that, if M is a topological manifold and if $m \in M$, then there exist exactly two direct orientations of M along $\{m\}$.

Definition. Let M be a topological manifold. For each $m \in M$, let Σ_m be the set of direct orientations of M along $\{m\}$. A **preorientation** of M is a map $\tau : M \to \bigcup_{m \in M} \Sigma_m$ such that, for all $m \in M$, we have $\tau(m) \in \Sigma_m$.

Definition. Let M be a topological manifold and let τ be a preorientation of M. Let $d := \dim(M)$ and let $V \subseteq M$. For all $v \in V$, let $\phi_v : H_d(M, M\backslash V) \to H_d(M, M\backslash\{v\})$ be the map induced by the inclusion $M\backslash V \subseteq M\backslash\{v\}$. We say that τ is **coherent** along V if there is a direct orientation $\mu : H_d(M, M\backslash V) \to \mathbb{Z}$ of M along V such that, for all $v \in V$, we have $(\tau(v)) \circ \phi_v = \mu$.

Definition. Let M be a topological manifold and let τ be a preorientation of M. We say that τ is an **orientation** of M if, for all $m \in M$, there is an open neighborhood V of m in M such that τ is coherent along V.

We will assume the next result without proof.

Theorem. Let M be a manifold and let A be an Abelian group. Let $d := \dim(M)$. Then, for all integers $k > d$, we have $H_k(M; A) = 0$. Moreover, if M is noncompact, then we have $H_d(M; A) = 0$.

EXERCISE 20C: Let M be a d-dimensional topological manifold and let $m \in M$. Assume that $H_d(M) \neq \{0\}$. Let $\phi : H_d(M) \to H_d(M, M\backslash\{m\})$ be the map induced by the inclusion $\emptyset \subseteq M\backslash\{m\}$. Show that $\phi : H_d(M) \to H_d(M, M\backslash\{m\})$ is an isomorphism. (*Hint:* Using long exact sequences and the preceding theorem, show that $\phi : H_d(M) \to H_d(M, M\backslash\{m\})$ is injective. Then, using Exercise 20A, show that $H_d(M)$ is isomorphic to $\mathbb{Z}$. Note that, if $\psi : \mathbb{Z} \to \mathbb{Z}$ is not surjective, then, for some finite cyclic group A, we have $\psi \otimes \mathrm{id}_A : \mathbb{Z} \otimes A \to \mathbb{Z} \otimes A$ is not injective. It therefore suffices to show, for all finite cyclic groups A, that $\phi \otimes \mathrm{id}_A : (H_d(M)) \otimes A \to (H_d(M, M\backslash\{m\})) \otimes A$ is injective. For any additive Abelian group A, let $\phi_A : H_d(M; A) \to H_d(M, M\backslash\{m\}; A)$ be the map induced by the inclusion $\emptyset \subseteq M\backslash\{m\}$. Using long exact sequences and the preceding theorem, show that $\phi_A : H_d(M; A) \to H_d(M, M\backslash\{m\}; A)$ is injective. Conclude, by naturality in the Universal Coefficients Theorem, that $\phi \otimes \mathrm{id}_A : (H_d(M)) \otimes A \to (H_d(M, M\backslash\{m\})) \otimes A$ is injective.)

We will assume the next result without proof.

Theorem. Let M be a compact, connected topological manifold. Then any orientation on M is coherent along M.

Corollary. Let M be a compact, connected, d-dimensional topological manifold. Then the following are equivalent:

(A) M is orientable;
(B) $H_d(M) \neq \{0\}$;
(C) $H_d(M)$ is isomorphic to $\mathbb{Z}$; and
(D) M is direct orientable along M.

Proof: By Exercise 20A, we see, for all $m \in M$, that $H_d(M, M\backslash\{m\}) \cong \mathbb{Z}$. For all $m \in M$, let $\phi_m : H_d(M) \to H_d(M, M\backslash\{m\})$ be the map induced by the inclusion $\emptyset \subseteq M\backslash\{m\}$.

The equivalence of (B) and (C) follows from Exercise 20C. The equivalence of (C) and (D) follows from the definition of "direct orientable".

If there is a orientation on M, then, by the theorem, it is coherent on M, and it follows from the definition of "coherent" that, for all $m \in M$, the map

$$\phi_m : H_d(M) \to H_d(M, M\backslash\{m\})$$

is an isomorphism. Thus (A) implies (C).

It remains to show that (C) implies (A). Let $\rho : H_d(M) \to \mathbb{Z}$ be an isomorphism. By Exercise 20C, we see, for all $m \in M$, that $\phi_m : H_d(M) \to H_d(M, M\backslash\{m\})$ is an isomorphism. Define an orientation τ by: for all $m \in M$, $\tau(m) := \rho \circ \phi_m^{-1} : H_d(M, M\backslash\{m\}) \to \mathbb{Z}$. Then τ is coherent along M, and so τ is an orientation. **QED**

Definition. Let M be a compact, connected, orientable, d-dimensional manifold. A **fundamental class** on M is a generator of $H_d(M)$.

We will assume the next result without proof.

Poincare Duality. Let M be a compact, connected, orientable, d-dimensional topological manifold. Let A be an additive Abelian group. Let z be a fundamental class on M. Let $q \in \mathbb{Z}$. Then the map $\omega \mapsto z \cap \omega : H^q(M; A) \to H_{d-q}(M; A)$ is an isomorphism.

The basic intuition behind Poincare Duality is as follows. Suppose M is a compact, connected, orientable, d-dimensional topological manifold. Let $q \in \mathbb{Z}$. Let $a \in H_q(M)$ and let $b \in H_{d-q}(M)$. At some intuitive level, because a and b have complementary dimensions, it should be true that there is an intersection of a with b that is finite in size. To compute the "size" n of this intersection, proceed as follows:

(1) Choose almost any q-cycle a_0 in a and almost any $(d-q)$-cycle b_0 in b; then
(2) take the union $A \subseteq M$ of the images of the parametric simplices in a_0; then
(3) take the union $B \subseteq M$ of the parametric simplices in b_0; then
(4) develop some careful bookkeeping system to tell the number n_p of times any point of $p \in A \cap B$ is covered by a simplex in a_0 and one in b_0; and then
(5) let $n := \sum n_p$.

Of course, even if we could do the careful bookkeeping required, we would need to show that n is independent of the choice of a_0 and b_0, so, to accomplish all this is a daunting program.

On the other hand, with Poincare Duality and the Kronecker pairing, we can obtain this number n more easily: Let $\Phi : H^\bullet(M) \times H_\bullet(M) \to \mathbb{Z}$ be the Kronecker pairing. Let $D : H^\bullet(M) \to H_{d-\bullet}(M)$ be the isomorphism defined by $D(\omega) = z \cap \omega$. Then we define the **intersection number** of a and b (with respect to z) to be $\Phi(D^{-1}(a), b)$. We will show below that this is not commutative in a and b, although its absolute value is. That is, we have $\Phi(D^{-1}(a), b) = \pm[\Phi(D^{-1}(b), a)]$. The absolute value of this number may be thought of, intuitively, as the "size" of the intersection of a with b.

For $p, q \in \mathbb{Z}$, given $a \in H_p(M)$ and $b \in H_q(M)$ we define

$$a \sqcap b := D((D^{-1}a) \cup (D^{-1}b)) \in H_{p+q-d}(M),$$

and thus give a sense of what it means to "intersect" a pair of homology classes. This kind of intersection theory is perhaps the primary motivation for Poincare Duality. Note that, because $H^\bullet(M)$ is a graded commutative ring, it follows that

$$a \sqcap b \quad := \quad (-1)^{(d-p)(d-q)}\ (b \sqcap a).$$

Let z is a fundamental class on a d-dimensional compact, orientable manifold M. Define an isomorphism $\Psi : H_0(M) \to \mathbb{Z}$ by $\Psi(c) = \Phi(D^{-1}(c), z)$. Let $p, q \in \mathbb{Z}$ and assume that $p+q = d$. Let $a \in H_p(M)$ and $b \in H_q(M)$. Then the intersection number of a and b is equal to $\Psi(a \sqcap b)$, because $\Phi(D^{-1}a, b) = \Phi(D^{-1}a, z \cap (D^{-1}b)) = \Phi((D^{-1}a) \cup (D^{-1}b), z) = \Phi(D^{-1}(a \sqcap b), z) = \Psi(a \sqcap b)$. Note that, as $a \sqcap b := (-1)^{(d-p)(d-q)}\ (b \sqcap a)$, this implies that

$$\Phi(D^{-1}a, b) \quad = \quad (-1)^{(d-p)(d-q)}\ [\Phi(D^{-1}b, a)].$$

That is, the intersection number only commutes up to sign.

We now go carefully through an example.

Let $M := \mathbb{R}^2/\mathbb{Z}^2$, the 2-torus. Then $d := \dim(M) = 2$. Let $p := 1$ and $q := 1$. Let $f : \mathbb{R}^2 \to M$ be the canonical map. Let $u := (0,0)$, $v := (0,1)$, $w := (1,1)$ and $x := (1,0)$.

Recall, for any $r, s, t \in \mathbb{R}^2$, that $[rst]$ denotes the affine parametric 2-simplex $\Delta^2 \to \mathbb{R}^2$ which maps e_0^2 to r, e_1^2 to s and e_2^2 to t. Similary $[rs] : \Delta^1 \to \mathbb{R}^2$ is the unique affine map sending e_0^1 to r and e_1^1 to s.

Let $\widehat{z} := f_*([uvw] - [uxw]) \in Z_2(M)$. Let z denote the image of $\widehat{z}$ under the canonical map $Z_2(M) \to H_2(M)$. Then z is a fundamental class of M.

Let $k := (u+v)/2$, $l := (v+w)/2$, $m := (w+x)/2$ and $n := (u+x)/2$. Let $\widehat{a} := f_*[nl] \in Z_1(M)$ and $\widehat{b} := f_*[km] \in Z_1(M)$.

We now posit the existence of $\widehat{\alpha} \in Z^1(M)$ such that, intuitively, for any parametric 1-simplex σ, $\widehat{\alpha}(\sigma)$ counts the number of times that σ crosses $\widehat{a}$ from left to right (after lifting from M into $[0,1] \times [0,1]$). All we will use is that $\widehat{\alpha}(\widehat{b}) = 1$, that $\widehat{\alpha}(f_*[vw]) = 1$ and that $\widehat{\alpha}(f_*[xw]) = 0$. We ask our reader to accept the existence of such an element $\widehat{\alpha} \in Z^1(M)$. Let α denote the image of $\widehat{\alpha}$ under the canonical map $Z^1(M) \to H^1(M)$.

We now see, from the definition of cap product, that

$$\widehat{z} \cap \widehat{\alpha} \quad = \quad (\widehat{\alpha}(f_*[uvw]_b))\ (f_*[uvw]_f)\ -\ (\widehat{\alpha}(f_*[uxw]_b))\ (f_*[uxw]_f).$$

From the definition of front and back faces, we have $[uvw]_b = [vw]$, $[uvw]_f = [uv]$, $[uxw]_b = [xw]$ and $[uxw]_f = [ux]$. Substituting, and using the assumed properties of $\widehat{\alpha}$, we get $\widehat{z} \cap \widehat{\alpha} = f_*[uv]$. Since $f_*[uv]$ is homologous to $\widehat{a}$, if we take the image of the equation $\widehat{z} \cap \widehat{\alpha} = f_*[uv]$ under the canonical map $Z_1(M) \to H_1(M)$, we get $z \cap \alpha = a$. That is, $D(\alpha) = a$. Then the intersection number of a and b is given by $\Phi(D^{-1}a, b) = \Phi(\alpha, b) = \widehat{\alpha}(\widehat{b}) = 1$. If one pictures a and b by picturing $\widehat{a}$ and $\widehat{b}$, then this is a reasonable result, since $\widehat{a}$ and $\widehat{b}$ do intersect in just one point.

Definition. Let X be a topological space and let $p \in \mathbb{Z}$. Let $\sigma \in \mathbb{Z}_p(X)$. Choose an integer $j \geq 0$, choose $c_1, \ldots, c_j \in \mathbb{Z}\backslash\{0\}$ and choose $\sigma_1, \ldots, \sigma_j \in C(\Delta^p, X)$ such that $\sigma = c_1\sigma_1 + \cdots + c_j\sigma_j$. Then we define $\mathrm{im}(\sigma) := (\sigma_1(\Delta^p)) \cup \cdots \cup (\sigma_j(\Delta^p))$.

The next result, which we state without proof, is based on the intuition that there is a connection between intersection number and actual intersection.

Theorem. Let M be a compact, connected, oriented manifold. Let $d := \dim(M)$ and let $k \in \mathbb{Z}$. Let $a_0 \in Z_k(M)$ and $b_0 \in Z_{d-k}(M)$. Let a and b, be, respectively, the images of a_0 and b_0 under the canonical maps $Z_k(M) \to H_k(M)$ and $Z_{d-k}(M) \to H_{d-k}(M)$. Fix a fundamental class z on M. Assume that the intersection number (with respect to z) of a and b is $\neq 0$. Then $(\mathrm{im}(a_0)) \cap (\mathrm{im}(b_0)) \neq \emptyset$.

Corollary. Let $f : S^2 \to S^2$ be homotopic to the identity map $S^2 \to S^2$. Then there exists $p_0 \in S^2$ such that $f(p_0) = p_0$.

Note 1: The proof given below works, not just for S^2, but for any compact, orientable topological manifold whose Euler number is nonzero.

Note 2: One may use this result to argue that if M is a smooth compact, orientable manifold with nonzero Euler number then, for any smooth vector field V on M, there exists $m_0 \in M$ such that $V_{m_0} = 0$. (This requires ideas from differential topology, which is covered *next*, so we are actually getting ahead of ourselves here.) Here's why: Let ϕ_t denote the time t flow of V. For all integers $j \geq 0$, let $M_j := \{m \in M \,|\, \phi_{1/2^j}(m) = m\}$. The preceding corollary (together with Note 1 above) shows, for all integers $n \geq 0$, that $M_n \neq \emptyset$. Moreover, we have $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$. Then, by the finite intersection property, there exists $m_0 \in \cap_{j=1}^{\infty} M_j$. Then, for all $i, j \in \mathbb{Z}$, if $j \geq 0$, then $\phi_{i/2^j}(m_0) = m_0$. Since $\{i/2^j \,|\, i, j \in \mathbb{Z}, i \geq 0\}$ is dense in $\mathbb{R}$, we conclude, for all $t \in \mathbb{R}$, that $\phi_t(m_0) = m_0$. Then $V_{m_0} = 0$.

We now proceed to the proof of the corollary.

Proof: Let $M := S^2 \times S^2$.

Let z be a fundamental class on S^2. Choose $z_0 \in Z_2(S^2)$ such that the image of z_0 under the canonical map $Z_2(S^2) \to H_2(S^2)$ is a fundamental class z of S^2.

Let $I : S^2 \to M$ be defined by $I(p) = (p,p)$ and let $F : S^2 \to M$ be defined by $F(p) = (p, f(p))$ As f is homotopic to the identity, it follows that I and F are homotopic. We wish to show that $(I(S^2)) \cap (F(S^2)) \neq \emptyset$.

Let $a_0 := I_*(z_0) \in Z_2(M)$ and let $b_0 := F_*(z_0) \in Z_2(M)$. Let $a := I_*(z) \in H_2(M)$ and let $b := F_*(z) \in H_2(M)$.

Since z is a fundamental class on S^2, it follows that $a \cup a$ is a fundamental class on M.

Since $\mathrm{im}(a_0) \subseteq I(S^2)$ and $\mathrm{im}(b_0) \subseteq F(S^2)$, it suffices to show $(\mathrm{im}(a_0)) \cap (\mathrm{im}(b_0)) \neq \emptyset$. Then, by the preceding theorem, it suffices to show that the intersection number of a and b (with respect to $a \cup a$) is nonzero.

However, as I and F are homotopic, it follows that $I_* : H_2(S^2) \to H_2(M)$ and $F_* : H_2(S^2) \to H_2(M)$ are equal. Then $a = I_*(z) = F_*(z) = b$.

Then the intersection number of a and b is equal to that of a and a. By the following theorem, this number is

$$\sum_i (-1)^i[\dim(H_i(S^2;\mathbb{R}))] = 1 - 0 + 1 - 0 + 0 - 0 + 0 - \cdots = 2. \qquad \textbf{QED}$$

Given a compact, connected, oriented topological manifold X, note that $M := X \times X$ is also compact, connected and oriented. Let $d := \dim(X)$. Let $I : X \to M$ be the diagonal map $I(x) = (x,x)$. Let z be a fundamental class on X and let $a := I_*(z) \in H_d(M)$. Then $a \cup a$ is a fundamental class on M. The preceding proof indicates that we will often be interested in calculating the intersection number of a with itself.

This is a well-known problem, but to give the answer, we need to first define the "Euler number" of a topological space. This, in turn, requires that we define Betti numbers.

Definition. Let X be a topological space and let $i \in \mathbb{Z}$. Then the ***i*th Betti number** of X is defined as $\beta_i(X) := \dim(H_i(X;\mathbb{R}))$.

For any Abelian group A, we have $\mathrm{Tor}(A, \mathbb{R}) = 0$, so, by the Universal Coefficients Theorem, $H_i(X;\mathbb{R}) \cong (H_i(X)) \otimes \mathbb{R}$. Note that, if $H_i(X) = \mathbb{Z}^k \oplus F$ and if F is finite, then $(H_i(X)) \otimes \mathbb{R} = \mathbb{R}^k$, which implies that $\dim(H_i(X;\mathbb{R})) = k$. Thus, if $H_i(X)$ is finitely generated, we have $\beta_i(X) = \mathrm{rank}(H_i(X))$.

For any mapping $i \mapsto a_i : \mathbb{Z} \to \mathbb{R}$, if $\{i \in \mathbb{Z} \,|\, a_i \neq 0\}$ is finite, then we define $\sum' a_i := \sum (-1)^i a_i$. This is called the **alternating sum** of the a_i.

We'll say that a topologial space **admits an Euler number** if $\{i \in \mathbb{Z} \,|\, \beta_i(X) \neq 0\}$ is finite and if, for all $i \in \mathbb{Z}$, $\beta_i(X)$ is finite. In this case, we define $\chi(X) := \sum' \beta_i(X)$. This is called the **Euler number** of X; it is the alternating sum of the Betti numbers of X.

We omit the proof of the next result.

Theorem. Let X be a compact, connected, oriented topological manifold X. and let $M := X \times X$. Let $d := \dim(X)$. Let $I : X \to M$ be the diagonal map $I(x) = (x,x)$. Let z be a fundamental class on X and let $a := I_*(z) \in H_d(M)$. Then the intersection number of a and a (with respect to the fundamental class $a \cup a \in H_{2d}(M)$) is equal to $\chi(X)$.

Some remarks on the calculations of Euler numbers: Let $G_\bullet$ be a graded group. We'll say that $G_\bullet$ **admits an Euler number** if $\{i \in \mathbb{Z} \,|\, G_i \otimes \mathbb{R} \neq \{0\}\}$ is finite and if, for all $i \in \mathbb{Z}$, $G_i \otimes \mathbb{R}$ is finite-dimensional. In this case, we define the **Euler number** of $G_\bullet$ to be $\chi(G_\bullet) := \sum' \dim(G_i \otimes \mathbb{R})$.

Note that if X is a topological space admitting an Euler number, then $H_\bullet(X)$ admits an Euler number and then $\chi(X) = \chi(H_\bullet(X))$.

Let $(C_\bullet, \partial)$ be a chain complex. We say that $(C_\bullet, \partial)$ **admits an Euler number** if its underlying graded group $C_\bullet$ does, and, in this case, we define its **Euler number** to be $\chi(C_\bullet, \partial) = \chi(C_\bullet)$. Then we have the following elementary lemma.

Lemma. Let $C_\bullet$ be a chain complex admitting an Euler number. Then $H_\bullet(C_\bullet)$ admits an Euler number and $\chi(C_\bullet) = \chi(H_\bullet(C_\bullet))$.

Proof: Let $H_\bullet := H_\bullet(C_\bullet)$. We wish to show that $\chi(C_\bullet) = \chi(H_\bullet)$.

Let $Z_\bullet := Z_\bullet(C_\bullet)$. Let $B_\bullet := B_\bullet(C_\bullet)$. For all $i \in \mathbb{Z}$, define $c_i := \dim(C_i \otimes \mathbb{R})$, $h_i := \dim(H_i \otimes \mathbb{R})$, $z_i := \dim(Z_i \otimes \mathbb{R})$, $b_i := \dim(B_i \otimes \mathbb{R})$.

For $i \in \mathbb{Z}$, since $H_i = Z_i/B_i$, we get $h_i = z_i - b_i$. For $i \in \mathbb{Z}$, since there is a surjection $C_i \to B_{i-1}$ with kernel Z_i, we have $C_i/Z_i \cong B_{i-1}$, and so $c_i = z_i + b_{i-1}$.

Then $\chi(C_\bullet) = \sum' c_i = (\sum' z_i) + (\sum' b_{i-1}) = (\sum' z_i) - (\sum' b_i) = \sum' h_i = \chi(H_\bullet)$. **QED**

Now assume that X is the realization of a finite two-dimensional simplicial complex. (For example, you might consider carefully the case where X is the surface of a tetrahedron, which is homeomorphic to S^2.) Let v be the number of vertices in the simplicial complex, let e be the number of edges and let f be the number of faces. The simplicial complex gives rise to a skeletal filtration on X, and let $C_\bullet$ denote the resulting cellular chain complex.

The wonderful thing about Euler numbers is that, to find $\chi(C_\bullet)$, we need only know the ranks of the terms of $C_\bullet$. The boundary maps (which are harder to analyze) can be ignored. In the case under discussion, the simplical complex is two-dimensional, so, for all integers $i \notin \{0, 1, 2\}$, we have that $C_i = \{0\}$. Moreover, we have $\mathrm{rank}(C_0) = v$, $\mathrm{rank}(C_1) = e$ and $\mathrm{rank}(C_2) = f$. Then $\chi(C_\bullet) = v - e + f$.

Via spectral sequences, we verified that $H_\bullet(X) \cong H_\bullet(C_\bullet)$. Then, by the preceding lemma, we have $\chi(H_\bullet(X)) = \chi(C_\bullet)$. Then $\chi(X) = \chi(H_\bullet(X)) = \chi(C_\bullet) = v - e + f$.

In particular, when X is the surface of a tetrahedron, we have $\chi(X) = 4 - 6 + 4 = 2$. So since X is homeomorphic to S^2, we see that $\chi(S^2) = 2$. Alternatively, viewing $\chi(S^2)$ as the alternating sum of the Betti numbers of S^2, we get $\chi(S^2) = 1 - 0 + 1 - 0 + 0 - 0 + \cdots = 2$.

Next topic: There exist two topological spaces X and Y such that $H_\bullet(X) \cong H_\bullet(Y)$ but such that the graded rings $H^\bullet(X)$ and $H^\bullet(Y)$ are not isomorphic. (Note that, by Universal Coefficients, the graded groups $H^\bullet(X)$ and $H^\bullet(Y)$ are isomorphic.) Specifically, let $X := \mathbb{C}P^2$ and let $Y := S^2 \vee S^4$. We leave it as an unassigned exercise to show, for all $k \in \{0, 2, 4\}$, that $H_k(X) \cong \mathbb{Z} \cong H_k(Y)$. We leave it as an unassigned exercise to show, for all integers $k \notin \{0, 2, 4\}$, that $H_k(X) \cong \{0\} \cong H_k(Y)$. Recall that, by the graded ring structure on $H^\bullet(X)$, we have $[H^2(X)][H^2(X)] \neq \{0\}$. However, if $x, y \in H^2(Y)$, then we can argue that $x \cup y = 0$, as follows: Let $\phi : H^2(Y) \to H^2(S^2)$ and $\psi : H^4(Y) \to H^4(S^2)$ be the maps induced by the inclusion $S^2 \hookrightarrow Y$. Let $\phi' : H^2(Y) \to$

$H^2(S^4)$ and $\psi' : H^4(Y) \to H^4(S^4)$ be the maps induced by the inclusion $S^4 \hookrightarrow Y$. Then $\psi'(x \cup y) = [\phi'(x)] \cup [\phi'(y)]$, by functoriality of $H^\bullet$ from topological spaces to graded rings. We have $\psi(x \cup y) \in H^4(S^2) = \{0\}$. Moreover, $\phi'(x), \phi'(y) \in H^2(S^4) = \{0\}$. Then $\psi(x \cup y) = 0$ and $\psi'(x \cup y) = [\phi'(x)] \cup [\phi'(y)] = 0$. By Mayer-Vietoris, the map $\psi \oplus \psi' : H^4(Y) \to [H^4(S^2)] \oplus [H^4(S^4)]$ is an isomorphism, and so $x \cup y = 0$.

Note that most of the algebraic topological tools we have described have trouble distinguishing between homotopy equivalent topological spaces. An amazing exception is the dimension of a topological manifold, which shows, for example that S^1 and $S^1 \times \mathbb{R}$ are not homeomorphic. This follows from the Invariance of Domain theorem, which implies that two manifolds of different dimension cannot be homeomorphic. Recall that this result is proved by algebraic topological techniques, something truly remarkable. Generally, however, when one is presented with to homotopy equivalent topological manifolds of the same dimension, it is hard to know if they are homeomorphic.

In this context, the following result is quite important:

Weak Version of the Mostow Rigidity Theorem. Let M and N be compact hyperbolic manifolds. If M and N are homotopy equivalent, then M is homeomorphic to M.

We do not define "hyperbolic" here, but note that the class of hyperbolic manifolds is very broad and is of interest to many manthematicians. You may want to talk to Prof. Al Marden if you'd like to hear more about this theorem.

Also, if the dimension of M or N is ≥ 3, then homotopy equivalent does not just imply homeomorphic but actually implies isometric (which means isomorphic in the category of hyperbolic manifolds). Finally, there are many other strengthenings of this theorem that have been noted over the years, and there is a related subject, "superrigidity", which is quite an active area of current research.

Our next topic is the *degree* of a map.

Definition. Let M be a compact, orientable, d-dimensional topological manifold. Let $f : M \to M$ be continuous. The **degree** of f, denoted $\deg(f)$, is the unique $n \in \mathbb{Z}$ such that, for all $x \in H_d M$, we have $f_*(x) = nx$.

EXERCISE 21A: Let $p := (0,1) \in S^1$. Define $R : S^1 \to S^1$ by $R(x,y) = (-x,y)$, so that $R(p) = p$. Let $R_* : \pi_1(S^1,p) \to \pi_1(S^1,p)$ be the induced map on fundamental groups. Let $\gamma \in \pi_1(S^1,p)$. Show that $R_*(\gamma) = \gamma^{-1}$.

Then, by the Hurewicz Theorem, it follows that $R_* : H_1 S^1 \to H_1 S^1$ is multiplication by -1, i.e., that $R : S^1 \to S^1$ has degree -1. Similarly, one can argue:

Example. The map $(cos(t), \sin(t)) \mapsto (\cos(nt), \sin(nt)) : S^1 \to S^1$ has degree n.

We collect, without proof, some basic (easily proved) properties of degree:

Lemma. Let M be a compact, orientable topological manifold. Then:

(1) If $f, g : M \to M$ are homotopic maps, then $\deg(f) = \deg(g)$.
(2) If $\text{id} : M \to M$ is the identity map, then $\deg(\text{id}) = 1$.
(3) If $f : M \to M$ is a constant map, then $\deg(f) = 0$.

(4) If $f, g : M \to M$, then $\deg(f \circ g) = [\deg(f)][\deg(g)]$.

Theorem. Let $d \geq 1$ be an integer. Then

(1) for any integer $n \geq 1$, there exists $f : S^d \to S^d$ such that $\deg(f) = n$; and
(2) for any $f, g : S^d \to S^d$, we have: f is homotopic to g iff $\deg(f) = \deg(g)$.

That is, the degree map gives a bijection between the set of integers and the set of homotopy classes of maps $S^d \to S^d$.

In what follows (using a process called "suspension"), we will show, for any integer $d \geq 2$ and any integer n, that if there is a continuous map $S^{d-1} \to S^{d-1}$ of degree n, then there is a continuous map $S^d \to S^d$ of degree n. By the preceding Example, for any integer n, there is a continuous map $S^1 \to S^1$ of degree n. Thus, by induction, Conclusion (1) of the preceding theorem is true. The "only if" part of (2) follows from (1) of the preceding lemma. The "if" part of (2) is hard and is, in fact, beyond the scope of this course.

We now define the suspension of a map. Let $J := [-1, 1]$. The **suspension functor** $\Sigma : \{\text{topological spaces}\} \to \{\text{topological spaces}\}$ is defined by letting ΣX be the quotient of $X \times J$ by the smallest equivalence relation R on X satisfying both

(a) $X \times \{0\}$ is an equivalence class of R; and
(b) $X \times \{1\}$ is an equivalence class of R.

That is, we form $X \times J$, then collapse $X \times \{0\}$ to a point, and then collapse $X \times \{1\}$ to a point. The result is ΣX. For spheres, note that ΣS^d is homeomorphic to S^{d+1}, as follows: Define $\phi : S^d \times J \to S^{d+1}$ by: $\phi(x, t) = (x\sqrt{1 - t^2}, t)$ and note that ϕ factors to a homeomorphism $\psi : \Sigma S^d \to S^{d+1}$. For any continuous map $f : S^{d-1} \to S^{d-1}$, we define $\Sigma_1 f : S^d \to S^d$ by $\Sigma_1 f := \psi \circ (\Sigma f) \circ \psi^{-1}$. For all $x \in S^{d-1}$, for all $t \in J$, we have $(\Sigma_1 f)(x\sqrt{1 - t^2}, t) = ((f(x))\sqrt{1 - t^2}, t)$. Sometimes $\Sigma_1 f$ is called the **suspension** of f.

Theorem. Let $d \geq 1$ be an integer and let $f : S^d \to S^d$ be continuous. Then we have: $\deg(\Sigma_1 f) = \deg(f)$.

Proof: For concreteness, we will prove this for the case $d = 1$. The proof for other dimensions is completely analogous.

Let $k := \deg(f)$. Then $f_* : H_1 S^1 \to H_1 S^1$ is multiplication by k.

Let $F := \Sigma_1 f$ and let $\Lambda := F_* : H_2 S^2 \to H_2 S^2$. We wish to show that Λ is multiplication by k.

Let E be the equator of S^2. Let

$$S^2_+ := \{(x, y, z) \in S^2 \mid z \geq 0\}, \qquad S^2_- := \{(x, y, z) \in S^2 \mid z \leq 0\}.$$

Let $F^0 := F|E : E \to E$, let $F^+ := F|S^2_+ : S^2_+ \to S^2_+$ and let $F^- := F|S^2_- : S^2_- \to S^2_-$. Note that $F^0 : E \to E$ is isomorphic (in the arrow category of the category of topological spaces) to $f : S^1 \to S^1$. It follows that $F^0_* : H_1 E \to H_1 E$ is isomorphic (in the arrow category of the category of additive Abelian groups) to $f_* : H_1 S^1 \to H_1 S^1$. Thus $F^0_* : H_1 E \to H_1 E$ is multiplication by k.

Let

$$\widetilde{S}^2 := (S^2, E), \qquad \widetilde{S}^2_+ := (S^2_+, E), \qquad \widetilde{S}^2_- := (S^2_-, E).$$

Let $\widetilde{F} := (F, F^0) : \widetilde{S}^2 \to \widetilde{S}^2$. Let

$$\widetilde{F}^+ := (F^+, F^0) : \widetilde{S}^2_+ \to \widetilde{S}^2_+, \qquad \widetilde{F}^- := (F^-, F^0) : \widetilde{S}^2_- \to \widetilde{S}^2_+.$$

By functoriality of long exact sequences, we see that $\widetilde{F}^+_* : H_2\widetilde{S}^2_+ \to H_2\widetilde{S}^2_+$ is isomorphic (in the arrow category of the category of additive Abelian groups) to $F^0_* : H_1E \to H_1E$ Thus $\widetilde{F}^+_* : H_2\widetilde{S}^2_+ \to H_2\widetilde{S}^2_+$ is multiplication by k. Similarly, $\widetilde{F}^-_* : H_2\widetilde{S}^2_- \to H_2\widetilde{S}^2_-$ is multiplication by k.

Let $A := H_2\widetilde{S}^2_+ \oplus H_2\widetilde{S}^2_-$ and $\Phi := \widetilde{F}^+_* \oplus \widetilde{F}^-_* : A \to A$. Then Φ is multiplication by k. Let $B := H_2\widetilde{S}^2$ and $\Psi := \widetilde{F}_* : B \to B$. Let $\beta^+ : \widetilde{S}^2_+ \to \widetilde{S}^2$ and $\beta^- : \widetilde{S}^2_- \to \widetilde{S}^2$ be the inclusion maps. Let $\alpha := \beta^+_* \oplus \beta^-_* : A \to B$.

We leave it as an unassigned exercise to show that $\alpha \circ \Phi = \Psi \circ \alpha$. (*Hint:* Note that $\Phi = (\widetilde{F}^+ \coprod \widetilde{F}^-)_*$ and $\alpha = (\beta^+ \coprod \beta^-)_*$, where $\coprod$ denotes disjoint union of topological pairs. Argue that there is a commutative diagram in the category of topological pairs whose image under H_2 yields $\alpha \circ \Phi = \Psi \circ \alpha$.)

We also leave it as an unassigned exercise to show that α is surjective. (*Hint:* In fact, S^2 is obtained from E by attaching two cells, namely S^2_+ and S^2_-. It therefore follows from our general theory about the cellular chain comples that $H_2(S^2, E) \cong \mathbb{Z}^2$. More specifically, we have that the inclusion maps $(S^2_+, E) \subseteq (S^2, E)$ and $(S^2_-, E) \subseteq (S^2, E)$ induce an isomorphism $(H_2(S^2_+, E)) \oplus (H_2(S^2_-, E)) \to H_2(S^2, E)$. This is exactly the statement that α is an isomorphism, and is therefore, in particular, surjective.)

Since Φ is multiplication by k, since $\alpha \circ \Phi = \Psi \circ \alpha$ and since α is surjective, it follows that $\Psi : B \to B$ is multiplication by k.

Let $\gamma : H_2S^2 \to H_2\widetilde{S}^2$ be induced by the inclusion $(S^2, \emptyset) \subseteq (S^2, E) = \widetilde{S}^2$. By the long exact sequence of (S^2, E), we see that γ is injective. Recall that $\Lambda := F_* : H_2S^2 \to H_2S^2$. By functoriality of long exact sequences, we have $\Psi \circ \gamma = \gamma \circ \Lambda$. So, since Ψ is multiplication by k, it follows that Λ is multiplication by k, as desired. **QED**

Corollary. Let $d \geq 1$ be an integer. Then any reflection of S^d has degree -1.

Proof: We leave it as an unassigned exercise to show that any two reflections of S^d are conjugate by a rotation. (In fact, just take any rotation that moves the fixed hyperplane of the first reflection to the fixed hyperplane of the second.) Consequently, in the arrow category of the category of topological spaces, any two reflections of S^d are isomorphic. Thus if one of them has degree -1, they all do.

By Exercise 21A and the Hurewicz Theorem, some reflection of S^1 has degree -1.

We now proceed by induction. Now assume that there is some reflection R of S^{d-1} which has degree -1. We wish to show that some reflection of S^d has degree -1.

We leave it as an unassigned exercise to verify that $\Sigma_1 R : S^d \to S^d$ is a reflection. By the preceding theorem, $\deg(\Sigma_1 R) = \deg(R) = -1$. **QED**

Corollary. Let $d \geq 1$ be an integer. Let $A : S^d \to S^d$ be the antipodal map defined by $A(p) = -p$. Then $\deg(A) = (-1)^{d+1}$.

Proof: For all integers $i \in [0, d]$, let $R_i : S^d \to S^d$ be the reflection defined by

$$R_i(x_0, \ldots, x_d) = (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_d);$$

by the preceding corollary, $\deg(R_i) = -1$.

Since $A = R_0 \circ \cdots \circ R_d$, $\deg(A) = [\deg(R_0)][\deg(R_1)] \cdots [\deg(R_d)] = (-1)^{d+1}$. **QED**

For integers $d \geq 1$, for $x, y \in \mathbb{R}^d$, we write $x \perp y$ to mean $x \cdot y = 0$, where $\cdot$ denote the Euclidean dot product.

Lemma. Let $n \geq 1$ be an integer and let $f : S^n \to S^n$ be continuous. Assume, for all $x \in S^n$, that $f(x) \neq -x$. Then f is homotopic to the identity map.

Proof: For all $x \in S^n$, let $x^\perp := \{y \in \mathbb{R}^{n+1} \mid x \perp y\}$. For all $x \in S^n$, let $\phi_x : x^\perp \to S^n \backslash \{-x\}$ be stereographic projection, given by $\phi_x(y) = [x + y]/[\|x + y\|]$. For all $t \in [0, 1]$, let $f_t : S^n \to S^n$ be defined by $f_t(x) = \phi_x(t \cdot \phi_x^{-1}(f(x)))$. Then $(t, x) \mapsto f_t(x) : I \times S^n \to S^n$ is a homotopy from the identity to f. **QED**

We now give a proof that you can't comb the hairs on a hedgehog:

Theorem. Let $f : S^2 \to \mathbb{R}^2$ be continuous and assume, for all $x \in S^2$, that $f(x) \perp x$. Then there exists $x_0 \in S^2$ such that $f(x_0) = 0$.

Proof: Assume, to the contrary, that, for all $x \in S^2$, we have $f(x) \neq 0$. Define $g : S^2 \to \mathbb{R}^2$ by $g(x) = [f(x)]/[\|f(x)\|]$. Let $I : S^2 \to S^2$ be the identity map, defined by $I(x) = x$. Let $A : S^2 \to S^2$ be the antipodal map, defined by $A(x) = -x$. Let $h := A \circ g : S^2 \to S^2$. Then, for all $x \in S^2$, we have $g(x) \notin \{x, -x\}$, which implies that $g(x) \neq -x$ and $h(x) \neq -x$. Then, by the preceding lemma, both g and h are homotopic to I. Then $A \circ h$ is homotopic to $A \circ I$. That is, g is homotopic to A. On the other hand g is homotopic to I. Then A is homotopic to I. However, the degree of A is -1, whereas the degree of I is 1, a contradiction. **QED**

Recall that the lemma preceding the Ham Sandwich Theorem asserts

Lemma. Let $\psi : S^3 \to S^3$ be odd. Then ψ is not homotopic to a constant map.

The proof given is valid all odd dimensional spheres, *i.e.*,

Lemma. Let $n > 0$ be an odd integer. Let $\psi : S^n \to S^n$ be odd. Then ψ is not homotopic to a constant map.

We now show that, by using suspension, we may extend the preceding lemma to *all* spheres, *i.e.*,

Lemma. Let $n > 0$ be an integer. Let $\psi : S^n \to S^n$ be odd. Then ψ is not homotopic to a constant map.

NOTE: Since, for any $x \in S^n$, $S^n \backslash \{x\}$ is contractible, it is a corollary of the above lemma that an odd map $S^n \to S^n$ must be surjective.

Proof: If n is odd, then we are done by the preceding lemma, so assume that n is even, and that ψ is homotopic to a constant map. We aim for a contradiction.

We leave it as an unassigned exercise to show that, because $\psi : S^n \to S^n$ is odd, it follows that $\Sigma_1 \psi : S^{n+1} \to S^{n+1}$ is odd. We leave it as an unassigned exercise to show that, because $\psi : S^n \to S^n$ is homotopic to thd a constant map, it follows that $\Sigma_1\psi : S^{n+1} \to S^{n+1}$ is homotopic to thd a constant map.

This then contradicts the preceding lemma (with n replaced by $n+1$ and with ψ replaced by $\Sigma_1\psi$). **QED**

Temperature Pressure Theorem. Let $f : S^2 \to \mathbb{R}^2$ be continuous. Then there exists $p \in S^2$ such that $f(p) = f(-p)$.

NOTE: This result implies that, at any moment in time, there is a point p on the Earth (which we assume to be a perfect sphere, centered at the origin in $\mathbb{R}^3$) such that:

(1) the temperature at p is the same as at $-p$; and

(2) the pressure at p is the same as at $-p$.

Proof: Define $g : S^2 \to \mathbb{R}^2$ by $g(p) = (f(p)) - (f(-p))$. Then g is odd. Assume for all $p \in S^2$, that $g(p) \neq 0$. We aim for a contradiction.

Let $h : S^2 \to S^1$ be defined by $h(p) = [g(p)]/[\|g(p)\|]$. Then h is odd. Let $i : S^1 \to S^2$ be the "equator inclusion map", given by $i(x) = (x, 0)$. Then the image of $i \circ h : S^2 \to S^2$ is contained in the equator, so $i \circ h : S^2 \to S^2$ is not surjective.

For any point $x \in S^2$, recall that $S^2 \backslash \{x\}$ is contractible. Thus any map $S^2 \to S^2$ which is not surjective is homotopic to a constant map. Consequently, $i \circ h : S^2 \to S^2$ is homotopic to a constant map.

However, h is odd and i is even, so $i \circ h$ is odd, so we have a contradiction to the preceding lemma. **QED**

Finally, we discuss the classification of surfaces.

Definition. Let $d \geq 1$ be an integer and let M be a topological d-manifold. For all $m \in M$, a map $f : \mathbb{R}^d \to M$ will be called **an m-centered chart** if

(1) f is open, continuous and injective; and

(2) $f(0) = m$.

Definition. Let $d \geq 1$ be an integer. Let M and N be connected topological d-manifolds. Let $m \in M$ and $n \in N$. Let $f : \mathbb{R}^d \to M$ be an m-centered chart and let $g : \mathbb{R}^d \to M$ be an n-centered chart. Define $h : (f(\mathbb{R}^d)) \backslash \{m\} \to (g(\mathbb{R}^d)) \backslash \{n\}$ by $h(f(x)) = g(x/\|x\|^2)$. Let R be the smallest equivalence relation on $M \coprod N$ whose graph contains the graph of h. Define $M \#_{f,g} N := (M \coprod N)/R$.

EXERCISE 21B: Show that $M \#_{f,g} N$ is a connected topological d-manifold.

Fact. Let $d \geq 1$ be an integer. Let M and N be connected topological d-manifolds. Let $m, m' \in M$ and $n, n' \in N$. Let $f : \mathbb{R}^d \to M$ be an m-centered chart and let $g : \mathbb{R}^d \to M$ be an n-centered chart. Let $f' : \mathbb{R}^d \to M$ be an m'-centered chart and let $g' : \mathbb{R}^d \to M$ be an n'-centered chart. Then $M \#_{f,g} N$ is homeomorphic to $M \#_{f',g'} N$.

Because of the preceding Fact, we often simply write $M \# N$, which is well-defined up to homeomorphism. We call $M \# N$ **the connected sum** of M and N. Note, *e.g.*, that $\mathbb{T}^2 \# \mathbb{T}^2$ is the genus two orientable surface.

We leave it as an unassigned exercise to show that # is associative and commutative. We leave it as an unassigned exercise to show, for any connected topological d-manifold M, that $S^d \# M$ is homeomorphic to M.

We will write $X \cong Y$ to mean that the topological spaces X and Y are homeomorphic. A **surface** is a 2-manifold. The following is difficult, and we present it without proof:

Theorem. For any compact connected surface X, either $X \cong S^2$, or there exist an integer $n \geq 1$ and $X_1, \ldots, X_n \in \{\mathbb{T}^2, \mathbb{R}P^2\}$ such that $X \cong X_1 \# \cdots \# X_n$.

To summarize the preceding theorem, we say that the #-closure of $\mathbb{T}^2$ and $\mathbb{R}P^2$ is the collection of all compact connected surfaces. One naturally wonders when two such expressions $X_1 \# \cdots \# X_n$ are homeomorphic, and we next devise a scheme to answer this.

Let S be a finite set and let w be an element of the free group $\langle S \rangle$ generated by S. We obtain a topological space $T(w)$ as follows: Let P be a regular polygon in $\mathbb{R}^2$ with as many edges as there are factors in w. (E.g., if $w = aba^{-1}b^{-1}$, then P should have 4 edges.) Pick one vertex as a starting vertex and label each edge, traveling clockwise, with one factor in w. Now let X be the precompact connected component of $\mathbb{R}^2 \backslash P$ and let $T(w)$ be the topological space obtained by identifying the edges of $X \cup P$ according to their labelings. (Note that $T(aba^{-1}b^{-1})$ is homemorphic to the 2-torus $\mathbb{T}^2 := S^1 \times S^1$.)

We have $T(aa) \cong \mathbb{R}P^2 \cong T(bb)$, so $\mathbb{R}P^2 \# \mathbb{R}P^2 \cong T(aabb)$. If we cut the $aabb$ polygon along a line going from

the vertex at the middle of the path of aa

to

the vertex at the middle of the path of bb,

then label the new edges along the cut both as c, and then glue the resulting two triangles (acb and abc^{-1}) along b, we find that $T(aabb) \cong T(aca^{-1}c)$.

Algebraically, this amounts to

(1) picking a two letter subword w' (specifically ab) in the word $aabb$; then
(2) replacing that subword w' with a new letter, say c, yielding a new word w_0 (in our case $w_0 = acb$); then
(3) picking one of the letters, say b, in that subword w'; then
(4) solving $c = w'$ for b, getting an equation of the form $b = w''$ (in our case, we solve $c = ab$ for b and get $b = a^{-1}c$); and then
(5) replacing every b in w_0 with w'' (in our case we get, in the end, $aca^{-1}c$).

Since $K := T(aca^{-1}c)$ is the Klein bottle, we have shown that the connected sum of two projective planes is the Klein bottle K. Remark: The projective plane $\mathbb{R}P^2$ is sometimes called a **cross cap**, so one says that the Klein bottle is a connected sum of two cross caps.

Lemma. We have $\mathbb{R}P^2 \# K \cong \mathbb{R}P^2 \# \mathbb{T}^2$.

Proof: Let $X := \mathbb{R}P^2 \# K$. Since $\mathbb{R}P^2 \cong T(aa)$ and $K \cong T(bcbc^{-1})$, we get $X \cong T(aabcbc^{-1})$. Replacing ab with d and a with db^{-1}, we obtain $X \cong T(db^{-1}dcbc^{-1})$. Replacing dc by e and c by $d^{-1}e$, we obtain $X \cong T(db^{-1}ebe^{-1}d)$.

A cyclic permutation of $db^{-1}ebe^{-1}d$ is $ddb^{-1}ebe^{-1}$, so we get $X \cong T(ddb^{-1}ebe^{-1})$.

Replacing b by f^{-1}, we get $X \cong T(ddfef^{-1}e^{-1})$. Then, as

$$\mathbb{R}P^2 \cong T(dd) \qquad \text{and} \qquad \mathbb{T}^2 \cong T(fef^{-1}e^{-1}),$$

we conclude that $X \cong \mathbb{R}P^2 \# \mathbb{T}^2$, as desired. **QED**

Let n be a positive integer and let $X_1, \ldots, X_n \in \{\mathbb{T}^2, \mathbb{R}P^2\}$. Let $X := X_1 \# \cdots \# X_n$. Since $K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$, the preceding lemma yields $T \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$. Thus, if $\mathbb{R}P^2 \in \{X_1, \ldots, X_n\}$, then X is homeomorpic to a connected sum of projective planes (because any $\mathbb{T}^2$ appearing in the connected sum $X_1 \# \cdots \# X_n$ can be replaced by $\mathbb{R}P^2 \# \mathbb{R}P^2$). Either $\mathbb{R}P^2 \notin \{X_1, \ldots, X_n\}$, in which case X is a connected sum of 2-tori, or else $\mathbb{R}P^2 \in \{X_1, \ldots, X_n\}$, in which case X is a connected sum of projective planes.

This observation, combined with the preceding theorem, shows that every compact connected surface is homeomorphic to one of

(1) S^2; or
(2) a connected sum of 2-tori; or
(3) a connected sum of projective planes.

We leave it as an unassigned exercise for the reader to use algebraic topological techniques developed in this course to show that, of the surfaces described in (1), (2) and (3) above, no two are homeomorphic. This then gives a complete classification of compact connected surfaces.

Question: Can one do something similar for compact connected 3-manifolds?

This is an extremely difficult question and has been worked on by many, many people. An attempt at answering this question is the **Thurston Geometrization Conjecture**, which is too complicated even to state here, but which would similarly break any compact connected 3-manifold up into pieces each of which belongs to a well-studied list.

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