MATH 4281 HOMEWORK 10 (SOLUTIONS)

17. Rings: Definitions and Elementary Properties

J3. Prove that in a commutative ring R with unity, a divisor of zero cannot be invertible.

Proof. Let $r \in R$ be a zero divisor, and suppose r is invertible. Then there exists $s \in R$ such that $rs = 1$. Since r is a zero divisor, there exists a nonzero element $t \in R$ such that $tr = 0$. Then multiplying the equation $rs = 1$ on the left by t, we have $trs = t \cdot 1 = t$, which implies $0 \cdot s = t$ so $0 = t$, a contradiction. □

J4. Suppose $ab \neq 0$ in a commutative ring R. If either a or b is a divisor of zero, so is ab.

Proof. Without loss of generality, suppose $a \in R$ is a zero divisor, then there exists a nonzero element $c \in R$ such that $ac = ca = 0$. Multiplying this by b and use the associativity we can get $c(ab) = (ca)b = 0 \cdot b = 0$. By assumption $ab \neq 0$, so we must have ab is a zero divisor of R. □

J5. Suppose a is neither 0 nor a divisor of zero. If $ab = ac$, then $b = c$.

Proof. Let R be the commutative ring. Suppose $ab = ac$, i.e., $ab - ac = 0$. Then by distributivity, $a(b - c) = 0$. Since $a \neq 0$, and a is not a zero divisor, we must have $b - c = 0$. Hence $b = c$. □

J6. $A \times B$ always has divisors of zero.

Proof. Let $a \in A$ and $b \in B$ be any two nonzero elements, then $(a, 0)(0, b) = (0, 0)$. □