MATH 4281 HOMEWORK 5

9. Isomorphism

D2. Find the isomorphism classes of: S_3, $\mathbb{Z}_6$, $\mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_7^\times$.

Claim: we will show that $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_7^\times$, and S_3 is isomorphic to non of the other three groups.

Proof. Since $\gcd(2,3) = 1$, $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$ by Chinese Remainder Theorem. Also, we know $\mathbb{Z}_7^\times$ is a cyclic group of order 6, and any cyclic group of order n is isomorphic to $\mathbb{Z}_n$. Hence $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_7^\times$. But S_3 is not abelian (In fact, it is the smallest nonabelian group.), so it is not a cyclic group, and hence is not isomorphic to any of the other groups. □

I3. If G is any group, and $a \in G$, prove that $f(x) = axa^{-1}$ is an automorphism of G.

Proof. Let $f : G \to G$ be the map defined by $f(x) = axa^{-1}$, we will show f is an automorphism.

To show f is a homomorphism, let $x, y \in G$. Then

$$f(xy) = axya^{-1} = axeya^{-1} = ax(a^{-1}a)ya^{-1} = (axa^{-1})(aya^{-1}) = f(x)f(y).$$

Also, note that $x \in \ker(f)$ if and only if $axa^{-1} = e$ for all a, and this happens if and only if $x = e$, so f has a trivial kernel. Thus it is injective. For the surjectivity, let $y \in G$ be given, let $x := a^{-1}yg$, then $f(x) = y$. □

I4. Prove the set $\mathrm{Aut}(G)$ of all automorphisms of G is a subgroup of S_G.

Proof. The identity map is an automorphism, so the set $\mathrm{Aut}(G) \neq \varnothing$. Now we show the composition of two automorphisms is an automorphism. Let $\varphi, \gamma \in \mathrm{Aut}(G)$, and let $\rho = \varphi \circ \gamma$. Let $g, h \in G$, then

$$\begin{aligned}\rho(gh) &= \varphi(\gamma(gh)) \\ &= \varphi(\gamma(g)\gamma(h)) \\ &= \varphi(\gamma(g))\varphi(\gamma(h)) \\ &= \rho(g)\rho(h).\end{aligned}$$

Thus ρ is a homomorphism. We've already known that ρ is a bijection, so that ρ is an automorphism.

Now we show the inverse of an automorphism is an automorphism. Let $k = gh$. Since φ is a permutation, we can find g', h' and k' so that $\varphi(g') = g$, $\varphi(h') = h$ and $\varphi(k') = k$. Since φ is a homomorphism, $\varphi(g'h') = gh = k$. Applying φ^{-1} to both sides, we get $g'h' = k'$, that is,

$$\varphi^{-1}(gh) = \varphi^{-1}(g)\varphi^{-1}(h),$$

so that φ^{-1} is an automorphism and so $\mathrm{Aut}(G) \leq S_G$. □

12. Partition and equivalence relations

D4. Let $a \sim b$ iff there is an integer k such that $a^k = b^k$.

Proof. *Reflexive:* $a^k = a^k$, so $a \sim a$.

Symmetric: Suppose $a \sim b$, there exists k so that $a^k = b^k$. So $b^k = a^k$, and $b \sim a$.

Transitive: Suppose $a \sim b$ and $b \sim c$, then there exists k_1, k_2 so that $a^{k_1} = b^{k_1}$ and $b^{k_2} = c^{k_2}$. Therefore we have $k_1 k_2$ satisfying $a^{k_1 k_2} = c^{k_1 k_2}$, hence $a \sim c$.

$[e] = \{g \in G : g^n = e \text{ for some } n \in \mathbb{Z}_{\geq 0}\}$. $\square$