MATH 4281 HOMEWORK 7 (SOLUTIONS)

13. Counting Cosets

C3. Let G have order 4. Either G is cyclic , or every element of G is its own inverse. Conclude that every group of order 4 is abelian.

Proof. By Lagrange's Theorem, every element in G has order $1, 2$ or 4. If there is an element $x \in G$ with order 4, then $G = \langle x \rangle$, which is a cyclic group, so certainly abelian.

Suppose there is no element in G has order 4, then every nonidentity element in G must have order 2, i.e., $g^2 = 1$ for all $g \in G$. Pick two nonidentity elements $x, y \in G$, so $x^2 = 1, y^2 = 1$ and $(xy)^2 = 1$. This implies $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$, so x and y commutes. Thus G is abelian. (We can further show in this case $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ via the map $\varphi : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to G$ defined by $\varphi(\bar{a}, \bar{b}) = x^a y^b$.)

*** The second paragraph above actually shows a stronger assertion: Any group in which all non-identity elements have order 2 is abelian.** □

H1. Let G be a group of order 8. If every $x \neq e$ in G has order 2, let a, b, c be three such elements. Prove that $G = \{e, a, b, c, ab, bc, ac, abc\}$. Conclude that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Suppose every nonidentity element in G has order 2. In other words, every element is its own inverse. Define a map $\varphi : G \to G$ by $\varphi(g) = g^{-1}$. This is an identity map and hence is an isomorphism. By the highlighted sentence in **C3**, we know that G is abelian. Let $a, b \in G$ be two elements of order 2, and pick $c \in G$ distinct from e, a, b, ab. We claim that

$$G = \{e, a, b, c, ab, ac, bc, abc\}.$$

By cancellation law, we can check these eight elements are distinct. For example, if $ab = bc = cb$, the cancellation law implies $a = c$, a contradiction.

Now the group G can be described as

$$G = \langle a, b, c \mid a^2 = b^2 = c^2 = e, ab = ba, ac = ca, bc = cb \rangle.$$

From this presentation, we can check

$$\varphi : G \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

is an isomorphism defined by

$$a \mapsto (1, 0, 0) \quad b \mapsto (0, 1, 0) \quad c \mapsto (0, 0, 1).$$

□

H2. Assume G has an element a of order 4. Let $H = \langle a \rangle = \{e, a, a^2, a^3\}$. If $b \neq H$, then the coset $Hb = \{b, ab, a^2b, a^3b\}$. By Lagrange's theorem, G is the union of $He = H$ and Hb, hence

$$G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

Assume there is in Hb an element of order 2. (Let b be this element.) If $ba = a^2b$, prove that $b^2a = a^2b^2$, hence $a = a^4$, which is impossible. (Why?) Conclude that either $ba = ab$ or $ba = a^3b$.

Proof. Define $H \coloneqq \{e, a, a^2, a^3\}$, and let b be an element in G that is not in H, so $bH \neq H$.

Now we show $ba = a^2b$. For the sake of contradiction, suppose $ba = a^2b$, then

$$ba^2b = baab = a^2bab = a^2a^2bb = a^4b^2 = ee = e.$$

But then

$$a = ea = b^2a = bba = ba^2b = e,$$

contradicting the fact that a is an element of order 4. Thus, we must have either $ba = ab$ or $ba = a^3b$. □

H3. Let b be as in part 2. Prove that if $ba = ab$, then $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

Proof. Suppose $ba = ab$, then G is abelian. We can easily check

$$\varphi : G \to \mathbb{Z}_4 \times \mathbb{Z}_2$$

is an isomorphism defined by

$$a \mapsto (1, 0) \quad b \mapsto (0, 1).$$

□