MATH 4281 HOMEWORK 8 (SOLUTIONS)

14. Homomorphisms

E2 Suppose $a \in G$ has order 2. Then $\langle a \rangle$ is a normal subgroup of G if and only if $a \in Z(G)$.

Proof. ($\Leftarrow$) Suppose $a \in Z(G)$ and $\text{ord}(a) = 2$. Then the subgroup $\langle a \rangle$ generated by a has only 2 elements, namely $1, a$. Since $a \in Z(G)$, and certainly 1 commutes with all elements in G. So for any $g \in G$, we have $g\langle a \rangle = \langle a \rangle g$.

($\Rightarrow$) Suppose $\langle a \rangle$ is an order 2 normal subgroup of G, then a is the only nonidentity element in $\langle a \rangle$. Then for all $g \in G$, we have $gag^{-1} = 1$ or $gag^{-1} = a$. If the latter case occurs, then $\langle a \rangle \subset Z(G)$, and we are done. So assume that $gag^{-1} = 1$. Then $ga = ag$, thus multiplying on the left by g^{-1} we can get $a = 1$, a contradiction. Thus $\langle a \rangle \subset Z(G)$. $\square$

E3. If a is an element of G, $\langle a \rangle$ is a normal subgroup of G if and only if a has the following property: For any $x \in G$ there is a positive integer k such that $xa = a^k x$.

Proof. Let $a \in G$, then

$$\begin{aligned}\langle a \rangle \trianglelefteq G &\iff x\langle a \rangle = \langle a \rangle x, \quad \forall x \in G \\ &\iff \text{there exists some } y \in \langle a \rangle \text{ such that } xa = yx \\ &\iff xa = a^k x \text{ for some } k.\end{aligned}$$

$\square$

E4. In a group G, a *commutator* is any product of the form $aba^{-1}b^{-1}$, where a and b are any elements of G. If a subgroup H of G contains all the commutators of G, then H is normal.

Proof. Suppose H is a subgroups of G containing all the commutators. Let $g \in G$ and $h \in H$. Then $ghg^{-1}h^{-1} = h'$ for some $h' \in H$ since H contains all the commutators. But then $ghg^{-1} = h'h \in H$, so $gHg^{-1} \subset H$ for all $g \in G$. Multiplying g^{-1} on the left and g on the right we can get $(g^{-1}g)h(g^{-1}g) \subset g^{-1}Hg \Rightarrow H \subset g^{-1}Hg$. Thus, H is normal in G. $\square$

E5. If H and K are subgroups of G, and K is normal, then HK is a subgroup of G.

Proof. Let H, K be subgroups of G, and suppose K is normal. Then we know $HK = KH$. Let $a, b \in HK$. We prove $ab^{-1} \in HK$ so HK is a subgroup by the subgroup criterion. Let

$$a = h_1k_1 \quad \text{and} \quad b = h_2k_2,$$

for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Thus $b^{-1} = k_2^{-1}h_2^{-1}$, so $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1}$. Let $k_3 = k_1k_2^{-1} \in K$ and $h_3 = h_2^{-1}$. Thus $ab^{-1} = h_1k_3h_3$. Since $HK = KH$,

$$k_3h_3 = h_4k_4, \quad \text{for some } h_4 \in H,\ k_4 \in K.$$

Thus $ab^{-1} = h_1h_4k_4 \in HK$, as desired. $\square$