MATH 4281 HOMEWORK 9 (SOLUTIONS)

16. Fundamental Homomorphism Theorem

E2 and 3. Let G and H be groups. Suppose J is a normal subgroup of G and K is a normal subgroup of H. Use the Fundamental Theorem of Homomorphism to conclude that $(G \times H)/(J \times K) \cong (G/J) \times (H/K)$.

Proof. Let $J \trianglelefteq G$ and $K \trianglelefteq H$. We first check $J \times K \trianglelefteq G \times H$ (so the quotient is indeed a group). Suppose $(j,k) \in J \times K$ and $(g,h) \in G \times H$. Then

$$(g,h)(j,k)(g,h)^{-1} = (g,h)(j,k)(g^{-1},h^{-1}) = (gjg^{-1}, hkh^{-1}) \in J \times K$$

since we are given $J \trianglelefteq G$ and $K \trianglelefteq H$. Hence $J \times K \trianglelefteq G \times H$.

Consider now the map

$$f : G \times H \to G/J \times H/K$$

$$f(x,y) = (f_J(x), f_K(y)) = (xJ, yK)$$

where f_J and f_K are the natural projection homomorphisms for G/J and H/K as described in part (2). The kernel of this map is the set

$$\ker(f) = \{(x,y) \in G \times H \mid xJ = J, yK = K\}.$$

The conditions $xJ = J$ and $yK = K$ can be rewritten to fond that

$$\ker(f) = \{(x,y) \in G \times H \mid x \in J, y \in K\} = J \times K.$$

Now by the Fundamental Homomorphism Theorem

$$G \times H/\ker(f) = (G \times H)/(J \times K) \cong (G/J) \times (H/K).$$

□

F 1-6. *We mix* $(1) - (6)$ *and prove the following:*

Let G be a group and $H, K \leq G$ and $H \leq N_G(K)$ (where $N_G(K) := \{g \in G \mid gK = Kg\}$, called the *normalizer* of K in G), then $HK \leq G$. In particular, if $K \trianglelefteq G$ then $HK \leq G$ for any $H \leq G$. Furthermore, $K \trianglelefteq HK$, $H \cap K \trianglelefteq H$ and $HK/K \cong H/H \cap K$.

Proof. We first prove $HK \leq G$ by showing that $HK = KH$. Let $h \in H$, $k \in K$. By assumption, $hkh^{-1} \in K$, hence

$$hk = (hkh^{-1})h \in KH.$$

This proves $HK \subseteq KH$. Similarly, $kh = (hkh^{-1})h \in HK$, proving the reverse containment. Hence $HK \leq G$ by the **Lemma 1** below.

Now we have $HK \leq G$. Since $H \leq N_G(K)$ by assumption and $K \leq N_G(K)$ trivially, it follows that $HK \leq N_G(K)$, i.e., K is a normal subgroup of the subgroup HK.

Since K is normal in HK, the quotient group HK/K is well defined. Define the map $\varphi : H \to HK/K$ by $\varphi(h) = hK$. Since the group operation in HK/K is well defined itis easy to see that φ is a homomorphism:

$$\varphi(h_1h_2) = (h_1h_2)K = h_1K \cdot h_2K = \varphi(h_1)\varphi(h_2).$$

Alternatively, the map φ is just the restriction to the subgroup H of the natural projection homomorphism $\varphi : HK \to HK/K$, so is also a homomorphism. It is clear from the definition of HK that φ is surjective. The identity in HK/K is the coset $1K$, so the kernel of φ consists of the elements $h \in H$ with $hK = 1K$, which are the elements $h \in K$, i.e., $\ker \varphi = H \cap K$. By the Fundamental Homomorphism Theorem, $H \cap K \trianglelefteq H$ and $K/H \cap K \cong HK/K$. □

Lemma 1. *If H and K are subgroups of a group, HK is a subgroup if and only if $HK = KH$.*

Proof. Assume first that $HK = KH$ and let $a, b \in HK$. We prove $ab^{-1} \in HK$ so HK is a subgroup by the subgroup criterion. Let

$$a = h_1 k_1 \quad \text{and} \quad b = h_2 k_2,$$

for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Thus $b^{-1} = k_2^{-1} h_2^{-1}$, so $ab^{-1} = h_1 k_1 k_2^{-1} h_2^{-1}$. Let $k_3 = k_1 k_2^{-1} \in K$ and $h_3 = h_2^{-1}$. Thus $ab^{-1} = h_1 k_3 h_3$. Since $HK = KH$,

$$k_3 h_3 = h_4 k_4, \quad \text{for some } h_4 \in H,\ k_4 \in K.$$

Thus $ab^{-1} = h_1 h_4 k_4 \in HK$, as desired.

Conversely, assume that HK is a subgroup of G. Since $K \leq HK$ and $H \leq HK$, by the closure property of subgroups, $KH \subseteq HK$. To show the reverse containment let $hk \in HK$. Since HK is assumed to be a subgroup, write $hk = a^{-1}$, for some $a \in HK$. If $a = h_1 k_1$, then

$$hk = (h_1 k_1)^{-1} = k_1^{-1} h_1^{-1} \in KH.$$

□