MATH 4281 HOMEWORK 6 (SOLUTIONS)

10. Order of Group Elements

D6. If a is the only element of order k in G, then a is in the center of G.

Proof. Let $a \in G$ and suppose a is the only element of order k in G. Recall that the order of every element in a group is equal to its conjugate, so $\text{ord}(a) = \text{ord}(gag^{-1}) = k$ for all $g \in G$. Therefore $a = gag^{-1}$, and hence $ag = ga$ for all g. □

11. Cyclic Groups

E7. $\langle a \rangle \times \langle b \rangle$ is cyclic if and only if $\text{ord}(a)$ and $\text{ord}(b)$ are relatively prime.

Proof. Suppose $\text{ord}(a), \text{ord}(b)$ are not relatively prime, and let $\text{ord}(a) = m, \text{ord}(b) = n$. From parts 5 and 6, we have

$$\begin{aligned} \gcd(m,n) > 1 &\iff \text{lcm}(m,n) < mn \\ &\iff \text{every element of } \langle a \rangle \times \langle b \rangle \text{ has order less than } mn \\ &\iff \langle a \rangle \times \langle b \rangle \text{ is not cyclic.} \end{aligned}$$

□

E8. Let G be an abelian group of order mn, where m and n are relatively prime. Prove: If G has an element of a of order m and an element b of order n, then $G \cong \langle a \rangle \times \langle b \rangle$.

Proof. Let G be an abelian group with an element a of order m and an element b of order n. Recall that for any abelian group, we have $\text{ord}(a) \cdot \text{ord}(b) = \text{ord}(ab)$ for any element a, b. (This is not true for nonabelian groups in general.) Hence there is an element $ab \in G$ with order mn. Therefore G is a cyclic group of order mn, i.e., $G \cong \mathbb{Z}/mn\mathbb{Z}$. But since m, n are coprime, from the Chinese Remainder Theorem $G \cong \mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \langle a \rangle \times \langle b \rangle$. □

E9. Let $\langle a \rangle$ be a cyclic group of order mn, where m and n are relatively prime, and prove that $\langle a \rangle \cong \langle a^m \rangle \times \langle a^n \rangle$.

Proof. This is a direct consequence of $E8$ above. From the question above, we know that $\langle a \rangle \cong \langle b \rangle \times \langle c \rangle$, where $b, c \in \langle a \rangle$ are two elements with order n and m, respectively. Thus $\langle a \rangle \cong \langle a^m \rangle \times \langle a^n \rangle$. □