MATH 4707, HOMEWORK I, SOLUTIONS TO GRADED PROBLEMS

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(1) 1.8.21

Solution: Let us count all possible subsets Y of A such that $|B \cap Y| = 1$ There are k different ways to select one element of Y from B. The remaining elements of Y come from $A \setminus B$ and these elements can be chosen from any subset of $A \setminus B$. Since $A \setminus B$ has $2^{|A \setminus B|} = 2^{n-k}$ subsets, we have $k2^{n-k}$ such possible subsets as Y.

(2) 1.8.33

1st Solution: We first give a combinatorial proof. Notice that the left hand side of the given identity counts the number of k-subsets of an n set X. We'll show that the right hand side counts the same thing. Let Y be a fixed subset of X with $|Y| = 2$. k-subsets of X can be divided into three disjoint classes: a) k-subset has 0 common element with Y, b) k-subset has 1 common element with Y, or c) k-subset has two common elements with Y. First, we count k-subsets of X that are disjoint from Y. There are $\binom{n-2}{k}$ such subsets. Next, we count k-subsets of X that have a 1 common element with Y. There are 2 ways to select one element from Y, and there are $\binom{n-2}{k-1}$ ways to select the remaining $k-1$ elements from $X \setminus Y$. Thus, there are $2\binom{n-2}{k-1}$ subsets in class b). Finally, we count k-subsets of X that have 2 elements from Y. There are $\binom{n-2}{k-2}$ such k-subsets. We counted three disjoint classes of k-subsets whose sum gives us exactly the right hand side of the given identity.

2nd Solution: We give an algebraic proof. Let us verify that the left-hand side and right-hand side of the given identity are equal.

$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$

$\frac{n!}{k!(n-k)!} = \frac{(n-2)!}{k!(n-2-k)!} + 2\frac{(n-2)!}{(k-1)!(n-k-1)!} + \frac{(n-2)!}{(k-2)!(n-k)!}$

Next, we multiply both sides with $\frac{k!(n-k)!}{(n-2)!}$.

$n(n-1) = (n-k)(n-k-1) + 2k(n-k) + k(k-1)$

$n^2 - n = n^2 - 2kn + k^2 - n + k + 2kn - 2k^2 + k^2 - k$

$n^2 - n = n^2 - n.$

3rd Solution: Using Pascal's identity, we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} = (\binom{n-2}{k} + \binom{n-2}{k-1}) + (\binom{n-2}{k-1} + \binom{n-2}{k-2}) = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$

(3) 2.5.1

Solution (with induction):

Let $s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$. By experiment, we see that $s_n = 1 - \frac{1}{n+1}$. Let us denote this statement by $P(n)$.

$P(1)$ holds: $s_1 = \frac{1}{1\cdot 2} = 1 - \frac{1}{1+1}$

Next, assume that $P(n-1)$ holds for $n \geq 1$, and we will show that $P(n)$ holds.

$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = s_{n-1} + \frac{1}{n(n+1)} = 1 - \frac{1}{(n-1)+1} + \frac{1}{n(n+1)} = 1 - \frac{1}{n} + (\frac{1}{n} - \frac{1}{(n+1)}) = 1 - \frac{1}{(n+1)}$.

2nd Solution: $s_n = \sum_{1}^{n} \frac{1}{k(k+1)} = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + ... + (\frac{1}{n-1} - \frac{1}{n}) + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$.

(4) 2.5.2

1st Solution:

Let $a_n = \sum_{k=0}^{n} k\binom{n}{k}$. By experiment, we conjecture that $a_n = n2^{n-1}$. We give an algebraic proof below

$a_n = 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + ... + k \cdot \binom{n}{k} + ... + (n-1) \cdot \binom{n}{n-1} + n \cdot \binom{n}{n} = 1 \cdot \frac{n!}{1!(n-1)!} + 2 \cdot \frac{n!}{2!(n-2)!} + 3 \cdot \frac{n!}{3!(n-3)!} + ... + k \cdot \frac{n!}{k!(n-k)!} + ... + n \cdot \frac{n!}{n!(n-n)!} = \frac{n!}{0!(n-1)!} + \frac{n!}{1!(n-2)!} + \frac{n!}{2!(n-3)!} + ... + \frac{n!}{(k-1)!(n-k)!} + ... + \frac{n!}{(n-1)!(n-n)!} = n(\frac{(n-1)!}{0!(n-1)!} + \frac{(n-1)!}{1!(n-2)!} + \frac{(n-1)!}{2!(n-3)!} + ... + \frac{(n-1)!}{(k-1)!(n-k)!} + ... + \frac{(n-1)!}{(n-1)!0!}) = n2^{n-1}$.

2nd Solution: Let's now give a combinatorial proof. Each number $k \cdot \binom{n}{k}$ counts the number of k-subsets of an n-set X with a distinguished element, that is, the pairs (Y, y) such that $|Y| = k$ and y belongs to Y. If we choose Y first and then y, we get the left hand side of the identity. If we choose y first, then there are 2^{n-1} subsets that can contain y. We get the right hand side of the identity.

3rd Solution: By Binomial Theorem, we have $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$. Take derivative from both sides, and let $x = 1$.

(5) 3.8.8

Solution:

a) Using Pascal's identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $k = 1, \cdots, m$, we have $\sum_{k=0}^{m}(-1)^k\binom{n}{k} = \binom{n}{0} - (\binom{n-1}{1} + \binom{n-1}{0}) + (\binom{n-1}{2} + \binom{n-1}{1}) - (\binom{n-1}{3} + \binom{n-1}{2}) + \cdots + (-1)^{m-1}(\binom{n-1}{m-1} + \binom{n-1}{m-2}) + (-1)^m(\binom{n-1}{m} + \binom{n-1}{m-1}) = (-1)^m(\binom{n-1}{m}$

b) *1st Solution:*

$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m} = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{k!}{m!(k-m)!} = \frac{n!}{m!(n-m)!}(\sum_{k=0}^{n}\frac{(n-m)!}{(n-k)!(k-m)!}) = \binom{n}{m}2^{n-m}$.

2nd Solution: We give a combinatorial proof. Notice that the right hand side of the given identity counts the number of pairs of subsets (A, B) of an n-set X, where $|A| = m$, $A \subset B \subset X$. To get the right hand side, we first choose an m-set A of an n-set X. This can be done in $\binom{n}{m}$ ways. Next, we choose the rest of the elements of B from any subset of $A \setminus B$. There are $2^{|A/B|} = 2^{n-m}$ such subsets. To get the left hand side, we notice that the each number $\binom{n}{k}\binom{k}{m}$ counts the number of ways to choose k-subsets B of an n-set X and then choose an m-set A out of that k-subset. In conclusion, we counted the number of the pairs (A, B) as above in two different ways obtaining the right and the left hand side of the equality, which proves that the given identity.

(6) 3.8.12

1st Solution: By Binomial Theorem, we have $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$. To get the given identity, let $x = 1$ and $y = 2$.

2nd Solution We give a combinatorial proof. There are 3^n trinary strings with length n. Partition these strings into disjoint groups depending on the number of 1s in the string: 0 ones, 1 one, 2 ones, 3 ones, ... n ones. The number of length n strings with exactly $n-k$ ones is $\binom{n}{n-k}2^k = \binom{n}{k}2^k$.