Name: ______________________________

Signature: ______________________________

Section and TA: ______________________________

**Math 1271. Lecture 060 (V. Reiner) Midterm Exam III**
**Tuesday, November 24, 2009**

This is a 50 minute exam. No books, notes, calculators, cell phones or other electronic devices are allowed. There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

| Problem | Score |
|---|---|
| 1. | ______ |
| 2. | ______ |
| 3. | ______ |
| 4. | ______ |
| Total: | ______ |

**Problem 1.** *(25 points total) Compute the following.*

a. (5 points)

$$\int (e^{-t} + t^4)dt$$

b. (5 points)

$$\int_1^3 (e^{-t} + t^4)dt$$

c. (5 points) *The Riemann sum approximating* $\int_1^3 (e^{-t}+t^4)dt$, *using* **two** *equal length subintervals and taking* **right endpoints** *of the subintervals as sample points. Since you cannot use a calculator, leave numerical answers unevaluated.*

c. (5 points)

$$\frac{d}{dx}\int_1^x (e^{-t}+t^4)dt$$

d. (5 points)

$$\frac{d}{dx}\int_1^{x^{10}} (e^{-t}+t^4)dt$$

**Problem 2.** *(15 points) Find the area of the bounded region lying above the $x$-axis and below the graph $y = 100 - x^2$.*

**Problem 3.** *(30 points total) Compute the following limits. Indicate which limit rules or laws you are using.*

a. (10 points)

$$\lim_{x \to \pi} \frac{\sin(2x)}{\sin(7x)}$$

b. (10 points)

$$\lim_{x \to \pm\infty} xe^{-x^2}$$

c. (10 points)

$$\lim_{x \to 0} xe^{-x^2}$$

**Problem 4.** *(30 points) Let* $f(x) = xe^{-x^2}$.

a. (10 points) *Find all critical points* $(c, f(c))$ *of* $f(x)$, *and indicate whether they are local maxima, local minima, or neither.*

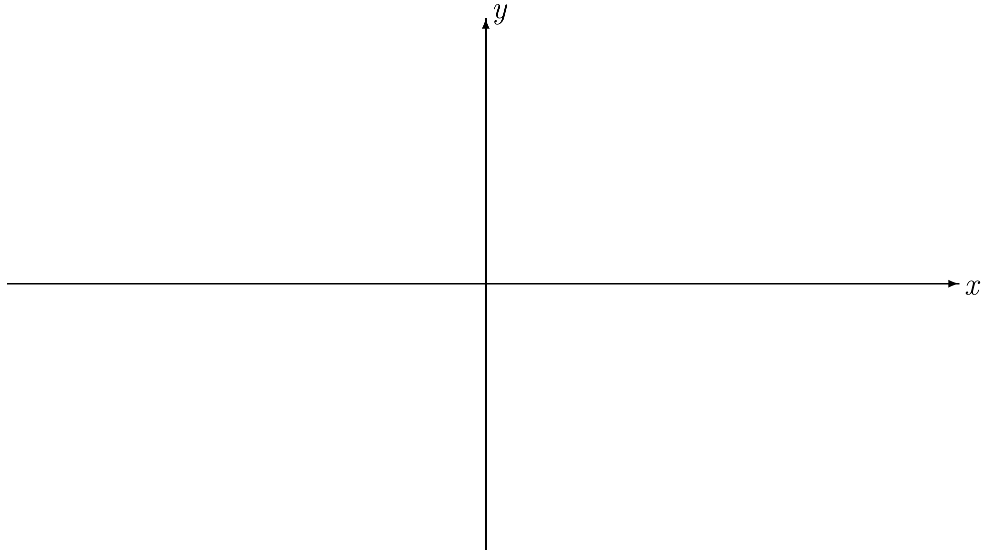


FIGURE 1. Axes for your sketch in part (c) of the graph $y = f(x) = xe^{-x^2}$

b. (10 points) *Find all inflection points* $(c, f(c))$ *of* $f(x)$.

c. (10 points) *On the axes above, give a rough sketch of the graph* $y = f(x) = xe^{-x^2}$*, indicating the features you discussed in parts (a),(b) of this problem, as well as those features found in Problem 3 parts (b),(c).*

**Brief solutions.**

1. a. (5 points)

$$\int (e^{-t}+t^4)dt = -e^{-t}+\frac{t^5}{5}$$

b. (5 points)

$$\int_1^3 (e^{-t}+t^4)dt = \left[-e^{-t}+\frac{t^5}{5}\right]_1^3 = -e^{-3}+\frac{3^5}{5}-(-e^{-1}+\frac{1^5}{5})$$

c. (5 points) The Riemann sum approximating $\int_1^3 (e^{-t}+t^4)dt$, using **two** equal length subintervals and taking **right endpoints** of the subintervals as sample points is

$$(2-1)(e^{-2}+2^4)+(3-2)(e^{-3}+3^4)$$

d. (5 points)

$$\frac{d}{dx}\int_1^x (e^{-t}+t^4)dt = (e^{-x}+x^4)$$

e. (5 points)

$$\frac{d}{dx}\int_1^{x^{10}} (e^{-t}+t^4)dt = (e^{-x^{10}}+(x^{10})^4)\cdot 10x^9$$

2. (15 points) The area of the bounded region lying above the $x$-axis and below the graph $y=100-x^2$ is

$$\int_{-10}^{10}(100-x^2)dx = \left[100x-\frac{x^3}{3}\right]_{-10}^{10} = (100\cdot 10-\frac{10^3}{3})-(100(-10)-\frac{(-10)^3}{3}).$$

3. (30 points total)
a. (10 points)

$$\lim_{x\to\pi}\frac{\sin(2x)}{\sin(7x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\pi}\frac{2\cos(2x)}{7\cos(7x)} \overset{\text{continuity of } \cos(2x),\cos(7x)}{=} \frac{2\cos(2\pi)}{7\cos(7\pi)} = \frac{2(1)}{7(-1)} = -\frac{2}{7}.$$

b. (10 points)]

$$\lim_{x\to\pm\infty} xe^{-x^2} = \lim_{x\to\pm\infty}\frac{x}{e^{x^2}} \overset{\text{L'Hôpital}}{=} \lim_{x\to\pm\infty}\frac{1}{2xe^{x^2}} = \frac{1}{\pm\infty} = 0.$$

c. (10 points)]

$$\lim_{x\to 0} xe^{-x^2} \overset{\text{product law}}{=} \lim_{x\to 0} x\cdot\lim_{x\to 0} e^{-x^2} \overset{\text{continuity of } x, e^{-x^2}}{=} 0\cdot e^{-0^2} = 0\cdot 1 = 0.$$

4. Let $f(x) = xe^{-x^2}$.
a. (10 points)] To find all critical points $(c, f(c))$ of $f(x)$, compute

$$f'(x) = 1 \cdot e^{-x^2} + x(-2x)e^{-x^2} = (1-2x^2)e^{-x^2}.$$

Since $e^{-x^2}$ is always positive, $f'(x) = 0$ if and only if $1 - 2x^2 = 0$, that is when $x = \pm\sqrt{\frac{1}{2}}$. Also note that $f'(x)$ is

- negative for $x < -\sqrt{\frac{1}{2}}$,
- positive for $-\sqrt{\frac{1}{2}} < x < +\sqrt{\frac{1}{2}}$,
- negative for $x > +\sqrt{\frac{1}{2}}$.

Hence at $x = -\sqrt{\frac{1}{2}}$ the function reaches a local minimum, and at $x = +\sqrt{\frac{1}{2}}$ a local maximum.
b. (10 points)] To find all inflection points $(c, f(c))$ of $f(x)$, compute

$$\begin{aligned} f''(x) &= -4xe^{-x^2} + (1-2x^2)(-2x)e^{-x^2}(-4x - 2x + 4x^3)e^{-x^2} \\ &= (-6x + 4x^3)e^{-x^2} \\ &= 2x(2x^2 - 3)e^{-x^2} \end{aligned}$$

Since $e^{-x^2} > 0$, one has $f''(x) = 0$ if and only if $2x(2x^2 - 3) = 0$, that is when $x = 0, \pm\sqrt{\frac{3}{2}}$. Also note that $f''(x)$ is

- negative for $x < -\sqrt{\frac{3}{2}}$,
- positive for $-\sqrt{\frac{3}{2}} < x < 0$,
- negative for $0 < x < +\sqrt{\frac{3}{2}}$,
- and negative for $x > +\sqrt{\frac{3}{2}}$.

Thus there is an inflection point above each of the three $x$-values $x = 0, \pm\sqrt{\frac{3}{2}}$.
c. (10 points) Here is what Maple's plotter gives:

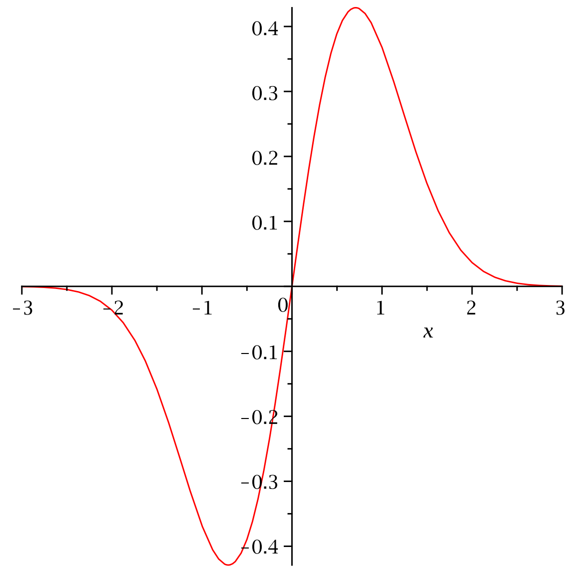


FIGURE 2. The graph of $y = f(x) = xe^{-x^2}$.