MATH 4707 FINAL with SOLUTIONS

DUE: May 10, 2010

INSTRUCTOR: Anar Akhmedov

Name: ______________________

Signature: ______________________

ID #: ______________________

Instructions:

Take home final is due on Monday, May 10th, during my office hours (2.00pm - 5.00pm). My office is in Vincent Hall, room 355. Late submission will not be accepted. You are NOT allowed to work on this with anyone else. If you have any questions, send me an email at: akhmedov@math.umn.edu.

Show all of your work. No credit will be given for an answer without some work or explanation.

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
Total (150 points)	

1. (20 points) Show that a finite simple graph with more than one vertex has at least two vertices with the same degree.

   Solution: Let G be any finite simple graph with more than one vertex and $|V_G| = n \geq 2$. First, we notice that the maximal degree of any vertex in G is less than equal $n-1$. Also, if our graph G is not connected, then the maximal degree is strictly less than $n-1$.

   Case 1: Assume that G is connected. We can not have a vertex of degree 0 in G, so the set of vertex degrees is a subset of $S = \{1, 2, \cdots, n-1\}$. Since the graph G has n vertices, by pigeon-hole principle we can find two vertices of the same degree in G.

   Case 2: Assume that G is not connected. G has no vertex of degree $n-1$, so the set of vertex degrees is a subset of $S' = \{0, 1, 2, \cdots, n-2\}$. By pigeon-hole principle again, we can find two vertices of the same degree in G.

2. (20 points) Find all values of $n \leq 2010$ such that $\phi(n) = 1000$.

Solution. $1000 = 2^3 5^3 = n(1-1/p_1)(1-1/p_2)(1-1/p_3)\cdots(1-1/p_k)$, where p_1, p_2, $\cdots$, p_k are the prime divisors of n. We can assume that p_i is odd for $2 \leq i \leq k$. Using the fact that $p_i - 1$ is an even divisors of 1000 for $i \geq 2$, we find the following possible values for $p_i - 1$: $2, 4, 8, 10, 20, 40, 50, 100, 200, 250, 500, 1000$. Since p_i is prime, we reduce the possible values for p_i : $3, 5, 11, 41, 101, 251$. If p_1 is not odd, then $p_1 = 2$. Now use the fact that $n \leq 2010$, to conclude that there are five such numbers: $1111 = 11 \times 101$, $1255 = 5 \times 251$, $1375 = 5^3 \times 11$, $1875 = 3 \times 5^4$, and $2008 = 2^3 \times 251$.

3. (20 points) Prove the following two identities

a) $\sum_{k=0}^{n} \binom{n}{k} F_{k-1} = F_{2n-1}$

b) $\sum_{k=0}^{n} \binom{n}{k} F_{3k-1} = 2^n F_{2n-1}$

Solution: We will use Binet's formula for the Fibonacci numbers and Binomial Theorem to prove a) and b). We have $F_n = \frac{1}{a-b}(a^n - b^n)$, where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Since a and b are solutions of the equation $x^2 - x - 1 = 0$, we have the following identities hold: $a^2 = a+1$, $b^2 = b+1$. Using these identities, we have $a^3 = a^2 a = (a+1)a = a^2 + a = 2a+1$, and $b^3 = b^2 b = (b+1)b = b^2 + b = 2b+1$.

a) $\binom{n}{0} F_{-1} + \binom{n}{1} F_0 + \binom{n}{2} F_1 + \cdots + \binom{n}{n} F_{n-1} = \binom{n}{0} \frac{1}{a-b}(a^{-1} - b^{-1}) + \binom{n}{1} \frac{1}{a-b}(a^0 - b^0) + \binom{n}{2} \frac{1}{a-b}(a^1 - b^1) + \cdots + \binom{n}{n} \frac{1}{a-b}(a^{n-1} - b^{n-1}) = \frac{1}{a-b}(\frac{1}{a}(\binom{n}{0} a^0 + \binom{n}{1} a^1 + \binom{n}{2} a^2 + \cdots + \binom{n}{n} a^n) - \frac{1}{b}(\binom{n}{0} b^0 + \binom{n}{1} b^1 + \binom{n}{2} b^2 + \cdots + \binom{n}{n} b^n)) = \frac{1}{a-b}(\frac{1}{a}(a+1)^n - \frac{1}{b}(b+1)^n) = \frac{1}{a-b}(\frac{1}{a}(a^2)^n - \frac{1}{b}(b^2)^n) = \frac{1}{a-b}(a^{2n-1} - b^{2n-1}) = F_{2n-1}$.

b) $\binom{n}{0} F_{-1} + \binom{n}{1} F_2 + \binom{n}{2} F_5 + \cdots + \binom{n}{n} F_{3n-1} = \binom{n}{0} \frac{1}{a-b}(a^{-1} - b^{-1}) + \binom{n}{1} \frac{1}{a-b}(a^2 - b^2) + \binom{n}{2} \frac{1}{a-b}(a^5 - b^5) + \cdots + \binom{n}{n} \frac{1}{a-b}(a^{3n-1} - b^{3n-1}) = \frac{1}{a-b}(\frac{1}{a}(\binom{n}{0} a^0 + \binom{n}{1} a^3 + \binom{n}{2} a^6 + \cdots + \binom{n}{n} a^{3n}) - \frac{1}{b}(\binom{n}{0} b^0 + \binom{n}{1} b^3 + \binom{n}{2} b^6 + \cdots + \binom{n}{n} b^{3n})) = \frac{1}{a-b}(\frac{1}{a}(a^3+1)^n - \frac{1}{b}(b^3+1)^n) = \frac{1}{a-b}(\frac{1}{a}(2a+2)^n - \frac{1}{b}(2b+2)^n) = \frac{1}{a-b}(\frac{1}{a}(2a^2)^n - \frac{1}{b}(2b^2)^n) = \frac{1}{a-b}(2^n a^{2n-1} - 2^n b^{2n-1}) = 2^n F_{2n-1}$.

4. (20 points) The number N is said to be *perfect* if $\sigma(N) = 2N$. Show that if $2^n - 1$ is prime, then $N = 2^{n-1}(2^n - 1)$ is perfect.

   Solution: Since $p = 2^n - 1$ is prime, all the divisors of $N = 2^{n-1}(2^n - 1)$ are as follows: 1, 2, 4, $\cdots$, 2^n, p, $2p$, $4p$, $\cdots$, $2^n p$. We have $\sigma(N) = (1+2+4+\cdots+2^n)+(p+2p+4p+\cdots+2^{n-1}p) = (2^n - 1)+p(2^n - 1) = (2^n - 1)(p+1) = (2^n - 1)2^n = 2N$. Thus, N is perfect.

   Alternatively, using the facts that σ is a multicplicative function, $\sigma(p) = p+1 = 2^n$, and $\sigma(2^{n-1} - 1) = (2^{(n-1)+1} - 1)/(2-1) = 2^n - 1$, we have $\sigma(N) = \sigma(2^n - 1)\sigma(p) = (2^n - 1)2^n = 2N$.

   Remark: The converse of this statement also holds. If N is an even perfect number, then $N = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime. Try to prove this fact.

5. (20 points) The *girth* of a graph is the length of the smallest polygon in the graph. Let G be a graph with girth 5 for which all vertices have degree $\geq d$. Show that G has at least $d^2 + 1$ vertices.

   Solution. Let fix a vertex v of G. Since each vertex of G has degree $\geq d$, there are at least d vertices v_1, v_2, ..., and v_d with distance 1 from v. Since the girth of G is 5, G has no 3 or 4-cycles. Using this fact and the vertices v_1, v_2, ..., v_d, we can construct at least $d(d-1)$ new vertices with distance two from v. We choose $d-1$ distance 1 vertices $v_{i1}, v_{i2}, \cdots v_{i(d-1)}$ from each vertex v_i (different than v) for $1 \leq i \leq d$. These new vertices have distance 2 from v. Thus, $|V_G| \geq 1 + d + d(d-1) = d^2 + 1$.

6. (15 points) Compute the chromatic polynomial of C_n, a cycle graph of lenght n.

Solution: We'll give an inductive proof. We'll denote a path of length n by P_n. First, we recall that the chromatic polynomial of any tree T_n with n vertices is $P_{T_n}(\lambda) = \lambda(\lambda-1)^{n-1}$. Since C_2 has only two vertices (and connected), it's chromatic polynomial is given by the degree two polynomial $P_{C_2}(\lambda) = \lambda(\lambda-1) = (\lambda-1)^2+(\lambda-1) = (\lambda-1)^2+(-1)^2(\lambda-1)$. Using the reduction formula for the chromatic polynomial, we have $P_{C_3}(\lambda) = P_{P_3}(\lambda) - P_{C_2}(\lambda) = \lambda(\lambda-1)^2 - \lambda(\lambda-1) = (\lambda-1)^3 - (\lambda-1) = (\lambda-1)^3 + (-1)^3(\lambda-1)$. Now assume that the formula $P_{C_k}(\lambda) = (\lambda-1)^k + (-1)^k(\lambda-1)$ holds for all $k \leq n-1$. We'll show that it holds for $k = n$ as well. $P_{C_n}(\lambda) = P_{P_n}(\lambda) - P_{C_{n-1}}(\lambda) = \lambda(\lambda-1)^{n-1} - ((\lambda-1)^{n-1} + (-1)^{n-1}(\lambda-1)) = (\lambda-1)^n + (-1)^n(\lambda-1)$.

7. (15 points) Let G be a complete graph on 17 vertices in which edge is coloured either red, blue or green. Show that G contains at least two monochromatic triangles.

Solution: We'll use the following fact that we proved in class: if the edges of the complete graph K_6 are coloured using two colors (say red or blue), then there will be at least two monochromatic triangles. We proved this result in class by counting the number of bi-chromatic triangles. There are $\binom{6}{3} = 20$ triangles in K_6 and at most 18 of them are bi-chromatic. We can also use the following more general theorem that we proved in class

Theorem: If the edges of K_n are colored red or blue, and r_i $i = 1, 2, \cdots, n$ denotes the number of red edges with vertex i as an endpoint, and if Δ denotes the number of monochromatic triangles, then $\Delta = \binom{n}{3} - 1/2 \sum_{i=1}^{n} r_i(n-1-r_i)$.

Let choose an arbitrary vertex v of the graph K_{17}. We have 16 edges incident to the vertex v, so by pigeonhole principle we have at least 6 edges of the same color (say green) from the vertex v. Let us denote the other endpoints of these edges by v_1, v_2, v_3, v_4, v_5, and v_6.

Case 1: If none of the edges connecting the vertices v_1, v_2, v_3, v_4, v_5, and v_6 are green, then we have 2-coloured K_6. We use the fact above to get two monochromatic triangles.

Case 2: If there are at least two green edges among the edges connecting the vertices v_1, v_2, v_3, v_4, v_5, and v_6, then we get at least two green triangles. Use the end points of the green edges and the vertex v.

Case 3: If there is only one green edge among the edges connecting the vertices v_1, v_2, v_3, v_4, v_5, and v_6, then we have one green triangle. Next, we apply the same argument that we used for the vertex v to other vertex w . By our careful choice of w (here we assume that w is not a vertex of the green triangle), we make sure that a new triangle is different than our first triangle.

Remark:. Try to generalize above Theorem for K_n with edges are colored red, blue and green. What is the minimal number of monochromatic triangles for K_{17}?

8. (20 points) Prove that the bipartite graph $K_{3,4}$ is not a planar graph. Show that $K_{3,4}$ can be drawn on the torus (the surface of a doughnut) without any crossings.

   Solution: First, recall that if a graph G is planar and has no 3-cycles, then $e_G \leq 2v_G - 4$. The bipartite graph $K_{3,4}$ has 7 vertices, 12 edges, and no 3 cycles. $K_{3,4}$ can not be a planar graph as it violates the inequality $e_G \leq 2v_G - 4$. The genus of the complete bipartite graph $K_{m,n}$ is given by $g(K_{m,n}) = \lceil (m-2)(n-2)/4 \rceil$. Using this formula, we can compute the genus: $g(K_{3,4}) = \lceil (3-1)(4-1)/4 \rceil = 1$.

   ADD: embedding.eps